Area growth and Green's function of Riemann surfaces

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0. Introduction

In this note we study the relationship between growth conditions of area of balls and the existence of Green's function on Riemann surfaces which we view as two-dimensional surfaces with a complete metric of constant negative curvature -1.

This is a well-known and classical issue. We refer, for instance, to [5], [7], [8], [10], [12], [17] and the references therein for some general (arbitrary Riemannian manifolds) geometric and topological conditions related to the existence of Green's function. Green's function exists if and only if there exists a positive non-constant superharmonic function or equivalently if Brownian motion on the surface is transient (see, e.g. [1, p. 204], [17]).

To fix notation let us denote by R a Riemann surface (whose universal covering space is the unit disk Δ) endowed with its Poincaré metric, i.e. the metric obtained by projecting the Poincaré metric of the unit disk, which is the metric given by $ds=2(1-|z|^2)^{-1}|dz|$, $z\in\Delta$. The only Riemann surfaces which are left out are the sphere, the plane, the punctured plane and the tori. If $p\in R$ and t is a positive number we denote by $A_R(p, t)$ the area of the ball of radius t centered at p. The precise question we address here is: which rate of growth of $A_R(p, t)$ as t tends to ∞ implies that R possesses a Green's function? Of course, $A_R(p, t)\leqq A_\Delta(0, t)=4\pi \sin h^2 (t/2)\approx \pi e^t$ as $t\to\infty$.

The following theorem is known.

Theorem A. (i) *If for a point* $p_0\in R$ *and constants* c_0, t_0

$$A_R(p_0, t) \geqq c_0 e^t, \quad \textit{for every} \quad t \geqq t_0,$$

then R has a Green's function.

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(ii) *Given a function* $\psi\colon (0, \infty)\to(0, \infty)$, *increasing, and such that*

$$\lim_{t\to\infty} \frac{\psi(t)}{e^t} = 0, \tag{0.1}$$

there exists a Riemann surface R *and a point* $p_0 \in R$ *so that*

$$A_R(p_0, t) \geqq \psi(t), \quad \text{for every} \quad t \geqq t_1,$$

but R *has no Green's function.*

Part (i) is elementary and part (ii) is due to Nicholls [13]. We will give a sketch of a simple proof of theorem A in section 2. We do remark now that in the example of (ii) of theorem A, R can be chosen to be planar.

It has been suggested by N. Varopoulos that in our situation, i.e., constant negative curvature, a *uniform exponential growth of the area* should imply that Green's function exists. More precisely, define

$$A_R(t) = \inf_{p\in R} A_R(p, t),$$

and assume that

$$\inf_{t\geqq t_0} \frac{\log A_R(t)}{t} > 0,$$

can we deduce that R has a Green's function?

This is the case for planar domains, and actually, as a corollary of the following theorem we have much more.

Theorem 1. *If there exists a radius* t_0 *and a constant* c_0 *so that*

$$A_R(p, t_0) \geqq c_0, \quad \text{for every} \quad p\in R, \tag{0.2}$$

then there exists a positive lower bound for the lengths of all closed curves which are not homotopic to zero. And conversely.

We deduce:

Corollary. *If* R *is a planar domain satisfying* (0.2) *then* R *has Green's function.*

Therefore Varopoulos' observation is verified at least in the case of planar Riemann surfaces. As a matter of fact the same argument works for Riemann surfaces of finite genus. The situation in general is radically different.

We shall show:

Theorem 2. *There exists a Riemann surface* R *so that*

$$A_R(t) \geqq e^{\alpha_0 t}, \quad \text{for every} \quad t \geqq t_0,$$

where α_0, t_0 *are positive numbers, and such that* R *does not have a Green's function.*

We believe that one can replace α_0 by any $\alpha \in (0, 1)$, and even obtain that $A_R(t) \geqq \psi(t)$ for ψ as in (0.1), but we have not been able to do so. If one keeps track of the constants appearing in the argument one sees that $\alpha_0 = 0.08$ works. A more sophisticated argument of this sort builds a similar example with $\alpha_0 = 1/3$.

Varopoulos in [18, p.271] gives an example like the one in theorem 2 with $\alpha_0 = 1$, but with variable curvature. Notice that topologically his example is the plane.

In section 1 and 3 we give the proofs of theorems 1 and 2, respectively, while the proof of theorem A is discussed in section 2. Section 4 contains the proof of some technical lemmas needed in the proof of theorem 2.

Notation. By c we will mean an absolute constant which can change its value from line to line, and even in the same line.

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1. Proof of Theorem 1

First we deal with the corollary.

Let R be a planar domain and assume that R satisfies (0.2). Then there exists a constant $c_0 > 0$ so that every homotopically non-trivial closed curve has length at least c_0. We deduce that R has uniformly perfect boundary (see [14]) and, in particular, that ∂R has positive capacity, and equivalently that R has a Green's function.

To prove the theorem we assume that R satisfies (0.2) and we show that if Γ is a Fuchsian group representing R then Γ has no parabolic elements and there exists $c_0 > 0$ so that the translation length of every hyperbolic element of Γ is larger than c_0.

If γ is a hyperbolic element of Γ then the axis of γ projects onto a closed geodesic σ whose length is the translation length of γ, L, say. Let C be the set of points whose distance to σ is less than t_0. If $p \in \sigma$ then

$$c_0 \leqq A_R(p, t_0) \leqq \text{area}(C) \leqq 2L \sinh(t_0).$$

The third inequality is obtained simply by lifting C to the unit disk.

We conclude that

$$L \geqq c_0 (2 \sinh(t_0))^{-1}.$$

If Γ has a parabolic element γ then as p "converges" to the puncture ([11, p. 52]) determined by γ, $A_R(p, t_0)$ tends to zero.

The converse follows easily because from the hypothesis one deduces that for some constant c_1, $B_R(p, c_1)$ is simply connected for every $p \in R$.

Remark. We should notice that if R is not compact then (0.2) improves itself automatically to

(1.1) $$A_R(p, t) \geqq ct \quad \text{for every} \quad t \geqq t_1.$$

A **Z**-cover of a genus 2 compact Riemann surface gives an example where $A_R(p, t) \asymp t$. We leave the details to the reader.

2. Proof of Theorem A

First we deal with part (i). So we assume that for some point $p_0 \in R$ and constant $c_0 > 0$

(2.1) $$A_R(p_0, t) \geqq c_0 e^t.$$

We shall prove that R has bounded non-constant harmonic functions and, in particular, that R has a Green's function. Let Π be a universal covering map from Δ onto R with $\Pi(0) = p_0$. Let Γ be the associated covering group and denote by D the Dirichlet region of Γ at 0, [2, p. 227]. It is easy to see (e.g., [16, p. 488]) that (2.1) implies that $\partial D \cap \partial \Delta$ has positive length. Let E be a Borel subset of $\partial D \cap \partial \Delta$ whose length is half the length of $\partial D \cap \partial \Delta$ and $F = \bigcup_{\gamma \in \Gamma} [\gamma(E)]$. If u is the Poisson extension of the characteristic function of F, then u is Γ-invariant, bounded and non-constant. Notice that R is accessible in the terminology of [15].

The argument above is standard.

The examples of (ii) can be constructed as follows. Let θ_n be positive numbers decreasing to zero slowly, $\theta_0 = \pi$. Let g_n be the geodesic joining $e^{i\theta_{n-1}}$ with $e^{i\theta_n}$, and h_n be its reflection on the real axis. Consider the Möbius transformations γ_n preserving Δ and such that $\gamma_n(g_n) = h_n$, so that if $p \in g_n$, $\gamma_n(p) = \bar{p}$, where "$-$" means complex conjugate. Then by Poincaré's theorem the group generated by the γ_n's is a Fuchsian group and the (infinite sided) polygon Q determined by the g_n's and the h_n's is a fundamental region. The quotient $R = \Delta / \Gamma$ is the plane with a sequence converging to ∞ removed. And therefore R has no Green's function. It is clear that given $\psi(t)$ satisfying (0.1) one can choose the sequence θ_n so that

$$\text{area}\,(Q \cap B_\Delta(0, t)) \geqq \psi(t), \quad \text{for every} \quad t \geqq t_1,$$

and consequently, if p_0 is the point of R represented by 0, then

$$A_R(p_0, t) \geqq \psi(t).$$

This is essentially the construction of Nicholls in [13].

3. Proof of Theorem 2

The proof is divided into two steps. In the first step we construct a graph G with two properties:

(i) for every vertex v the number of vertices at graph-distance less than t is at least $ce^{\alpha_0 t}$.

(ii) the random walk in the graph (with equal probability of jumping from a vertex to any of its adjacent vertices) is recurrent.

The second step consists in making a Riemann surface out of this discrete model.

The basic building block is a Y-piece. That is the graph with 4 vertices and 3 edges which is represented by the letter Y. The central vertex is called a node. The other three vertices are dots. The graph will be obtained by adjoining Y-pieces by the dots. A node has then 3 neighbours which are dots, and a dot has 2 neighbours which are nodes.

Consider an increasing sequence of positive integers $\{k_j\}_{j=1}^{\infty}$.

We start with two Y-pieces joined by a dot. Denote this common dot by p_0. At each dot (there are four) we adjoin another Y-piece by a dot. So that after this we have 2^3 free dots. At each one of those we adjoin a Y-piece, and we keep doing this (k_1-2) times. Now we have 2^{k_1} free dots. Now we reverse the process. These free dots come in pairs (having the same father). At each pair we adjoin a Y-piece. Now we have 2^{k_1-1} free dots. Again they come in pairs (having the same grandfather), and we adjoin a Y-piece to each such pair. Now we have 2^{k_1-2} free dots. We repeat this until finally we have 1 free dot left. We call this free dot p_1; and start again by adjoining a Y-piece, and so on until there are 2^{k_2} free dots. Reverse again until there is one free dot left; call it p_2 and continue.

Any edge in this graph connects a dot with a node. The distance between adjacent node and dot is declared to be $1/2$.

No matter how the k_n are chosen the random walk on the graph is recurrent. One way of verifying this is by means of the well-known electrical network analogy, which is carefully described in the delightful book of Doyle and Snell [6]. We refer to this book for definitions and results. Consider the graph as an electrical network and consider each edge as having conductance 1 [6, p. 40]. (This is the condition that from a vertex the random walker has equal probability to jump to any of its neighbours.) It is easy to see that the effective resistance ([6, p. 53]) of the piece of graph "between" p_n and p_{n+1} is essentially constant and consequently the effective resistance of the whole graph from p_0 to ∞ is infinite. (See p. 55 and chapter 6.5 of [6] for the standard tricks of evaluating the effective resistance of a graph.) Therefore the random walker starting from p_0 has zero probability of escaping to ∞.

Nonetheless, we shall verify directly later that the Riemann surface we are going to build from the graph has no Green's function.

Choose the k_n's so that $k_{n+1}=2^n k_1$.

For a vertex p denote by $a(p, m)$ the number of Y-pieces in the graph contained in the graph-ball of center p and radius the positive integer m. Elementary counting arguments show first that

$$a(p_j, m) \geqq c_1 2^{Bm}, \tag{3.1}$$

for each j and each positive integer m, where $B=1/10$ works, and from that it follows easily that

$$a(p, m) \geqq c_1 2^{(B/2)m}, \tag{3.2}$$

for each vertex p and each positive integer m.

We assume now (3.2) and continue with the rest of the argument. The verification of (3.1) and (3.2) is dealt with in the next section.

To build up our Riemann surface modelled upon the graph above we will substitute the Y-pieces of the graph by the so-called Löbell Y-pieces, which are a standard tool for constructing Riemann surfaces. A clear description of these Y-pieces and their use is given in [4, chapter X.3].

A Löbell Y-piece is a three-hole sphere, endowed with a metric of constant negative curvature -1, so that the boundary curves are geodesics. We also require that the lengths of the boundary curves are the same, say 2α, and that the distance between any two of these boundary curves is β, say. Then α and β are related by

$$\sinh(\alpha/2)\sinh(\beta/2) = 1/2.$$

(This is the only restriction on α and β. See [3], [4, p. 248] for details.)

Fix $\alpha_0=\beta_0$ satisfying the relation above.

If we now put together these Löbell Y-pieces following the combinatorial design of our graph, by identifying corresponding boundary curves, we obtain a complete surface R of constant negative curvature -1. The dots are the boundary curves of the Löbell Y-pieces which themselves are represented by the nodes.

Let Q_1 and Q_2 be two non-adjacent Löbell Y-pieces of R, and let q_1 and q_2

denote the corresponding nodes. Then, if dist_G is the distance in the graph G,

$$A^{-1} \leqq \frac{\mathrm{dist}_R(Q_1, Q_2)}{\mathrm{dist}_G(q_1, q_2)} \leqq A,$$

where A is a fixed constant.

Since our Löbell Y-pieces have the same area 2π we immediately deduce that if $p\in R$ then

$$A_R(p, t) \geqq c_1 e^{c_2 t}, \quad \text{for every} \quad t \geqq t_0,$$

where c_1 and c_2 are some fixed constant.

Finally, we must check that R has no Green's function.

The two Y-pieces used to start up our construction have a common boundary curve which is represented by the dot-vertex p_0. Let us call the domain they determine by D. We simply have to check that the extremal length $\lambda(\Gamma)$ of the family of curves Γ "joining" ∂D with the Alexandrov $-\infty$ of R is infinite [1, p. 229]. Now, the geodesics g_j represented by the dot-vertices p_j have all length α_0. Let C_j denote the collar around g_j of width d, where d is an appropriate fix constant (i.e. $\sinh(d) = \mathrm{cosech}(\alpha_0/2)$). Let Γ_j denote the family of curves joining the boundary components of C_j. Then the extremal length $\lambda(\Gamma_j)$ is a fixed constant λ_0. Since every curve in Γ contains a subcurve in every Γ_j we have, by the composition laws ([1, p. 222]) that

$$\lambda(\Gamma) \geqq \sum_{j=1}^{n} \lambda(\Gamma_j) = n\lambda_0.$$

We deduce that $\lambda(\Gamma)=\infty$, as desired.

In any case the graph and the Riemann surface are clearly roughly isometric in the sense of Kanai [9]; consequently since the random walk in the graph is recurrent the Brownian motion in the Riemann surface is also recurrent, i.e. it does not possess Green's function.

4. Proof of (3.1) and (3.2)

Lemma 1. *There is a universal constant* $c>0$ *such that*

$$a(p_j, m) \geqq c2^{m/10}$$

for every j, *for every positive integer* m.

Proof. First we deal with p_0. There are three cases.

(1) If $0 < m \leqq k_1 - 1$, then

$$a(p_0, m) = 2^{m+1} - 2 \geqq 2^m.$$

(2) If $-1+2\sum_1^N k_j < m \leqq -1+2\sum_1^N k_j + k_{N+1}$, then

$$a(p_0, m) = -1+2\sum_1^N (2^{k_j}-1) + 2^{m+1-2\sum_1^N k_j} - 1 \geqq c(2^{k_N} + 2^{m-2\sum_1^N k_j}).$$

We use the inequality

$$(4.1) \qquad x+y \geqq x^{1-u}y^{u} \quad \text{for every} \quad x, y \in \mathbf{R}^+, u \in (0,1),$$

with $u=1/5$. We have

$$a(p_0, m) \geqq c2^{m/5}$$

because $\frac{4}{5}k_N - \frac{2}{5}\sum_1^N k_j \geqq 0$.

(3) If $-1+2\sum_1^N k_j + k_{N+1} < m \leqq -1+2\sum_1^{N+1} k_j$, then

$$a(p_0, m) \geqq a\left(p_0, -1+2\sum_1^N k_j + k_{N+1}\right) \geqq c2^{m/10}.$$

The proof is finished in the case $j=0$.

For arbitrary j, simply do not count the Y-pieces of the generations preceding p_j and obtain the same inequality with the same universal constants.

Lemma 2. *Let p be a dot between p_{j-1} and p_j. If d is the graph-distance between p and p_j, then*

$$a(p, m) \geqq c2^{m/3}, \quad \text{if} \quad 1 \leqq m \leqq d,$$

where c is an absolute constant.

Proof. First we observe that $a(p, m) = a(p', m)$ if p and p' are dots of the same generation.

If p is such that $d \leqq k_j$, then

$$a(p, m) \geqq 2(2^{m/2}-1) \geqq c2^{m/2}, \quad \text{if} \quad 1 \leqq m \leqq d.$$

We do not need to count the Y-pieces of the generations preceding p.

If p is such that $d > k_j$, then

$$a(p, m) \geqq 2^m, \quad \text{if} \quad 1 \leqq m \leqq d-k_j,$$

and

$$a(p, m) \geqq 2^{d-k_j} + c2^{(m-d+k_j)/2} \geqq c2^{m/3}, \quad \text{if} \quad d-k_j < m \leqq d,$$

where we have used (4.1) with $u=2/3$. Then

$$a(p, m) \geqq c2^{m/3} \quad \text{if} \quad 1 \leqq m \leqq d.$$

Finally,

Lemma 3. *There is a universal constant $c>0$ such that*

$$a(p, m) \geqq c2^{m/13}$$

for every dot p, and positive integer m.

Proof. We fix a dot p. Then, if $1 \leqq m \leqq d$, we have

$$a(p, m) \geqq c2^{m/3}.$$

And if $m \geqq d$, then

$$a(p, m) \geqq c(2^{d/8} + 2^{(m-d)/10}) \geqq c2^{m/13},$$

where we have used (4.1) with $u = 10/13$. Then

$$a(p, m) \geqq c2^{m/13} \quad \text{for every positive integer } m.$$

With a suitable choice of the $\{k_j\}_{j=1}^{\infty}$, we can replace the exponent $1/13$ by any number less than $1/9$.

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