On the distribution of Sidon series

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1. Introduction

Suppose that G is a compact abelian group with dual group Γ. Denote the normalized Haar measure on G by μ. Let $\mathcal{C}(G)$ be the Banach space of continuous complex-valued functions on G. If $S \subset \Gamma$, a function $f \in L^1(G)$ is called S-*spectral* whenever $\hat{f}$ is supported in S, where here and throughout the paper $\hat{\ }$ denotes taking the Fourier transform. The collection of S-spectral functions that belong to a class of functions $\mathcal{W}$ will be denoted by $\mathcal{W}_S$.

Definition 1.1. A subset E of Γ is called a Sidon set if there is a constant $c>0$, depending only on E, such that

$$(1) \qquad \sum_{\gamma \in \Gamma} |\hat{f}(\gamma)| \leq c \, \|f\|_\infty$$

for every $f \in \mathcal{C}_E(G)$. The smallest constant c such that (1) holds is denoted by $S(E)$ and is called the constant of sidonicity of E, or the Sidon constant of E.

If $E = \{\gamma_j\} \subset \Gamma$ is a Sidon set and $\{a_j\}$ is a sequence in a Banach space B, then the formal series $\sum a_j \gamma_j$ will be referred to as a B-*valued Sidon series*. The norm on a given Banach space B will be denoted by $\|\cdot\|$, or, sometimes, by $\|\cdot\|_B$.

It is well-known that Sidon series share many common properties with Rademacher series. The following theorem of Pisier illustrates this fact and will serve as a crucial tool in our proofs.

Theorem 1.2 [Pi1, Théorème 2.1]. *Suppose that $E = \{\gamma_n\} \subset \Gamma$ is a Sidon set, that B is a Banach space, and that $a_1, ..., a_N \in B$. There is a constant c_1, depending*

only on the Sidon constant $S(E)$, such that, for every $p \in [1, \infty[$, we have

$$(2) \quad c_1^{-1} \left(\mathbf{E} \left\| \sum_{n=1}^{N} a_n r_n \right\|^p \right)^{1/p} \leq \left(\int_G \left\| \sum_{n=1}^{N} a_n \gamma_n \right\|^p d\mu \right)^{1/p} \leq c_1 \left(\mathbf{E} \left\| \sum_{n=1}^{N} a_n r_n \right\|^p \right)^{1/p}.$$

In view of this similarity between Sidon series and Rademacher series, it is natural to ask how the distribution function of a Sidon series compares to the distribution function of a Rademacher series. Our main result provides an answer to this question.

Theorem 1.3. *Suppose that $E = \{\gamma_n\} \subset \Gamma$ is a Sidon set, and let B denote an arbitrary Banach space. There is a constant $c > 0$ that depends only on the Sidon constant $S(E)$, such that for all $a_1, ..., a_N \in B$, and all $\alpha > 0$, we have*

$$(3) \quad c^{-1} P \left[\left\| \sum_{n=1}^{N} a_n r_n \right\| \geq c\alpha \right] \leq \mu \left[\left\| \sum_{n=1}^{N} a_n \gamma_n \right\| \geq \alpha \right] \leq c P \left[\left\| \sum_{n=1}^{N} a_n r_n \right\| \geq c^{-1}\alpha \right].$$

Thus the distribution functions of Sidon and Rademacher series are equivalent. Our proof of this result combines well-known properties of Sidon series with Lemma 2.3 below. This Lemma provides sufficient conditions for the equivalence of distribution functions. It applies as well in the setting of noncommutative groups yielding an analogue of Theorem 1.3. Using the estimates of [Mo], we obtain sharp lower bound estimates on the distribution of scalar-valued Sidon series on compact abelian groups.

We should note that the topological implication of our main result is much easier to show, as is done in [Pi1], that is, the measure topology on the spaces of Rademacher series and the Sidon series are equivalent.

2. A principle for the equivalence of distribution functions

All random variables are defined on some probability space $(\Omega, \mathcal{M}, dP)$. We denote the set of positive integers by $\mathbf{N}$, the set of integers by $\mathbf{Z}$, and the circle group by $\mathbf{T}$. All other notation is as in Section 1. We start with a couple of preliminary lemmas.

Lemma 2.1. *Suppose that $f_1, f_2, ..., f_N$ are independent identically distributed random variables, and let f be a function with the same distribution as the f_j's such that*

$$P[|f| \geq \alpha] \geq \frac{\theta}{N}$$

where α and θ are positive numbers. Then

$$P\Big[\sup_{1\le j\le N}|f_j|\ge\alpha\Big]\ge\frac{\theta}{1+\theta}.$$

Proof. We first show that, for $\theta>0$ and $N\in\mathbf{N}$, we have

(4)
$$\Big(1-\frac{\theta}{N}\Big)^N\le\frac{1}{(1+\theta)}.$$

This follows from the inequalities

$$\Big(1-\frac{\theta}{N}\Big)\Big(1+\frac{\theta}{N}\Big)\le 1$$

and

$$\Big(1+\frac{\theta}{N}\Big)^N\ge 1+\theta.$$

Using (4) and independence, we get:

$$P\big[\sup_j|f_j|\ge\alpha\big]=1-P\big[\sup_j|f_j|<\alpha\big]=1-\prod_j P\big[|f_j|<\alpha\big]$$

$$\ge 1-\Big(1-\frac{\theta}{N}\Big)^N\ge 1-\frac{1}{(1+\theta)}=\frac{\theta}{(1+\theta)}.$$

Lemma 2.2. *Suppose that $f_1,f_2,...,f_N$ are independent identically distributed random variables, and let f be a function with the same distribution as the f_j's such that*

$$P[|f|\ge\alpha]\le\frac{\theta}{N}$$

where α and θ are positive numbers. Then, we have

(5)
$$P\Big[\sup_j|f_j|\ge\alpha\Big]\le\theta.$$

Proof.

(6)
$$P\Big[\sup_j|f_j|\ge\alpha\Big]=P\bigcup_j[|f_j|\ge\alpha]\le\sum_j P\big[|f_j|\ge\alpha\big]\le\theta.$$

Before stating our main Lemma, we recall two well-known inequalities. Let X denote a random variable on a probability space $(\Omega,\mathcal{M},dP)$, then, for all $y>0$, we have:

(7) $P[|X|\ge y\mathbf{E}|X|]\le y^{-1}$ (Chebychev's Inequality);

and, for $0<y<1$, we have

(8) $P[|X|\ge y\mathbf{E}|X|]\ge(1-y)^2\dfrac{\mathbf{E}^2|X|}{\mathbf{E}|X|^2}$ (Paley–Zygmund Inequality).

See [Ka, Ineq. II, p. 8]. The statement of our main Lemma now follows.

Lemma 2.3. *Suppose that X and Y are two Banach valued random variables that are not identically zero, and suppose that $\{X_n\}$ and $\{Y_n\}$ are two sequences of independent Banach valued random variables such that X_n is identically distributed with X, and Y_n is identically distributed with Y. Suppose that there are constants c_1 and c_2 such that, for all positive integers N, we have:*

$$(9) \qquad c_1^{-1}\Big\|\sup_{1\le j\le N}\|X_j\|\Big\|_1 \le \Big\|\sup_{1\le j\le N}\|Y_j\|\Big\|_1 \le c_1\Big\|\sup_{1\le j\le N}\|X_j\|\Big\|_1;$$

$$(10) \qquad \frac{\big\|\sup_{1\le j\le N}\|X_j\|\big\|_1^2}{\big\|\sup_{1\le j\le N}\|X_j\|\big\|_2^2} \ge c_2;$$

and

$$(11) \qquad \frac{\big\|\sup_{1\le j\le N}\|Y_j\|\big\|_1^2}{\big\|\sup_{1\le j\le N}\|Y_j\|\big\|_2^2} \ge c_2.$$

Then there is a constant c, depending only on c_1 and c_2, such that, for all $\alpha>0$, we have:

$$(12) \qquad c^{-1}P[\|X\|\ge c\alpha] \le P[\|Y\|\ge \alpha] \le cP[\|X\|\ge c^{-1}\alpha].$$

Proof. We start with the second inequality in (12). Given an arbitrary $\alpha=\alpha_1>0$ with

$$(13) \qquad 0 < P[\|Y\|\ge \alpha_1],$$

choose ν to be the smallest positive integer satisfying:

$$(14) \qquad \frac{1}{2\nu} \le P[\|Y\|\ge\alpha_1] \le \frac{1}{\nu}.$$

From Lemma 2.1, it follows that

$$(15) \qquad \tfrac{1}{3} \le P\Big[\sup_{1\le j\le \nu}\|Y_j\|\ge\alpha_1\Big].$$

Chebychev's Inequality (7), and (15) imply that

$$(16) \qquad \tfrac{1}{3} \le \frac{1}{\alpha_1}\Big\|\sup_{1\le j\le\nu}\|Y_j\|\Big\|_1.$$

From (9) and (16), we have

(17)
$$\tfrac{1}{3} \leq \frac{c_1}{\alpha_1} \Big\| \sup_{1 \leq j \leq \nu} \|X_j\| \Big\|_1.$$

Hence, for any $\alpha_2 > 0$, (17) implies that

(18)
$$\frac{\alpha_1}{3\alpha_2 c_1} \leq \frac{1}{\alpha_2} \Big\| \sup_{1 \leq j \leq \nu} \|X_j\| \Big\|_1.$$

In particular, if $\alpha_2 = \alpha_1/6c_1$, we get from (18)

(19)
$$2 \leq \frac{1}{\alpha_2} \Big\| \sup_{1 \leq j \leq \nu} \|X_j\| \Big\|_1.$$

Now we go back to the Paley–Zygmund Inequality (8) and apply it to

$$\sup_{1 \leq j \leq \nu} \|X_j\| \quad \text{with } y = \frac{\alpha_2}{\big\| \sup_{1 \leq j \leq \nu} \|X_j\| \big\|_1}.$$

Taking into account (10) and noticing from (19) that $y \leq \tfrac{1}{2}$, we get

(20)
$$\tfrac{1}{4}c_2 \leq P\Big[\sup_{1 \leq j \leq \nu} \|X_j\| \geq \alpha_2 \Big] = P\Big[\sup_{1 \leq j \leq \nu} \|X_j\| \geq d\alpha_1 \Big]$$

where $d = 1/6c_1$. Lemma 2.2, (14), and (20) imply that

(21)
$$P[\|X\| \geq d\alpha_1] \geq \frac{c_2}{4\nu}$$

(22)
$$\geq \frac{c_2}{4} P[\|Y\| \geq \alpha_1].$$

Take $c^{-1} = \min(d, c_2/4)$, then (22) shows that

(23)
$$P[\|X\| \geq c^{-1}\alpha_1] \geq c^{-1} P[\|Y\| \geq \alpha_1].$$

Note that (23) holds for all α_1 for which (13) is true. For all other values of α_1, inequality (23) holds trivially. Thus (23) holds for all $\alpha_1 > 0$. Now we repeat the proof with X and Y interchanged. From (23) we get:

(24)
$$P[\|Y\| \geq c^{-1}\alpha_1] \geq c^{-1} P[\|X\| \geq \alpha_1],$$

for all $\alpha_1 > 0$. Combining (23) and (24), we obtain

(25)
$$c^{-1} P[\|X\| \geq \alpha_1] \leq P[\|Y\| \geq c^{-1}\alpha_1] \leq c P[\|X\| \geq c^{-2}\alpha_1]$$

for all $\alpha_1 > 0$. Equivalently, we have

(26)
$$c^{-1} P[\|X\| \geq c\alpha] \leq P[\|Y\| \geq \alpha] \leq c P[\|X\| \geq c^{-1}\alpha]$$

for all $\alpha > 0$, which proves (12).

3. The distribution of Banach valued Sidon series

To prove Theorem 1.3, Lemma 2.3 suggests that we consider independent copies of the given Sidon series. The construction of independent copies of a given trigonometric polynomial on a group G is easily done on the product group. The spectra of the resulting polynomials are supported in a subset of the product of the character group. Our first goal in this section is to study properties of this set. To simplify the presentation, we will treat the commutative and noncommutative cases separately. Throughout this section G will denote a compact abelian group with character group Γ and Haar measure μ. Similar meanings are attributed to G_j, Γ_j and μ_j, respectively.

Definition 3.1. Suppose that $E_j \neq \emptyset$ is a subset of Γ_j, for $j=1,...,n$. The n-fold join of the sets E_j is a subset of $\prod_{j=1}^n \Gamma_j$, denoted by $\bigvee_{j=1}^n E_j$, and defined by:

$$\bigvee_{j=1}^n E_j = \left\{ \gamma = (\gamma_1, ..., \gamma_n) \in \prod_{j=1}^n \Gamma_j : \text{all but one } \gamma_j \in E_j \text{ are } 0 \right\}.$$

If $E \neq \emptyset \subset \Gamma$, the n-fold join of E, denoted by $\bigvee_{j=1}^n E$, is the set $\bigvee_{j=1}^n E_j$, where $E_j = E$ for all $j = 1, ..., n$.

Thus a generic element γ of $\bigvee_{j=1}^n E_j$ is of the form $\gamma = (0, 0, ..., \gamma_j, 0, ..., 0)$ where $\gamma_j \in E_j$. When γ is evaluated at $x = (x_1, ..., x_j, ..., x_n) \in G^n$ we get:

$$\gamma(x) = \gamma_j(x_j).$$

Suppose that $S \subset \Gamma$. For the sake of our proof of Theorem 1.3 it turns out that it is sufficient to study the n-fold join of the set $\{1\} \times S \subset \mathbf{Z} \times \Gamma$. What is needed is the following simple result. A more general result concerning joins of Sidon sets is presented following the proof of Theorem 1.3.

Lemma 3.2. *Suppose $E \subset \Gamma$ is a Sidon set. Let $T = \{1\} \times E \subset \mathbf{Z} \times \Gamma$, and let n be an arbitrary positive integer. The n-fold join $\bigvee_{j=1}^n T$ is a Sidon subset of $\mathbf{Z}^n \times \Gamma^n$ with Sidon constant $S\left(\bigvee_{j=1}^n T\right)$ equal to $S(E)$.*

Proof. It is clear that $S(T) = S(E)$ and that $S\left(\bigvee_{j=1}^n T\right) \geq S(E)$. Let F be a trigonometric polynomial with spectrum supported in $S\left(\bigvee_{j=1}^n T\right)$. We can write F as:

$$F = \sum_{j=1}^n f_j(t_j, x_j)$$

where

$$f_j = \sum_{l=1}^{k_j} a_{jl} e^{it_j} \chi_{jl}(x_j) = e^{it_j} \sum_{l=1}^{k_j} a_{jl} \chi_{jl}(x_j).$$

For each $j = 1, ..., n$, pick x_j so that

$$\left| \sum_{l=1}^{k_j} a_{jl} \chi_{jl}(x_j) \right| = \|f_j\|_\infty,$$

and then pick t_j so that

$$e^{it_j} \sum_{l=1}^{k_j} a_{jl} \chi_{jl}(x_j) = \|f_j\|_\infty.$$

We have

$$f_j(t_j, x_j) = \|f_j\|_\infty,$$

and so

$$F(t_1, t_2, ..., t_n, x_1, x_2, ..., x_n) = \sum_{j=1}^{n} \|f_j\|_\infty \geq (S(E))^{-1} \sum_{j=1}^{n} \sum_{l=1}^{k_j} |a_{jl}|.$$

Hence (1) holds for F with $c = S(E)$, and the proof is complete.

We still need one ingredient for the proof of Theorem 1.3. This is the Khintchin–Kahane theorem.

Theorem 3.3 ([Pi1, Théorème K]). *If $0 < p < q < \infty$, there is a constant $K_{p,q}$ such that*

(27) $$\left(\mathbf{E} \left\| \sum_{j=1}^{n} a_j r_j \right\|^q \right)^{1/q} \leq K_{p,q} \left(\mathbf{E} \left\| \sum_{j=1}^{n} a_j r_j \right\|^p \right)^{1/p}$$

for any sequence $\{a_j\}$ in a Banach space B.

Proof of Theorem 1.3. Let $f(x) = \sum_{n=1}^{N} a_n \gamma_n(x)$ $(x \in G)$ be a trigonometric polynomial with $\gamma_n \in S$, and let $Y = e^{it} \sum_{n=1}^{N} a_n \gamma_n(x)$ where $(t, x) \in \mathbf{T} \times G$. Clearly, f and Y have the same distribution functions. We apply Lemma 2.3 with

$$X = \sum_{n=1}^{N} a_n r_n \quad \text{and} \quad Y = e^{it} \sum_{n=1}^{N} a_n \gamma_n(x).$$

Given $\nu \in \mathbf{N}$, we construct a sequence of independent random variables $\{Y_j\}$ on $\mathbf{T}^\nu \times G^\nu$, identically distributed with Y, in the obvious way: for

$$(t, x) = (t_1, t_2, ..., t_\nu, x_1, ..., x_\nu) \in \mathbf{T}^\nu \times G^\nu,$$

let

$$(28) \qquad\qquad Y_j(t, x) = Y(t_j, x_j) = e^{it_j} \sum_{n=1}^{N} a_n \gamma_n(x_j).$$

Write γ_{nj} $(n=1, ..., N; j=1, ..., \nu)$ for the character in Γ^ν given by:

$$\gamma_{nj}(x) = \gamma_n(x_j)$$

for all $x \in G^\nu$. Let $l_\infty^\nu(B)$ denote the Banach space consisting of the vectors $a = (a_1, ..., a_\nu)$, where $a_j \in B$, equipped with the norm

$$\|a\|_{l_\infty^\nu(B)} = \sup_{1 \le j \le \nu} \|a_j\|.$$

Let $a_{nj} \in l_\infty^\nu(B)$ be the vector whose components are all zero except that the j-th component is equal to a_n: $a_{nj} = a_n(\delta_{ij})_{i=1}^\nu$, $n=1, ..., N$; $j=1, ..., \nu$. Let

$$(29) \qquad \mathcal{Y}(t, x) = \sum_{j=1}^{\nu} \sum_{n=1}^{N} a_{nj} e^{it_j} \gamma_{nj}(x) \quad ((t, x) \in \mathbf{T}^\nu \times G^\nu).$$

We clearly have:

$$(30) \qquad\qquad \mathcal{Y}(t, x) = (Y_1(t, x), Y_2(t, x), ..., Y_\nu(t, x))$$

for all $(t, x) \in \mathbf{T}^\nu \times G^\nu$. For $j = 1, ..., \nu$, let

$$X_j(t) = \sum_{n=1}^{N} a_n r_{nj}(t).$$

Corresponding to $\mathcal{Y}$, construct a Rademacher sum $\mathcal{X}$ with values in $l_\infty^\nu(B)$:

$$(31) \qquad\qquad \mathcal{X}(t) = \sum_{j=1}^{\nu} \sum_{n=1}^{N} a_{nj} r_{nj}(t) \quad (t \in [0, 1]),$$

where $(r_{nj})_{n=1\,j=1}^{N\quad\nu}$ is an enumeration of distinct Rademacher functions, and a_{nj} is as above. Note that $\mathcal{Y}$ is a Sidon series with spectrum supported in the join $\bigvee_{j=1}^{\nu}\{1\}\times E=\bigvee_{j=1}^{n}T$ of $T=\{1\}\times E$. Pisier's Theorem 1.2 implies that

$$(32) \qquad c_1^{-1}\left(\mathbf{E}\sup_{1\le j\le\nu}\|X_j\|\right)\le\int_{G^\nu}\sup_{1\le j\le\nu}\|Y_j\|\le c_1\left(\mathbf{E}\sup_{1\le j\le\nu}\|X_j\|\right)$$

where c_1 depends only on $S(\bigvee T)$, and hence only on $S(E)$, by Lemma 3.2. We have thus obtained (9). It remains to prove (10) and (11). These are consequences of (27) and Pisier's Theorem 1.2 applied to the random variables $\mathcal{X}$ and $\mathcal{Y}$ above. Indeed, applying (27) with $p=1$ and $q=2$, we obtain:

$$(33) \qquad \left(\mathbf{E}\left(\sup_{1\le j\le\nu}\|X_j\|\right)^2\right)^{1/2}\le K_{1,2}\,\mathbf{E}\sup_{1\le j\le\nu}\|X_j\|$$

which proves (10). To get (11), we apply Pisier's Theorem 1.2, with $p=2$ to the functions $\mathcal{X}$ and $\mathcal{Y}$, then use (32) and (33) again.

Remark 3.4. It is worth noting that either of the inequalities (3) characterizes Sidon sets. For the first inequality, the assertion follows from the definition of Sidon sets. For the second inequality, this follows by Pisier's characterization of Sidon sets [Pi3].

In our original proof of Theorem 1.3, we worked directly with the function $f=\sum a_j\gamma_j$ whose spectrum is supported in a Sidon set E. The result that we needed concerned the Sidon constant of the n-fold join of E. We present this result in the next theorem, because of its interest in its own right. As far as we know, the best constant in the theorem below is not known.

Theorem 3.5. *Suppose that E is a Sidon subset of Γ and $n\ge 1$. Then $\bigvee_{j=1}^{n}E$ is a Sidon subset of Γ^n with Sidon constant $S(\bigvee_{j=1}^{n}E)\le 2\pi S(E)+1$. In particular, $S(\bigvee_{j=1}^{n}E)$ is independent of n.*

The proof below was kindly communicated to us by D. Ullrich and other people after him. It is an easy consequence of the following lemma which, as D. Ullrich also remarked, may be well-known to probabilists.

Lemma 3.6. *For $j=1,...,n$, let K_j denote a compact topological space. Suppose that $f_j\in\mathcal{C}(K_j)$, $j=1,...,n$. Suppose further that 0 is in the convex hull of $f_j(K_j)$, for each j. Define $f\in\mathcal{C}\left(\prod_{j=1}^{n}K_j\right)$ by $f(x)=\sum_{j=1}^{n}f_j(x_j)$. Then*

$$\pi\|f\|_\infty\ge\sum_{j=1}^{n}\|f_j\|_\infty.$$

Moreover, the constant π is best possible.

Proof. For all $\theta \in [-\pi, \pi[$, and all $x \in \prod_{j=1}^{n} K_j$, we clearly have:

$$\|f\|_\infty \geq \operatorname{Re}(e^{i\theta} f(x)) = \sum_{j=1}^{n} \operatorname{Re}(e^{i\theta} f_j(x_j)).$$

Choose $b_j \in K_j$ such that $|f_j(b_j)| = \|f_j\|_\infty$. Write $|f_j(b_j)| = e^{i\psi_j} f_j(b_j)$ for $\psi_j \in [-\pi, \pi[$. Now, for each $\theta \in [-\pi, \pi[$, $\operatorname{Re}(e^{i\theta} f_j(b_j))$ may, or may not, be nonnegative. If $\operatorname{Re}(e^{i\theta} f_j(b_j)) \geq 0$, set $a_j(\theta) = b_j$. If $\operatorname{Re}(e^{i\theta} f_j(b_j)) < 0$, define $a_j(\theta) \in K_j$ to be any element of K_j such that $\operatorname{Re}(e^{i\theta} f_j(a_j(\theta))) \geq 0$. This is possible since the convex hull of $f_j(K_j)$ contains 0. Thus, for all $\theta \in [-\pi, \pi[$, we have:

$$\operatorname{Re}(e^{i\theta} f_j(a_j(\theta))) \geq \max\{0, \cos(\theta - \psi_j)\|f_j\|_\infty\}.$$

Hence,

$$\|f\|_\infty \geq \sum_{j=1}^{n} \max\{0, \cos(\theta - \psi_j)\|f_j\|_\infty\}.$$

Now integrating both sides of the last inequality with respect to $\theta \in [-\pi, \pi[$ we obtain

$$2\pi \|f\|_\infty \geq 2 \sum_{j=1}^{n} \|f_j\|_\infty$$

from which the desired inequality follows.

To see that we have the best constant in the statement of the Lemma, we consider the following example. For each $j = 0, 1, \ldots, 2n-1$, we let $K_j = \{0, 1\}$, say. Define $f_j(0) = 0$ and $f_j(1) = e^{ij\pi/n}$. Clearly we have $\|f_j\|_\infty = 1$. To compute $\|f\|_\infty$, notice that

$$\|f\|_\infty = \operatorname{Re}\left(e^{i\theta} \sum_{j=0}^{2n-1} f_j(x_j)\right)$$

for some $0 \leq \theta < 2\pi$ and some $x_j \in K_j$. Thus it is clear that

$$\|f\|_\infty = \sum_{j=0}^{n-1} \sin\left(j\frac{\pi}{n} + \theta\right),$$

where $0 \leq \theta \leq \pi/n$. Thus, as $n \to \infty$,

$$\|f\|_\infty \approx n \int_0^1 \sin(\pi x)\, dx = \frac{2n}{\pi} = \frac{1}{\pi} \sum_{j=0}^{2n-1} \|f_j\|_\infty.$$

Corollary 3.7. *Suppose $E_j \subset \Gamma_j$, $S(E_j) \leq \varkappa$, for each $j=1,...,N$, and $0 \notin E_j$. Then $S\left(\bigvee_{j=1}^{N} E_j\right) \leq \pi \varkappa$.*

Proof. Let $E = \bigvee_{j=1}^{N} E_j$, and suppose that $f \in \mathcal{C}_E\left(\prod_{j=1}^{N} G_j\right)$. Then $f = \sum_{j=1}^{N} f_j$ with $f_j \in \mathcal{C}_{E_j}(G_j)$. The fact that $0 \notin E_j$ shows that f_j has mean 0, so Lemma 3.6 may be applied:

$$\pi \|f\|_\infty \geq \sum \|f_j\|_\infty \geq \varkappa^{-1} \sum_j \sum_{\gamma \in E_j} |\hat{f}_j(\gamma)|$$

$$= \varkappa^{-1} \sum_{\gamma \in E} |\hat{f}(\gamma)|.$$

To prove Theorem 3.5, apply Corollary 3.7, after removing 0 from E and putting it back, if necessary.

The following is a typical application of Theorem 1.3. It amounts to transferring, via Theorem 1.3, a known result about scalar valued Rademacher series to Sidon series. We need a definition.

Definition 3.8. Suppose that $a = (a_n)_{n=1}^{\infty} \in l^2$. Define the sequence $(a_n^*)_{n=1}^{\infty}$ to be the nondecreasing rearrangement of the terms $|a_n|$. If $0 < t < \infty$, we define

$$K_{1,2}(a,t) = \sum_{n=1}^{[t^2]} a_n^* + t\left(\sum_{[t^2]+1}^{\infty} (a_n^*)^2\right)^{1/2}$$

where $[t]$ denotes the integer part of t.

The next result follows from Theorem 1.3 and [Mo]. The latter is the following result for Rademacher series.

Theorem 3.9. *Suppose that $E = (\gamma_n) \subset \Gamma$ is a Sidon set. There is a constant $c > 0$ that depends only on $S(E)$ such that for all $a = (a_n)_{n=1}^{\infty} \in l^2$ we have*

$$\mu\left[\left|\sum_{n=1}^{\infty} a_n \gamma_n\right| \geq c K_{1,2}(a,t)\right] \leq c e^{-c^{-1} t^2}$$

and

$$\mu\left[\left|\sum_{n=1}^{\infty} a_n \gamma_n\right| \geq c^{-1} K_{1,2}(a,t)\right] \geq c^{-1} e^{-ct^2}$$

for all $t > 0$.

Thus it is possible to calculate rearrangement invariant norms of scalar valued Sidon series in the spirit of Rodin and Semyonov [RS].

We note that these results generalize to noncommutative compact groups with no further difficulties. We follow the notation of [Pi2], Section 5: G is a compact group; Σ is the dual object of G; μ is the normalized Haar measure on G. Thus Σ is the set of equivalence classes of the irreducible representations of G. For each $\iota \in \Sigma$, we let U_ι denote a representing element of the equivalence class. Thus $U_\iota(x)$ is, for each $x \in G$, a unitary operator on a fixed finite dimensional Hilbert space H_ι. The dimension of H_ι will be denoted by d_ι. For further details, we refer the reader to [Pi2], and [HR].

In analogy with Definition 1.1, a subset $S \subset \Sigma$ is called a *Sidon set* if there is a constant c such that

$$\sum d_\iota \operatorname{tr} |\hat{f}(\iota)| \leq c \|f\|_\infty$$

for every $f \in \mathcal{C}_S(G)$, where $\mathcal{C}_S(G)$ is defined as in the abelian case.

Let I be a countable indexing set. An analogue of the Rademacher functions is defined as a sequence, $\{\varepsilon_\iota\}_{\iota \in I}$, of independent random variables, each ε_ι being a random $d_\iota \times d_\iota$ orthogonal matrix, uniformly distributed on the orthogonal group $\mathcal{O}(d_\iota)$. These functions are studied in [Pi2] and [MP]. Note that we have the analogue of the Khintchin–Kahane inequality due to Pisier–Marcus, [MP, Corollary 2.12, p. 91].

The analogue of Pisier's Theorem 1.2 can be easily established in this setting by repeating the proof in [Pi1] and making use of the properties of Sidon sets on noncommutative groups. All of these properties are found in [HR, Theorem (37.2)]. For ease of reference, we state the result below, and omit the proof.

Theorem 3.10. *Suppose that $P=(\iota) \subset \Sigma$ is a Sidon set, that B is a Banach space, and that M_ι is a $d_\iota \times d_\iota$ matrix with values in B. Then there is a constant c_1, depending only on the Sidon constant $S(P)$, such that, for every $p \in [1, \infty[$, we have*

$$c_1^{-1} \left(\mathbf{E} \left\| \sum_{\iota \in F} d_\iota \operatorname{tr}(\varepsilon_\iota M_\iota) \right\|^p \right)^{1/p} \leq \left(\int_G \left\| \sum_{\iota \in F} d_\iota \operatorname{tr}(U_\iota M_\iota) \right\|^p \right)^{1/p}$$

$$\leq c_1 \left(\mathbf{E} \left\| \sum_{\iota \in F} d_\iota \operatorname{tr}(\varepsilon_\iota M_\iota) \right\|^p \right)^{1/p}$$

for any finite subset F of P.

The result concerning the join of Sidon sets in duals of compact groups is a straight analogue of Theorem 3.5. We omit even the statement. We have thus all the necessary ingredients to prove a noncommutative version of our main Theorem 1.3.

Theorem 3.11. *Let G be a compact group with dual object Σ. Suppose that $P = \{\iota\} \subset \Sigma$ is a Sidon set. For $n = 1, ..., N$, let a^n denote a $d_{\iota_n} \times d_{\iota_n}$ matrix with*

entries in a Banach space B. There is a constant c>0 such that, for all α>0, we have:

$$c^{-1}P\left[\left\|\sum_{n=1}^{N}d_{\iota_n}\operatorname{tr}(\varepsilon_{\iota_n}a^n)\right\|\geq c\alpha\right]\leq\mu\left[\left\|\sum_{n=1}^{N}d_{\iota_n}\operatorname{tr}(U_{\iota_n}a^n)\right\|\geq\alpha\right]$$
$$\leq cP\left[\left\|\sum_{n=1}^{N}d_{\iota_n}\operatorname{tr}(\varepsilon_{\iota_n}a^n)\right\|\geq c^{-1}\alpha\right].$$

We close our paper by pointing out to the interested reader that, under suitable conditions, these methods also apply to commensurate sets of characters of Pelczyński [Pe], as indeed they do also to topological Sidon sets, also described by Pelczyński loc. cit.

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