Extremal mappings for the Schwarz lemma

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Introduction

Part of the classical Schwarz lemma states that if $f\colon D\to D$ is a holomorphic mapping of the open unit disc D in the complex plane $\mathbf{C}$ to itself, and if $f(0)=0$, then

(i) $|f'(0)|\leqq 1$,

(ii) $|f'(0)|=1$ if and only if f has the form $f(z)=\lambda z$ for some constant λ with $|\lambda|=1$.

There are now many extensions of this result to higher dimensions (see Dineen [5]). An early result was the following one due to Carathéodory [3] and Cartan [4].

Theorem 0.1. *If $\mathscr{D}$ is a bounded domain in $\mathbf{C}^n$, if $f\colon \mathscr{D}\to\mathscr{D}$ is holomorphic and if $f(a)=a$ (a, a point in $\mathscr{D}$), then*

(i) $|\det f'(a)|\leqq 1$,

(ii) *the following are equivalent:*

(a) $|\det f'(a)|=1$;

(b) *the eigenvalues of $f'(a)$ have modulus* 1;

(c) *f is a biholomorphic automorphism of $\mathscr{D}$.*

A similar result has been obtained for holomorphic mappings from bounded symmetric domains into the (Euclidean) ball by Kubota [10, 12] and Travaglini [18]. Our purpose is to generalize these results as well as results of Alexander [1] and Lempert [13]. Our methods lead us to connections with ideas in the geometry of (finite dimensional) Banach spaces, such as the Banach—Mazur distance and minimal volume ellipsoids. A central rôle is played by the existence of a unique invariant inner product on the spaces we consider.

1. Invariant Inner Products

All Banach spaces we consider will be over the complex field $\mathbf{C}$, unless otherwise stated. The open unit ball of a Banach space X will be denoted by B_X.

Definition 1.1. An inner product $\langle \cdot , \cdot \rangle$ on a Banach space X is called *invariant* if $\langle Sz, Sw\rangle = \langle z, w\rangle$ for every isometric isomorphism $S: X \to X$.

Proposition 1.2. *Every finite dimensional Banach space X admits an invariant inner product.*

Proof. We denote by $I(X)$ the (compact Lie) group of linear isometries of X and by μ normalized Haar measure on $I(X)$. Let $\langle \cdot , \cdot \rangle$ denote some inner product on X and define

$$\langle x, y\rangle_i = \int_{I(X)} \langle Sx, Sy\rangle \, d\mu(S) \quad (x, y \in X) \tag{1.1}$$

It is easy to check that $\langle \cdot , \cdot \rangle_i$ is an invariant inner product on X. ▮

Definition 1.3. A Banach space X is said to have the *unique invariant inner product property* if there exists an invariant inner product $\langle \cdot , \cdot \rangle$ on X such that
(i) $\langle \cdot , \cdot \rangle$ generates the topology of X, and
(ii) if $\langle \cdot , \cdot \rangle_1$ is any invariant inner product on X with property (i), then there is a constant $c>0$ such that

$$\langle x, y\rangle_1 = c\langle x, y\rangle.$$

(Of course condition (i) is automatically satisfied if X is finite-dimensional.)

Example 1.4. ([6]) If X is a finite rank JB^*-triple (finite- or infinite-dimensional) which is irreducible (that is not expressible as a nontrivial direct sum $X_1 \oplus_\infty X_2$), then every invariant inner product on X is a multiple of the algebraic inner product of Harris.

Definition 1.5. Let $e_1, e_2, \ldots, e_n$ be a basis for a finite-dimensional normed space E. The basis is called *1-unconditional* if

$$\left\|\sum_{j=1}^n \lambda_j a_j e_j\right\| = \left\|\sum_{j=1}^n a_j e_j\right\|$$

holds for all choices of scalars a_j and λ_j satisfying $|\lambda_j| = 1$ $(1 \leq j \leq n)$.

Given a norm $\|\cdot\|_E$ on $\mathbf{C}^n$ for which the standard basis is 1-unconditional and normed spaces X_j $(1 \leq j \leq n)$ we can define a norm on $X = \bigoplus_{j=1}^n X_j$ by

$$\|(x_1, x_2, \ldots, x_n)\| = \big\|(\|x_1\|, \|x_2\|, \ldots, \|x_n\|)\big\|_E.$$

We refer to this as an *absolute norm* on the direct sum and say that X has an *absolutely normed direct sum decomposition.*

Proposition 1.6. *Suppose a Banach space X has an absolutely normed direct sum decomposition $\bigoplus_{j=1}^{n} X_j$ such that each X_j has the unique invariant inner product property. Let $\langle\cdot,\cdot\rangle_j$ denote an invariant inner product on X_j. Then every invariant inner product on X has the form*

$$(1.2)\qquad \langle(x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n)\rangle = \sum_{j=1}^{n} c_j \langle x_j, y_j\rangle_j$$

for some $c_j > 0$ $(1 \leq j \leq n)$.

Proof. Fix $1 \leq j \leq n$. Let T_θ denote the isometry of X which multiplies the jth coordinate by $e^{i\theta}$. Invariance of the inner product implies that, for $k \neq j$,

$$\langle x_j + x_k, x_j + x_k\rangle = \langle e^{i\theta} x_j + x_k, e^{i\theta} x_j + x_k\rangle$$

from which $\langle x_j, x_k\rangle = 0$ follows easily.

Because the norm on X is absolute, every isometry S of X_j extends to an isometry T of X by

$$T(x_1, x_2, \dots, x_n) = (x_1, \dots, x_{j-1}, S(x_j), x_{j+1}, \dots, x_n).$$

From this and the unique invariant inner product property of X_j, it follows that

$$\langle x_j, x_j\rangle = c_j \langle x_j, x_j\rangle_j$$

for some $c_j > 0$.

The result follows by combining these two observations. ∎

Proposition 1.7. *Let X be a finite-dimensional Banach space with a symmetric basis $(e_j)_{j=1}^{n}$ — that is one satisfying*

$$\left\|\sum_{j=1}^{n} \lambda_j a_j e_{\pi(j)}\right\| = \left\|\sum_{j=1}^{n} a_j e_j\right\|$$

for all permutations π of the indices $1, 2, \dots, n$ and all scalars a_j, λ_j with $|\lambda_j| = 1$ $(1 \leq j \leq n)$. Then X has the unique invariant inner product property and

$$\left\langle \sum_{j=1}^{n} a_j e_j, \sum_{j=1}^{n} b_j e_j\right\rangle = \sum_{j=1}^{n} a_j \bar{b}_j$$

is an invariant inner product on X.

Proof. The same argument as that used for the proof of proposition 1.6 shows that an invariant inner product on a space with an unconditional basis must have the form

$$\left\langle \sum_{j=1}^{n} a_j e_j, \sum_{j=1}^{n} b_j e_j\right\rangle = \sum_{j=1}^{n} c_j a_j \bar{b}_j.$$

Invariance of the inner product under permutations of the indices forces $c_1 = c_2 = \dots = c_n$. ∎

Examples 1.8.

(i) If a Banach space X has the unique invariant inner product property, then X is necessarily reflexive and it is not hard to check that its dual space X' must also have the same property.

(ii) Let X denote the space of all $n\times m$ complex matrices with a norm satisfying $\|Z\|=\|UZV\|$ for all $n\times n$ unitary matrices U and all $m\times m$ unitary V. Such a norm is called an *ideal norm* and the normed space is called an *operator ideal.*

Then X has the unique invariant inner product property and an invariant inner product is given by

$$\langle Z, W\rangle = \sum_{j,k} z_{jk}\bar{w}_{jk}.$$

2. The Banach—Mazur Distance

The Banach—Mazur distance between two isomorphic Banach spaces X and Y is defined to be

$$d(X, Y) = \inf\{\|T\|\,\|T^{-1}\|\colon T\colon X\to Y \text{ a linear isomorphism}\}.$$

A more geometric description follows from the observation that if $T\colon X\to Y$ is a linear isomorphism between Banach spaces X and Y (and $\varrho>0$), then the following are equivalent:

$$\|T\| \leqq 1 \quad\text{and}\quad \|T^{-1}\| \leqq \varrho \tag{2.1}$$

$$T(B_X) \subset B_Y \subset \varrho T(B_X) \tag{2.2}$$

$$\frac{1}{\varrho} B_Y \subset T(B_X) \subset B_Y \tag{2.3}$$

$$\frac{1}{\varrho}\|x\| \leqq \|Tx\| \leqq \|x\|. \tag{2.4}$$

Hence

$$d(X, Y) = \inf\{\varrho \geqq 1\colon (1/\varrho)B_Y \subset T(B_X) \subset B_Y,\ T\colon X\to Y \text{ a linear isomorphism}\}.$$

From the definition, it is clear that $d(X, Y)=d(Y, X)$. The infimum (in the definition) is certainly attained by some T when X and Y are finite-dimensional. We use the notation $e(X)$ to denote the Banach—Mazur distance from X to an isomorphic Hilbert space (e for eccentricity). We let $e(X)=\infty$ if X is not isomorphic to a Hilbert space.

Proposition 2.1. *If X is a finite-dimensional Banach space, then there exists an invariant inner product $\langle\cdot,\cdot\rangle$ on X which satisfies*

$$\frac{1}{e(X)^2}\|x\|^2 \leqq \langle x, x\rangle \leqq \|x\|^2 \quad (x\in X). \tag{2.5}$$

Proof. Let $T\colon X\to H$ be an isomorphism from X to a Hilbert space H (with the same dimension as X) which satisfies $\|T\|=1$ and $\|T^{-1}\|=e(X)$. The inner product $\langle\cdot,\cdot\rangle$ defined on X by

$$\langle x, y\rangle = \langle T(x), T(y)\rangle_H$$

satisfies

$$\frac{1}{e(X)^2}\|x\|^2 = \frac{1}{e(X)^2}\|T^{-1}\circ Tx\|^2$$

$$\leqq \frac{1}{e(X)^2}\|T^{-1}\|^2\|Tx\|_H^2 = \|Tx\|_H^2 = \langle x, x\rangle \leqq \|x\|^2.$$

Now the averaging process (1.1) produces an invariant inner product $\langle\cdot,\cdot\rangle_i$ on X and it is not hard to check that

$$\frac{1}{e(X)^2}\|x\|^2 \leqq \langle x, x\rangle_i \leqq \|x\|^2. \quad ∎$$

Corollary 2.2. *If X is a finite-dimensional Banach space with the unique invariant inner product property and $\langle\cdot,\cdot\rangle$ is any invariant inner product on X, then*

$$e(X)^2 = \sup_{\|x\|=1}\langle x, x\rangle \Big/ \inf_{\|x\|=1}\langle x, x\rangle,$$

and there is a unique invariant inner product on X satisfying (2.5).

Examples 2.3.

(i) For $X=\ell_p^n$ (that is $\mathbf{C}^n$ with the norm $\|(x_j)_j\|_p=(\sum_j |x_j|^p)^{1/p}$, $1\leqq p<\infty$ and $\|(x_j)_j\|_\infty=\sup_j|x_j|$) proposition 1.7. and corollary 2.2 easily imply the well-known result that

$$e(\ell_p^n) = n^{|1/2-1/p|} \quad (1\leqq p\leqq\infty).$$

(ii) Let E denote $\mathbf{C}^n$ with a symmetric norm and let X be a finite-dimensional Banach space with the unique invariant inner product property. Let $E(X)$ denote the direct sum of n copies of X, absolutely normed by E. From proposition 1.6 and the fact that permutations of the coordinates give isometries of $E(X)$, it follows that $E(X)$ has the unique invariant inner product property. Then we can calculate (using corollary 2.2) that

$$e\big(E(X)\big) = e(E)\,e(X).$$

(iii) Let $X_1, X_2, \ldots, X_n$ denote finite-dimensional Banach spaces with the unique invariant inner product property and suppose $1 \leqq p < \infty$. Using propositions 1.6 and 2.1 and Lagrange multipliers, one can show that

$$e(X_1 \oplus_p X_2 \oplus_p \ldots \oplus_p X_n)^2 = \left\| (e(X_1)^2, e(X_2)^2, \ldots, e(X_n)^2) \right\|_r$$

where $1/r = |1-2/p|$. In particular, if $p \neq 2$,

$$e(\ell_q^n \oplus_p \ell_s^m)^2 = (n^{p/q \cdot |q-2|/|p-2|} + m^{p/s \cdot |s-2|/|p-2|})^{|p-2|/p}.$$

Example 2.4. If X is a JB^*-triple of rank r, then the algebraic inner product (see [6] and example 1.4) satisfies

$$\|x\|^2 \leqq \langle x, a \rangle_a \leqq r \|x\|^2$$

and equality occurs in both inequalities for nonzero values of x. A rescaling of (2.4) shows that $e(X) \leqq \sqrt{r}$. Since X contains ℓ_∞^r isometrically (see [6]) we have, by example 2.3(i), $e(X) \geqq e(\ell_\infty^r) \geqq \sqrt{r}$. Hence $e(X) = \sqrt{r}$.

3. Determinants and Ellipsoids

Let X and Y be Banach spaces of the same finite dimension n. Let $(e_j)_{j=1}^n$ and $(f_n')_{j=1}^n$ denote fixed bases for X and Y'. Then the *determinant* of a linear operator $T: X \to Y$ is defined as the determinant of the $n \times n$ matrix $(f_k'(Te_j))_{jk}$. Changing the bases changes the value of the determinant $\det(T)$ by a multiplicative factor independent of T.

Remarks 3.1.

(i) Let $(e_j')_j$ be the biorthogonal basis for X' (that is, $e_j'(e_k) = 0$ if $j \neq k$ and $e_j'(e_j) = 1$). If $S: X \to X$ is an operator and its determinant is taken with respect to $(e_j)_j$ and $(e_j')_j$, then

$$\det TS = \det T \det S.$$

(ii) If S is an isometry of X, then $\|S^n\| = 1$ for all $n \in \mathbf{Z}$. Since the unit ball of the space of operators on X is compact and the determinant is a continuous function, we can conclude

$$\sup_{n \in \mathbf{Z}} |\det(S^n)| = \sup_{n \in \mathbf{Z}} |\det(S)|^n < \infty.$$

Hence $|\det(S)| = 1$.

(iii) For mappings $T: H \to Y$ with domain a Hilbert space, we will generally choose an orthonormal basis for H when taking determinants. Using a different orthonormal basis would not affect the absolute value $|\det(T)|$. Similar remarks apply when the range is a Hilbert space.

The following proposition is due to John [9]. Generalizations are given by Lewis [14] (see also [16]).

Proposition 3.2. *Let X be an n-dimensional Banach space and let H be an n-dimensional Hilbert space. Then there exists an operator T_0: $X \to H$ such that $\|T_0\| = 1$ and*

$$|\det(T_0)| = \sup\{|\det(T)|: T: X \to H \text{ linear}, \|T\| \leqq 1\}.$$

The operator T_0 has this property for any choice of determinant function — that is for any choice of basis in X.

Moreover, if T: $X \to H$ has $|\det(T)| = |\det(T_0)|$ and $\|T\| \leqq 1$, then $T = UT_0$ for some unitary operator U on H.

Remark 3.3. We may use the chosen basis in X to transfer the standard volume on $\mathbf{C}^n = \mathbf{R}^{2n}$ to X. Then the ellipsoid $T_0^{-1}(B_H)$ has volume $|\det(T_0)|^{-2}\operatorname{Vol}(B_H)$. (This is just the change of variables formula, taking into account that the matrix of T_0^{-1} is a matrix with respect to a basis over $\mathbf{C}$.) It follows that $T_0^{-1}(B_H)$ is an ellipsoid in X containing B_X of minimal volume. The conclusion of proposition 3.2 means that there is a unique *minimal volume ellipsoid* containing B_X.

Corollary 3.4. *With the notation of proposition* 3.2, *the inner product on X*

$$\langle x, y\rangle = \langle T_0 x, T_0 y\rangle_H$$

is an invariant inner product.

Moreover, $\{x \in X: \langle x, x\rangle \leqq 1\}$ is the minimal volume ellipsoid containing B_X,

$$\langle x, x\rangle \leqq \|x\|^2 \quad (x \in X), \tag{3.1}$$

and

$$\{x \in X: \|x\|^2 = \langle x, x\rangle = 1\} \neq \emptyset. \tag{3.2}$$

Proof. If S: $X \to X$ is any isometry of X, then $T_0 S$ has norm 1 and $|\det(T_0 S)| = |\det(T_0)|$ by remark 3.1. Hence $T_0 S = UT_0$ for some unitary operator U on H, and

$$\langle x, y\rangle = \langle T_0 x, T_0 y\rangle_H = \langle UT_0 x, UT_0 y\rangle_H$$

$$= \langle T_0 Sx, T_0 Sy\rangle_H = \langle Sx, Sy\rangle. \quad ∎$$

Remark 3.5. We will use the notation $C(X)$ for the set in (3.2), as this is the set of *contact points* between the unit sphere of X and the surface of the minimal volume ellipsoid containing B_X. It is known that $C(X)$ spans X in general (see [14]).

The dual versions of proposition 3.2 and corollary 3.4 are also valid and we have the following result.

Proposition 3.6. *Let X be an n-dimensional normed space and H an n-dimensional Hilbert space. If T_1: $H \to X$ satisfies $\|T_1\|=1$ and*

$$|\det(T_1)| = \sup\{|\det(T)|: T: H \to X \text{ linear}, \|T\| \leq 1\}$$

then the inner product

$$\langle x, y\rangle = \langle T_1^{-1}x, T_1^{-1}y\rangle_H$$

is an invariant inner product on X.

Moreover $\{x \in X: \langle x, x\rangle < 1\}$ is the maximal volume ellipsoid contained in B_X,

$$\|x\|^2 \leq \langle x, x\rangle \quad (x \in X), \tag{3.3}$$

and

$$\{x \in X: \|x\|^2 = \langle x, x\rangle = 1\} \neq \emptyset. \tag{3.4}$$

We now consider these results when X has the unique invariant inner product property. Let $\langle\cdot,\cdot\rangle_1$, $\langle\cdot,\cdot\rangle_2$ and $\langle\cdot,\cdot\rangle_3$ denote the invariant inner products considered in proposition 2.1, corollary 3.4 and proposition 3.6 respectively. In view of the inequalities satisfied by these inner products (which are sharp in the sense that equality holds for some nonzero $x \in X$) it is easy to see that

$$\langle x, x\rangle_1 = \langle x, x\rangle_2 = \frac{1}{e(X)^2}\langle x, x\rangle_3 \quad (x \in X).$$

Let $\mathcal{E} = \{x \in X: \langle x, x\rangle_1 \leq 1\}$. Then $\mathcal{E}$ is the minimal volume ellipsoid containing B_X and $(1/e(X))\mathcal{E}$ is the maximal volume ellipsoid contained in $\bar{B}_X$. The fact that the maximal and minimal volume ellipsoids are dilations of one another is a consequence of the unique invariant inner product property, and is not true in general finite-dimensional spaces (see corollary 3.10).

A further simplification occurs as a result of the unique invariant inner product property — the Banach—Mazur distance $e(X)$ from X to a Hilbert space is uniquely achieved (modulo rescaling and a unitary change of variables on the Hilbert space) by the same operator which realises the minimal (or maximal) volume ellipsoid. (This strengthens corollary 2.2.)

Theorem 3.7. *Let X be a finite-dimensional Banach space with the unique invariant inner product property and let T: $X \to H$ be a linear operator from X to a Hilbert space H with the same dimension as X. Suppose $\|T\|=1$ and $\|T^{-1}\|=e(X)$. Then there exists a unitary operator U on H such that $T=UT_0$ and $T^{-1}(B_H)$ is the minimal volume ellipsoid containing B_H.*

Proof. Let $\langle x, y\rangle = \langle Tx, Ty\rangle_H$ for $x, y \in X$. By (2.4),

$$\frac{1}{e(X)^2}\|x\|^2 \leqq \|Tx\|_H^2 = \langle x, x\rangle \leqq \|x\|^2 \quad (x\in X).$$

If $\langle\cdot,\cdot\rangle_i$ is the invariant inner product got from $\langle\cdot,\cdot\rangle$ as in (1.1), then

$$\frac{1}{e(X)^2}\|x\|^2 \leqq \langle x, x\rangle_i \leqq \|x\|^2 \quad (x\in X).$$

As in the comments preceding the theorem, $\mathcal{E}_i = \{x\in X\colon \langle x, x\rangle_i \leqq 1\}$ is the minimal volume ellipsoid containing B_X and $(1/e(X))\mathcal{E}_i$ is the maximal volume ellipsoid contained in $\bar{B}_X$.

Let $\mathcal{E} = \{x\in X\colon \langle x, x\rangle \leqq 1\}$. By (2.2), $B_X \subset T^{-1}(B_H) = \mathcal{E}$ and hence

$$\operatorname{Vol}\mathcal{E} \geqq \operatorname{Vol}\mathcal{E}_i.$$

Again by (2.2),

$$\frac{1}{e(X)}B_X \supset \frac{1}{e(X)}T^{-1}(B_H) = \frac{1}{e(X)}\mathcal{E}_i$$

and therefore

$$\operatorname{Vol}\frac{1}{e(X)}\mathcal{E} \leqq \operatorname{Vol}\frac{1}{e(X)}\mathcal{E}_i.$$

Hence

$$\operatorname{Vol}\mathcal{E}_i = \operatorname{Vol}\mathcal{E}.$$

and proposition 3.2 implies the desired result. ∎

Theorem 3.8. *Let X be the direct sum $(X_1 \oplus X_2 \oplus \ldots \oplus X_n)_{\ell_p}$ of n finite-dimensional Banach spaces each with the unique invariant inner product property. Suppose $2 \leqq p \leqq \infty$, $\frac{1}{r}+\frac{2}{p}=1$, $d_j = \dim(X_j)$, $d = \sum_j d_j = \dim(X)$ and $\langle\cdot,\cdot\rangle_j$ is the invariant inner product on X_j which satisfies*

$$\|x_j\|^2 \leqq \langle x_j, x_j\rangle_j \quad (x_j \in X_j)$$

with equality for some nonzero x_j.

For an n-tuple $c = (c_1, c_2, \ldots, c_n)$ of positive numbers let

(3.5) $$\left\langle \textstyle\sum_j x_j, \sum_j y_j\right\rangle_c = \textstyle\sum_j c_j \langle x_j, y_j\rangle_j.$$

(i) *The unit ball for* (3.5), $c_j = 1$ *(all j), is the maximal volume ellipsoid contained in B_X.*

(ii) *The unit ball for* (3.5),

$$c_j = \left(\frac{d_j}{d}\right)^{1/r}\frac{1}{e(X_j)^2}$$

is the minimal volume ellipsoid containing B_X.

(iii) *For the map* T_0 *of proposition* 3.2 *we have*

$$|\det(T_0)| = d^{-d/2r} \prod_{j=1}^n \left(\frac{d_j^{1/r}}{e(X_j)^2}\right)^{d_j/2},$$

where we are taking as basis for X *one which is orthonormal with respect to* $\langle\cdot,\cdot\rangle_c$, $c_j=1$ *all* j.

(iv) *The inner product* (3.5), $c_j=1/e(X)^2$ *all* j, *is the only invariant inner product on* X *which satisfies*

$$\frac{1}{e(X)^2}\|x\|^2 \leqq \langle x, x\rangle \leqq \|x\|^2 \quad (x\in X). \tag{3.6}$$

Proof. Let H denote the vector space X endowed with the inner product $\langle\cdot,\cdot\rangle_c$ and let T denote the identity mapping from X to H. By proposition 1.6 all invariant inner products on X are of the form $\langle\cdot,\cdot\rangle_c$ for some c. By corollary 3.4 and proposition 3.6 it suffices to consider operators of the form T and T^{-1} in calculating the ellipsoids of minimal and maximal volumes.

A calculation using Lagrange multipliers shows that

$$\|T\|^2 = \left\|(c_1e(X_1)^2, c_2e(X_2)^2, \dots, c_ne(X_n)^2)\right\|_r \tag{3.7}$$

$$\|T^{-1}\| = \sup_{1\leqq j\leqq n} c_j^{-1}. \tag{3.8}$$

Take a fixed basis for X to be of the form $\{e_{jk}\colon 1\leqq j\leqq n,\ 1\leqq k\leqq d_j\}$ where $(e_{jk})_{k=1}^{d_j}$ is orthonormal for X_j with respect to $\langle\cdot,\cdot\rangle_j$. (This choice will not affect the absolute values of determinants in (iii).) Then

$$\frac{e_{jk}}{\sqrt{c_j}} \quad (1\leqq j\leqq n,\ 1\leqq k\leqq d_j)$$

is an orthonormal basis for H. With these bases we have

$$|\det(T)| = \prod_j (c_j)^{d_j/2} \tag{3.9}$$

$$|\det(T^{-1})| = \prod_j (c_j)^{-d_j/2} \tag{3.10}$$

To check (i) it suffices, by proposition 3.6, to find the value of c which maximises $|\det(T^{-1})|$ for $\|T^{-1}\|\leqq 1$. By (3.8) and (3.10) this clearly occurs when $c_1=c_2=\dots=c_n=1$.

Some calculations, using Lagrange multipliers, show that

$$\sup\{|\det(T)|\colon \|T\|\leqq 1\}$$

occurs when $c_j=(d_j/d)^{1/r}e(X_j)^{-2}$ for all j. This proves (ii) and (iii) then follows from (3.9).

Since every invariant inner product $\langle\cdot,\cdot\rangle$ on X is $\langle\cdot,\cdot\rangle_c$ for some c, we can rephrase (iv) to say that if $\|T\|\leqq 1$ and $\|T^{-1}\|\leqq e(X)^2$ then $c_j=1/e(X)^2$, all j. This is true by example 2.3(iii), (3.7) and (3.8). ∎

Remark 3.9. The case $1\leqq p<2$ can be handled in a similar fashion. We keep the same notation and normalisations as in theorem 3.8, except that $\dfrac{1}{r}=\dfrac{2}{p}-1$. Then the minimal volume ellipsoid containing B_X is the unit ball for $\langle\cdot,\cdot\rangle_c$ with $c_j=1/e(X_j)^2$ all j. The maximal volume ellipsoid contained in B_X is the unit ball for $\langle\cdot,\cdot\rangle_c$ with $c_j=(d/d_j)^{1/r}$. Hence

$$|\det(T_0)| = \Pi_{j=1}^n e(X_j)^{-d_j/2}$$

$$|\det(T_1)| = \Pi_{j=1}^n \left(\frac{d}{d_j}\right)^{d_j/2r}.$$

In the setting of theorem 3.8, where $p\geqq 2$, $|\det(T_1)|=1$ because we chose a basis for X which turned out to be orthonormal for the inner product giving the maximal volume ellipsoid contained in B_X. The ratio of the volume of the minimal volume ellipsoid containing B_X to the maximal volume ellipsoid inside B_X is $|\det(T_0)|^{-2}\,|\det T_1|^2$ which is

$$d^{d/r}\,\Pi_{j=1}^n\left(\frac{e(X_j)^2}{d_j^{1/r}}\right)^{d_j}$$

whether $p>2$ or $p<2$, provided we take $1/r=|1-2/p|$.

Corollary 3.10. *For* $X=(X_1\oplus X_2\oplus\ldots\oplus X_n)_p$ *the minimal volume ellipsoid containing* B_X *is a dilation of the maximal volume ellipsoid contained in* B_X *if and only if*

$$(d_j)^{1/r}\,e(X_k)^2 = (d_k)^{1/r}\,e(X_j)^2$$

for all j and k (where $1/r=|1-2/p|$*).*

Proof. This follows immediately from (i) and (ii) of theorem 3.8 for $p\geqq 2$. The preceding remarks show that it also holds for $1\leqq p<2$. ∎

4. Extremal Problems

We now consider nonlinear versions of the Banach—Mazur distance and the minimal volume ellipsoid, where holomorphic mappings f with $f(0)=0$ replace linear operators. We show that the linear and holomorphic concepts are closely related and show that in certain circumstances the extremal holomorphic mappings are necessarily linear operators.

We restrict our attention to domains $\mathscr{D}\subset X$ of complex Banach spaces X which are *balanced* — that is if $z\in\mathscr{D}$, $\lambda\in\mathbf{C}$ and $|\lambda|\leqq 1$, then $\lambda z\in\mathscr{D}$. The following generalizes to Banach spaces a version of the Schwarz lemma due to Sadullaev [17].

Proposition 4.1. *Suppose $\mathscr{D}_1$ and $\mathscr{D}_2$ are bounded balanced domains in Banach spaces X and Y and that $\mathscr{D}_2$ is pseudoconvex. If f: $\mathscr{D}_1\to\mathscr{D}_2$ is a holomorphic mapping with $f(0)=0$, then*

(i) $$f'(0)(\mathscr{D}_1)\subset\mathscr{D}_2.$$

(ii) $$f(\varrho\mathscr{D}_1)\subset\varrho\mathscr{D}_2 \quad (0<\varrho<1).$$

Proof. Fix $z\in\mathscr{D}_1$. Since the domains are balanced and $f(0)=0$, the function of one variable

$$\varphi(\lambda)=\frac{f(\lambda z)}{\lambda}$$

is analytic from a neighbourhood of the closed unit disc in $\mathbf{C}$ to Y.

Let $R_{\mathscr{D}_2}(y)$ denote the gauge (or Minkowski functional) of $\mathscr{D}_2$, that is

$$R_{\mathscr{D}_2}(y)=\inf\{\varrho>0\colon (1/\varrho)y\in\mathscr{D}_2\}\quad (y\in Y).$$

Since $\mathscr{D}_2$ is balanced pseudoconvex domain, $R_{\mathscr{D}_2}$ is plurisubharmonic on Y ([2]) and consequently $R_{\mathscr{D}_2}\circ\varphi$ is subharmonic on a neighbourhood of the unit disc. Since $\{\varphi(\lambda)\colon |\lambda|=1\}$ is a compact subset of $\mathscr{D}_2$,

$$R_{\mathscr{D}_2}(\varphi(\lambda))<1 \quad \text{for} \quad |\lambda|<1.$$

Hence $\varphi(\lambda)\in\mathscr{D}_2$ for all $|\lambda|<1$.

In particular

$$\varphi(0)=f'(0)(z)\in\mathscr{D}_2$$

and, for $0<\varrho<1$,

$$f(\varrho z)=\varrho\varphi(\varrho)\in\varrho\mathscr{D}_2. \quad ∎$$

Definition 4.2. Let $\mathscr{D}_1\subset X$ and $\mathscr{D}_2\subset Y$ be balanced bounded domains in finite dimensional Banach spaces X and Y of the same dimension n. A holomorphic mapping f: $\mathscr{D}_1\to\mathscr{D}_2$ with $f(0)=0$ is called *C-maximal* if

$$(4.1)\quad \left|\det(f'(0))\right|=\sup\left\{\left|\det(g'(0))\right|\colon g\colon \mathscr{D}_1\to\mathscr{D}_2 \text{ holomorphic, } g(0)=0\right\}.$$

Suppose that f: $\mathscr{D}_1\to\mathscr{D}_2$ is a holomorphic embedding (by which we mean that f is a biholomorphic mapping from $\mathscr{D}_1$ onto its range). To say that f is C-maximal is then equivalent to asserting that if h: $f(\mathscr{D}_1)\to\mathscr{D}_2$ is holomorphic and $h(0)=0$, then $|\det(h'(0))|\leqq 1$. The latter states that $f(\mathscr{D}_1)$ is a maximal part of $\mathscr{D}_2$ according to the definition of Carathéodory [3].

If $f\colon \mathscr{D}_1 \to \mathscr{D}_2$ is holomorphic, $\mathscr{D}_2$ pseudoconvex, $f(0)=0$ and f is $\mathscr{D}$-maximal, then proposition 4.1 implies that $f'(0)$ is also C-maximal. Moreover, we have the following result.

Proposition 4.3. *If $\mathscr{D}_1 \subset X$ and $\mathscr{D}_2 \subset Y$ are bounded balanced domains in finite dimensional Banach spaces X and Y of the same dimension n and $\mathscr{D}_2$ is pseudoconvex, then*

$$\begin{aligned} &\sup\{|\det(f'(0))|;\ f\colon \mathscr{D}_1 \to \mathscr{D}_2 \textit{ holomorphic},\ f(0)=0\} \\ &= \sup\{|\det(f'(0))|;\ f\colon \mathscr{D}_1 \to \mathscr{D}_2 \textit{ a holomorphic embedding},\ f(0)=0\} \\ &= \sup\{|\det T|;\ T\colon X \to Y \textit{ linear},\ T(\mathscr{D}_1) \subset \mathscr{D}_2\}. \end{aligned}$$

Proof. It follows immediately from proposition 4.1 that the last supremum is at least as big as the first. Clearly the first supremum is bigger than the second and the second is in turn larger than the last. ▮

Now we consider a nonlinear version of the Banach—Mazur distance which has been considered implicitly in [1, 13] (see also (2.3)).

Definition 4.4. For $\mathscr{D}_1 \subset X$ and $\mathscr{D}_2 \subset Y$ bounded balanced domains in Banach spaces X and Y, the *holomorphic Banach—Mazur distance* $d_h(\mathscr{D}_1, \mathscr{D}_2)$ between $\mathscr{D}_1$ and $\mathscr{D}_2$ is the infimum of all $\varrho > 0$ such that there exists a holomorphic embedding $f\colon \mathscr{D}_1 \to \mathscr{D}_2$ with $f(0)=0$ and

$$\frac{1}{\varrho}\mathscr{D}_2 \subset f(\mathscr{D}_1) \subset \mathscr{D}_2. \tag{4.2}$$

The infimum is interpreted as ∞ if the set of such $\varrho > 0$ is empty.

A holomorphic embedding f of $\mathscr{D}_1$ into $\mathscr{D}_2$ which satisfies (4.2) for $\varrho = d_h(\mathscr{D}_1, \mathscr{D}_2)$ is called *extremal* for $d_h(\mathscr{D}_1, \mathscr{D}_2)$.

Proposition 4.5. *If $\mathscr{D}_1 \subset X$ and $\mathscr{D}_2 \subset Y$ are bounded balanced pseudoconvex domains in Banach spaces X and Y, then*

(i)
$$\begin{aligned} &d_h(\mathscr{D}_1, \mathscr{D}_2) \\ &= \inf\left\{\varrho > 0\colon \textit{there exists } T\colon X \to Y \textit{ a linear isomorphism with } \frac{1}{\varrho}\mathscr{D}_2 \subset T(\mathscr{D}_1) \subset \mathscr{D}_2\right\}; \end{aligned}$$

(ii)
$$d_h(\mathscr{D}_1, \mathscr{D}_2) = d_h(\mathscr{D}_2, \mathscr{D}_1);$$

(iii) *if all extremal holomorphic maps for $d_h(\mathscr{D}_1, \mathscr{D}_2)$ are linear, then all extremal holomorphic maps for $d_h(\mathscr{D}_2, \mathscr{D}_1)$ are linear.*

(iv) *For any Banach spaces X and Y,*

$$d_h(B_X, B_Y) = d(X, Y) = d_h(B_Y, B_X).$$

Proof. (i) Suppose f: $\mathscr{D}_1 \to \mathscr{D}_2$ is a holomorphic embedding $f(0)=0$ and

$$\frac{1}{\varrho}\mathscr{D}_2 \subset f(\mathscr{D}_1) \subset \mathscr{D}_2.$$

By proposition 4.1,

$$f'(0)(\mathscr{D}_1) \subset \mathscr{D}_2. \tag{4.3}$$

Since f is an embedding, $f'(0)$ is a linear isomorphism from X to Y. Since $(1/\varrho)\mathscr{D}_2 \subset f(\mathscr{D}_1)$, the restricted inverse map

$$f^{-1}\colon \frac{1}{\varrho}\mathscr{D}_2 \to \mathscr{D}_1$$

is holomorphic and satisfies $f^{-1}(0)=0$. By proposition 4.1

$$(f^{-1})'(0)\left(\frac{1}{\varrho}\mathscr{D}_2\right) \subset \mathscr{D}_1.$$

Since $(f^{-1})'(0)=(f'(0))^{-1}$, this implies

$$\frac{1}{\varrho}\mathscr{D}_2 \subset f'(0)(\mathscr{D}_1). \tag{4.4}$$

Combining (4.3) and (4.4), we get (i).

Since (4.3) and (4.4) are together equivalent to

$$\frac{1}{\varrho}\mathscr{D}_1 \subset S(\mathscr{D}_2) \subset \mathscr{D}_1,$$

where S denotes $(\varrho f'(0))^{-1}=(1/\varrho)(f^{-1})'(0)$, we see that (i) implies (ii). Clearly (i) also implies (iv).

To prove (iii), let $r=1/d_h(\mathscr{D}_2, \mathscr{D}_1)=1/d_h(\mathscr{D}_1, \mathscr{D}_2)$ and suppose that G: $\mathscr{D}_2 \to \mathscr{D}_1$ is extremal for $d_h(\mathscr{D}_2, \mathscr{D}_1)$. Then $r\mathscr{D}_1 \subset G(\mathscr{D}_2)$. We define F: $\mathscr{D}_1 \to \mathscr{D}_2$ by

$$F(z) = G^{-1}(rz).$$

By proposition 4.1, $G(r\mathscr{D}_2) \subset r\mathscr{D}_1$ and hence $r\mathscr{D}_2 \subset G^{-1}(r\mathscr{D}_1)$. This means that $r\mathscr{D}_2 \subset F(\mathscr{D}_1) \subset \mathscr{D}_2$ and thus F is extremal for $d_h(\mathscr{D}_1, \mathscr{D}_2)$. By hypothesis, F is linear and consequently G must be linear. ∎

We now consider the problem of when the C-maximal holomorphic functions of Definition 4.2 and the extremal functions in Definition 4.4 must be linear mappings. We need several restrictions to be able to make progress on these problems. We will restrict our attention to the case where one of the two domains $\mathscr{D}_1$ or $\mathscr{D}_2$ is the open unit ball of a Hilbert space H. In the case when $Y=H$ and $\mathscr{D}_2=B_H$ we will not need any restriction on $\mathscr{D}_1$ except a mild regularity assumption on the boundary of $\mathscr{D}_1$ — that the gauge $R_{\mathscr{D}_1}(x)$ is continuous on X. The idea is that, by changing to an equivalent norm on X, we can assume that B_X is the convex hull of $\mathscr{D}_1$. Then we can apply the results concerning minimal volume ellipsoids outlined in section 3.

When $X=H$ and $\mathcal{D}_1=B_H$, we will need to assume that $\mathcal{D}_2$ is convex, which means that we may as well assume $\mathcal{D}_2=B_Y$.

We will use the notation $H^\infty(\mathcal{D}_1, Y)$ for the space of all bounded holomorphic mappings from $\mathcal{D}_1$ to Y (normed by the supremum norm). The following result due to Harris [8, Proposition 2] gives a general criterion which can be applied when the linear solutions to the extremal problems are understood.

Proposition 4.6. *Let $\mathcal{D}_1$ be a balanced bounded domain in a Banach space X and let $T\colon X\to Y$ be a linear map from X to a Banach space Y with $T(\mathcal{D}_1)\subset\bar{B}_Y$. Then every holomorphic map $f\colon B_X\to\bar{B}_Y$ with $f(0)=0$ and $f'(0)=T$ is linear if and only if T is a complex extreme point of the closed unit ball of $H^\infty(\mathcal{D}_1, Y)$.*

Proof. The proof given by Harris [8] for the case when $\mathcal{D}_1=B_X$ needs no modification for this slightly more general setting. ∎

Theorem 4.7. *Let X be a finite-dimensional Banach space with the property that the set $C(X)$, the set of contact points between the unit sphere of X and the minimal volume ellipsoid containing B_X, is not contained in the zero set of any nonzero polynomial on X. Let $\mathcal{D}\subset X$ be a balanced pseudoconvex domain with a continuous gauge which has convex hull equal to B_X and let $Y=H$ be a Hilbert space with the same dimension as X. Then every C-maximal holomorphic mapping $f\colon\mathcal{D}\to B_H$ is linear.*

Proof. Let f be a C-maximal map. From propositions 3.2 and 4.3 we can assume that $f'(0)=T_0$ is also C-maximal. To show that f must coincide with $f'(0)$ we apply proposition 4.6. To this end suppose that $g\colon\mathcal{D}\to H$ is a bounded analytic function and $\|T_0+\lambda g\|_\infty\leqq 1$ for $|\lambda|\leqq 1$.

Let $x\in C(X)$ be fixed. Then $\|x\|=\|T_0x\|_H=1$. Observe that the closed unit ball $\bar{B}_X$ is the convex hull of the closure of $\mathcal{D}$. We claim that x must be in the boundary of $\mathcal{D}$. If not, we could write $x=(x_1+x_2)/2$ for some $x_1, x_2\in\bar{B}_X\setminus\{x\}$, hence $T_0x=(T_0x_1+T_0x_2)/2$, which would contradict strict convexity of $\bar{B}_H$. Since $\mathcal{D}$ has a continuous gauge, $\zeta x\in\mathcal{D}$ for $\zeta\in\mathbf{C}$, $|\zeta|<1$.

Now $g(\zeta x)$ is a bounded analytic function on the unit disc $\{|\zeta|<1\}$ with values in the finite-dimensional space H. Consequently the boundary values $g(e^{i\theta}x)$ are defined (as radial limits) for almost all values of $\theta\in[0, 2\pi]$. For the values of θ,

$$e^{i\theta}T_0(x)+\lambda g(e^{i\theta}x)\in\bar{B}_H$$

for $|\lambda|\leqq 1$. Hence $g(e^{i\theta}x)=0$ (for almost all θ). We deduce $g(\zeta x)=0$ for $|\zeta|<1$. Now consider the Taylor series expansion about the origin

$$g(\zeta x) = \textstyle\sum_n \zeta^n g_n(x) = 0 \quad (|\zeta| < 1).$$

Hence $g_n(x)=0$ for all n. Since $g_n(x)$ is a homogeneous polynomial of degree n, the hypotheses now imply $g_n=0$ for all n and hence $g\equiv 0$.

This shows that T_0 is a complex extreme point of the unit ball of $H^\infty(\mathcal{D}, H)$. By proposition 4.6, $f=T_0$. ∎

Proposition 4.8. *Let* $X=(X_1\oplus X_2\oplus\ldots\oplus X_n)_{\ell_p}$ *where* $p\geqq 2$ *and each* X_j *is a finite-dimensional Banach space. Suppose*

(i) *each* X_j *has the unique invariant inner product property, and*

(ii) $C(X_j)$ *is not contained in the zero set of any nonzero polynomial on* X_j.

Then $C(X)$ *is not contained in the zero set of any nonzero polynomial on* X.

Proof. We refer to theorem 3.8 for notation. For each j, let $\langle\cdot,\cdot\rangle_j$ denote the invariant inner product on X_j satisfying

$$\|x_j\|^2 \leqq \langle x_j, x_j\rangle_j \leqq e(X_j)^2\|x_j\|^2 \quad (x_j\in X_j).$$

Then

$$C(X_j) = \left\{x_j\in X_j\colon 1 = \|x_j\|^2 = \frac{1}{e(X_j)^2}\langle x_j, x_j\rangle_j\right\}.$$

By Hölder's inequality (with exponents r and $p/2$),

$$(4.5)\quad \sum_{j=1}^n \left(\frac{d_j}{d}\right)^{1/r}\frac{1}{e(X_j)^2}\langle x_j, x_j\rangle_j \leqq \sum_{j=1}^n\left(\frac{d_j}{d}\right)^{1/r}\|x_j\|^2 \leqq \left(\sum_{j=1}^n \|x_j\|^p\right)^{2/p}$$

and the points $\sum_j x_j\in X$ where both the left and the right sides of (4.5) are equal to 1 are precisely the points in $C(X)$ (by theorem 3.8(ii)). Suppose $\sum_j x_j\in C(X)$. Then from (4.5) we see that $\frac{1}{e(X_j)^2}\langle x_j, x_j\rangle_j=\|x_j\|^2$ for $1\leqq j\leqq n$. For $p>2$, the condition for equality in Hölder's inequality implies that $\|x_j\|^2=t(d_j/d)^{2/p}$ for some $t>0$. Since $\sum_j\|x_j\|^p=1$, we have $t=1$.

Let $\alpha_j=(d_j/d)^{1/p}$. We have shown that

$$C(X) = \alpha_1 C(X_1)\times\alpha_2 C(X_2)\times\ldots\times\alpha_n C(X_n).$$

In the case $p=2$, the second inequality in (4.5) is an equality and

$$C(X) = \bigcup\left\{\beta_1 C(X_1)\times\beta_2 C(X_2)\times\ldots\times\beta_n(X_n)\colon \sum_j\beta_j^2 = 1\right\}.$$

Now suppose $P(x)$ is a polynomial on X which vanishes on $C(X)$ and let x_j and y_j denote arbitrary points in X_j and $\alpha_j C(X_j)$ respectively. The mapping $x_1\mapsto P(x_1+y_2+\ldots+y_n)$ is a polynomial on X_1 which vanishes on $\alpha_1 C(X_1)$ and therefore vanishes identically on X_1. Considering next the polynomial on X_2 given by $x_2\mapsto P(x_1+x_2+y_3+\ldots+y_n)$, we see that it must also be identically zero. By induction it follows that $P\equiv 0$. ∎

We now present our generalization of theorem 0.1.

Theorem 4.9. *Let* $X=(X_1\oplus X_2\oplus\ldots\oplus X_n)_{\ell_p}$ *where each* X_j *is a finite-dimensional Banach space and* $p\geqq 2$. *Suppose each* X_j *has the unique invariant inner product property and* $C(X_j)$ *is not contained in the zero set of any nonzero polynomial on* X_j. *For each* j *let* $\langle\cdot,\cdot\rangle_j$ *denote the unique invariant inner product on* X_j *which satisfies*

$$\|x_j\|^2 \leqq \langle x_j, x_j\rangle_j \leqq e(X_j)^2\|x_j\|^2 \quad (x_j\in X_j)$$

(where $e(X_j)$ *is as defined in section 2). Let* d_j *denote the dimension of* X_j, $d=\sum_j d_j$ *the dimension of* X *and* $1/r=1-2/p$.

Let

$$\langle \textstyle\sum_j x_j, \sum_j y_j\rangle = \sum_j \langle x_j, y_j\rangle_j$$

for $\sum_j x_j$, $\sum_j y_j\in X$ *and let* $H=(X,\langle\cdot,\cdot\rangle)$.

Let $f\colon B_X\to B_H$ *be a holomorphic mapping with* $f(0)=0$. *Then, using a basis for* X *which is orthonormal for* $\langle\cdot,\cdot\rangle$,

(i)

$$|\det f'(0)| \leqq d^{-d/2r}\prod_{j=1}^{n}\left(\frac{d_j^{1/r}}{e(X_j)^2}\right)^{d_j/2}. \tag{4.6}$$

(ii) *the following are equivalent*

(a) *equality holds in* (4.6);

(b) f *is a linear map* T *where* T *satisfies*

$$\langle T(x_1+x_2+\ldots+x_n), T(x_1+x_2+\ldots+x_n)\rangle_H = \langle \textstyle\sum_j \lambda_j x_j, \sum_j \lambda_j x_j\rangle$$

for $x_j\in X_j$ *and* $\lambda_j=(d_j/d)^{1/2r}e(X_j)^{-1}$.

Proof. (i) follows from theorem 3.8 and proposition 4.3.

If (ii)(a) is true, then proposition 4.8 and theorem 4.7 imply that $f=f'(0)$ is a linear mapping which we can denote by T. By proposition 3.2,

$$\langle Tx, Ty\rangle = \langle T_0x, T_0y\rangle$$

and thus theorem 3.8(ii) shows that (ii)(a) implies (ii)(b).

To show that (ii)(b) implies (ii)(a) is a simple computation (see the proof of theorem 3.8). ∎

Examples 4.10. We give various examples of finite-dimensional Banach spaces X with the unique invariant inner product such that $C(X)$ is not contained in the zero set of a nonzero polynomial.

(i) Clearly $X=\mathbf{C}$ satisfies both conditions required of the component spaces in theorem 4.9. By examples 2.3(ii) and proposition 4.8, ℓ_p^n $(p\geqq 2)$ and $(\ell_p^n\oplus\ell_p^n\oplus\ldots\oplus\ell_p^n)_q$ $(p\geqq 2,\ q\geqq 2)$ also satisfy the conditions.

(ii) If X is an irreducible finite-dimensional JB^*-triple system then (see example 1.4) X has the unique invariant inner product property. The algebraic inner product

$\langle\cdot,\cdot\rangle_a$ satisfies

$$\|x\|^2 \leqq \langle x, x\rangle_a \leqq e(X)^2\|x\|^2$$

(note that the rank $r=r(X)=e(X)^2$). Moreover

$$C(X) = \left\{\sum_{j=1}^r \lambda_j e_j \colon |\lambda_j| = 1 \text{ all } j \text{ and } e_j \text{ orthogonal minimal tripotents}\right\}.$$

Suppose P is a polynomial on X and P vanishes on $C(X)$. If $x\in X$, then there exist orthogonal minimal tripotents $(e_j)_{j=1}^r$ such that $x=\sum_{j=1}^r c_j e_j$ for some scalars c_j. The distinguished boundary of the polydisc

$$R = \left\{\sum_j \lambda_j e_j \colon |\lambda_j| < 1\right\}$$

is a subset of $C(X)$. Thus P vanishes on R and on the linear span of the e_j. Hence $P(x)=0$. Since x was arbitrary, we have $P\equiv 0$.

Taking the X_j of theorem 4.9 to be irreducible JB^*-triples and $p=\infty$, we recover the results of Kubota [10, 12] and Travaglini [18].

(iii) Suppose X has a symmetric basis $(e_j)_{j=1}^n$ and $C(X)$ contains one point $x=\sum_j x_j e_j$ which has $x_j\neq 0$ for $1\leqq j\leqq n$. By proposition 1.7, X has the unique invariant inner product property. Since $C(X)$ is invariant under isometries of X (see corollary 3.4)

$$\sum_j e^{i\theta_j} x_j e_j \in C(X)$$

for all choices of θ_j. Thus we can argue as in (ii) to show that $C(X)$ is not in the zero set of any nonzero polynomial on X.

There is a dual version of theorem 4.7 which applies to holomorphic mappings $f\colon B_H\to B_Y$ from the unit ball of a finite-dimensional Hilbert space to the unit ball of a Banach space Y of the same dimension. As before we consider C-maximal mappings f (with $f(0)=0$), but $C(X)$ must be replaced by the set of contact points between the unit sphere of Y and the boundary of the maximal volume ellipsoid contained in B_Y. Moreover we need to assume that all boundary points of the unit ball of Y are complex extreme points.

This argument applies in particular to the case where $Y=\ell_p^n$ with $1\leqq p\leqq 2$. Thus every C-maximal mapping $f\colon B_H\to B_Y$ is linear in this case.

Examples 4.11.

(i) For $1\leqq p<2$ there are nonlinear C-maximal mappings from the unit ball of ℓ_p^n to the unit ball of ℓ_2^n.

In this situation T_0 is the map

$$T_0\colon (z_j)_{j=1}^n \mapsto (z_j)_{j=1}^n$$

and a holomorphic map f of $B_{\ell_p^n}$ to $B_{\ell_2^n}$ is C-maximal if and only if $f'(0)=UT_0$

for some unitary U. An example of a nonlinear C-maximal f is

$$f((z_j)_j) = (z_1 + \varepsilon z_1 z_2^2, z_2, \dots, z_n)$$

with $0<\varepsilon<2^{4/p}(1-2^{1-2/p})/3$. To see this, when $n=2$, let $x=|z_1|$, $y=|z_2|$, and note that

$$(x+\varepsilon xy^2)^2+y^2 = x^2+2\varepsilon x^2y^2+\varepsilon^2x^2y^4+y^2 < x^2+3\varepsilon x^2y^2+y^2.$$

By using Lagrange multipliers one finds that $x^2+3\varepsilon x^2y^2+y^2$ has its maximum on the surface $x^p+y^p=1$, $0\leqq x, y\leqq 1$, when $x=y$ or when $x=0$ or $y=0$. When $x=y=2^{-1/p}$, we have

$$x^2+3\varepsilon x^2y^2+y^2 = 2(2^{-2/p})+3\varepsilon 2^{-4/p} < 1.$$

Hence the maximum does not occur when $x=y$.

We thank the referee for noticing an error in our original version of this example and for suggesting the appropriate correction.

(ii) For $2<p\leqq\infty$ there are nonlinear C-maximal maps from the unit ball of ℓ_n^2 to the ball of ℓ_n^p.

Proposition 4.12. *Let X be an n-dimensional Banach space and $Y=H$ an n-dimensional Hilbert space. Suppose that every C-maximal embedding of B_X in B_H is known to be linear and suppose in addition that X has the unique invariant inner product property. Then every extremal function for $d_h(B_X, B_H)$ is linear.*

Proof. By proposition 4.5(i) and theorem 3.7 the map $T=f'(0)$ has the property that $T^{-1}(B_H)$ is the minimal volume ellipsoid containing B_X. Thus, by proposition 4.3, f is C-maximal. Hence f is linear by hypotheses. ∎

Example 4.13. For $1<p\leqq\infty$, $X=\ell_p^n$, $H=\ell_2^n$, the extremal maps for $d_h(B_X, B_Y)$ are linear. For $2\leqq p\leqq\infty$ this follows from proposition 4.12, example 4.10(i) and theorem 4.7. For $1<p<2$, the result follows from the remarks following examples 4.10.

For $p=\infty$ this is due to Alexander [1] and Lempert [13].

So far all our examples have been finite-dimensional and of course C-maximal mappings are only defined in a finite-dimensional setting. Our final example concerns the holomorphic Banach—Mazur distance which can be discussed in an infinite dimensional context.

Proposition 4.13. *Let X be a finite rank JB^*-triple (finite or infinite-dimensional) and H an isomorphic Hilbert space. Then every extremal function for $d_h(B_X, B_H)$ is linear.*

Proof. If X is irreducible and finite-dimensional then this result already follows from proposition 4.6, theorem 4.7, proposition 4.12 and example 4.10(ii).

The algebraic inner product $\langle\cdot,\cdot\rangle_a$ satisfies

$$\|x\|^2 \leqq \langle x, x\rangle_a \leqq r\,\|x\|^2$$

where $r=e(X)^2$ is the rank of X. Suppose $\langle\cdot,\cdot\rangle$ is any other inner product on X which satisfies an inequality

$$\|x\|^2 \leqq \langle x, x\rangle \leqq C\,\|x\|^2 \tag{4.7}$$

with equality for some nonzero vectors on both sides. By the definition of the Banach—Mazur distance, $C\geqq r$. Let $(e_j)_{j=1}^r$ be a set of r mutually orthogonal minimal tripotents. The finite-dimensional subspace E spanned by $(e_j)_j$ is isometrically isomorphic to ℓ_∞^r. If $C=r$, then $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_a$ coincide on E. Since X is the union of subspaces of this kind, we find that the algebraic inner product is the only inner product on X satisfying (4.7) with $C=e(X)^2$. We can now apply the method of theorem 4.7 — the one difference is that we use the fact that a Hilbert space H has the analytic Radon—Nikodym property (see for instance [5]) to obtain the existence of (almost everywhere) radial limits for g. The approach of example 4.10(ii) completes the proof. ▮

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