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A division problem in the space of entire functions of exponential type

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Introduction

The surjectivity of convolution operators and in particular of linear partial differential operators with constant coefficients is often equivalent to a problem of division in an appropriate space of entire functions of exponential type. In this frame Malgrange cite13 proved that for each convex domain G in $\mathbf{R}^N$ each partial differential operator on $\mathcal{E}(G)$ and each convolution operator on $A(\mathbf{C}^N)$ is surjective. Ehrenpreis [4] characterized the surjective convolution operators on $\mathcal{E}(\mathbf{R}^N)$ by means of lower bounds of its symbol. Hörmander [7] extended this result to convolution operators $\mathcal{E}(G_2)\to\mathcal{E}(G_1)$ where G_i, $i=1,2$ are convex domains in $\mathbf{R}^N$. Martineau [14] extended Malgrange's result proving that for each convex domain G in $\mathbf{C}^N$ each partial differential operator of infinite order on $A(G)$ is surjective. After these early papers, in the sequel several authors extended Ehrenpreis' result to convolution operators on spaces of ultradifferentiable functions $\mathcal{E}_\omega(\mathbf{R}^N)$, Chou [2], Cioranescu [3], Meise, Taylor, Vogt [15], and Momm [18]. Others considered convolution operators $A(G_2)\to A(G_1)$ for convex domains G_i, $i=1,2$ in $\mathbf{C}^N$, Korobeĭnik [9], Epifanov [5], [6], Tkačenko [29], [30], Napalkov [23], [24], Meril, Struppa [16], Morzhakov [21], [22], Sigurdsson [28], and recently Krivosheev [11] with the complete solution of this problem. There are many further results on this subject which cannot be mentioned here.

The aim of this paper is to present a systematic approach to division problems which applies in particular to all cases mentioned above (see [19] and [20]). We do this by extending Martineau's result to partial differential operators of infinite order on weighted spaces of analytic functions. Let G be a bounded convex domain of $\mathbf{C}^N$. For $\beta>0$ let A be the Frèchet space of all analytic functions f on G such that for each $k \in \mathbf{N}$ there exists $C_k > 0$ with

$$|f(z)| \leq C_k \exp\left(\frac{1}{k}\left(\frac{1}{d(z)}\right)^{\beta}\right), \quad z \in G,$$

where $d(z)$ is the distance of z to the boundary of G. If $F(z) = \sum_{\alpha \in \mathbf{N}_0^N} a_\alpha z^\alpha$ is an entire function such that for each $k \in \mathbf{N}$ there is $C_k > 0$ with

$$|F(z)| \leq C_k \exp\left(\frac{1}{k}|z|^{\beta'}\right), \quad z \in \mathbf{C}^N,$$

$(\beta' = \beta/(\beta+1))$ then a continuous linear differential operator of infinite order $L_F\colon A \to A$ is defined by $L_F(f) = \sum_\alpha a_\alpha f^{(\alpha)}$. In this paper in a general setting we characterize whether L_F is surjective. If H denotes the support function of G, then by the Fourier–Borel transform, the dual space A' of A can be identified with the space of all entire functions g satisfying

$$|g(z)| \leq C \exp\left(H(z) - \frac{1}{m}|z|^{\beta'}\right), \quad z \in \mathbf{C}^N,$$

for some $C, m > 0$ (Napalkov [25]). Let Γ_H denote the support of the Monge–Ampère measure $(dd^c H)^N$, i.e., $\mathbf{C}^N \setminus \Gamma_H$ is the maximal open cone with vertex in the origin on which H is an extremal plurisubharmonic function.

Theorem. *If G satisfies a technical condition, which is in particular fulfilled if G is a polyhedron, if ∂G is of Hölder class $C^{1,\lambda}$ for some $0 < \lambda \leq 1$, or if G is the Cartesian product of such domains, then for $F \not\equiv 0$ the following assertions are equivalent:*

(i) *$L_F\colon A \to A$ is surjective.*

(ii) *FA' is a closed subspace of A'.*

(iii) *Whenever $g \in A'$ and g/F is entire, then $g/F \in A'$.*

(E) *For each $k \in \mathbf{N}$ there is $R > 0$ such that for each $z \in \Gamma_H$ with $|z| \geq R$ there exists $w \in \mathbf{C}^N$ with $|w - z| \leq |z|^{\beta'}/k$ and $|F(w)| \geq \exp(-|w|^{\beta'}/k)$.*

The organization of the paper is as follows: In a preliminary section we introduce the spaces involved and the differential operator. Here we state the equivalence of (i) and (ii). In the second section we proceed as follows. By the open mapping theorem, (ii) implies certain norm estimates, i.e., the validity of a quantitative version of (iii). A careful use of the solution of the Dirichlet problem for the complex Monge–Ampère equation on balls $U(z, R)$ with boundary values $H(\zeta)$, $\zeta \in \partial U(z, R)$, and the application of Hörmander's $\bar{\partial}$-technique shows that this norm estimates imply a weak version (E′) of (E). In the case of a polyhedron both coincide. In the other cases we use the technical condition of the theorem's hypothesis to prove that (E′) implies (E). It is standard that (E) implies (ii) and (iii), since by [19] (see Krivosheev [11]), an appropriate Phragmén–Lindelöf theorem is available. In section three we evaluate the technical condition on G and apply the results of the previous sections to characterize the surjective differential operators of infinite order. Section four is devoted to a discussion of the criterion (E). In particular we prove that for the space A presented above there are also non surjective differential operators on A.

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1. Preliminaries

For the sequel let $N\in\mathbf{N}$ be fixed. We will use the abbreviation $\mathbf{R}_+:=\{x\in\mathbf{R}\mid x\geq 0\}$. For $z\in\mathbf{C}^N$ we put $|z|:=(\sum_{j=1}^N |z_j|^2)^{1/2}$. For $a\in\mathbf{C}^N$ and $R>0$ we put $U(a,R):=\{z\in\mathbf{C}^N\mid |z-a|<R\}$, $B(a,R):=\overline{U(a,R)}$, $S:=\{z\in\mathbf{C}^N\mid |z|=1\}$.

Fix a bounded convex domain G in $\mathbf{C}^N$. Let $H\colon\mathbf{C}^N\to\mathbf{R}$ be its support function

$$H(z):=\sup_{w\in G}\operatorname{Re}\langle z,w\rangle,\quad z\in\mathbf{C}^N,$$

where $\langle z,w\rangle:=\sum_{j=1}^N z_jw_j$, $z,w\in\mathbf{C}^N$. Note that H satisfies a Lipschitz condition $|H(z)-H(w)|\leq L|z-w|$, $z,w\in\mathbf{C}^N$, for some $L>0$. We put $d(z):=\inf\{\,|z-w|\mid w\in\partial G\,\}$, $z\in G$, which is concave as an infimum of affine functions.

For each open set $\Omega\subset\mathbf{C}^N$ let $A(\Omega)$ denote the space of all analytic functions on Ω. Furthermore we will apply the usual multiindex notation.

1.1. *Definition.* Let $\Psi=(\psi_k)_{k\in\mathbf{N}}$ be a sequence of continuous nonnegative non-decreasing unbounded functions on $]0,\infty[$ such that $\psi_k(e^x)$ is a convex function of $x\in\mathbf{R}$ for all $k\in\mathbf{N}$, and which satisfy the following conditions: For each $n\in\mathbf{N}$ there are $k\in\mathbf{N}$ and $x_0>0$ such that for all $x\geq x_0$

(α) $\psi_{k+1}(x)\leq\psi_k(x)$

(β) $2\psi_k(x)\leq\psi_n(x)$

(γ) $\psi_k(2x)\leq\psi_n(x)$

(δ) $\log x=o(\psi_n(x))$ as $x\to\infty$.

We put $\mathbf{P}:=(p_k)_{k\in\mathbf{N}}$ where

$$p_k(z):=\psi_k\left(\frac{1}{d(z)}\right),\quad z\in G,\ k\in\mathbf{N},$$

and define

$$A_{\mathbf{P}}^0 := \{ f \in A(G) \mid \|f\|_k = \sup_{z \in G} |f(z)| \exp(-p_k(z)) < \infty \text{ for each } k \in \mathbf{N} \}.$$

Endowed with the topology induced by the norms $(\| \ \|_k)_{k \in \mathbf{N}}$, the algebra $A_{\mathbf{P}}^0$ is a Fréchet space.

Remark. The Fréchet space $A_{\mathbf{P}}^0$ is nuclear. This useful information will not be applied in the sequel. Since the proof is folklore, we will only sketch it: Because of 1.1 (β) and (δ), $A_{\mathbf{P}}^0$ is the projective limit of the Banach spaces

$$A_k := \{ f \in A(G) \mid \|f\|_k < \infty \text{ and } \lim_{z \in G, z \to \partial G} |f(z)| \exp(-p_k(z)) = 0 \}.$$

Fix $n \in \mathbf{N}$. By 1.1, there are $k \in \mathbf{N}$ and $x_0 > 0$ such that

$$\psi_k(2x) + 2N \log x \le \psi_n(x) \quad \text{for all } \ x \ge x_0.$$

For each $z \in G$ let δ_z be the functional in A_k' defined by $\delta_z f = f(z)$, $f \in A_k$. The set

$$M := \{0\} \cup \{ \delta_z \exp(-p_k(z)) \mid z \in G \} \subset A_k'$$

is obviously essential in A_k' (see Pietsch [26, 2.3.1]). By the definition of A_k', the mapping $z \mapsto \delta_z \exp(-p_k(z))$ for $z \in G$ and $\infty \mapsto 0$ maps the one point compactification of G continuously onto M endowed with the weak* topology. Hence M is weak* compact. Since G is bounded, a Radon measure $\mu \in C(M)'$ is defined by

$$\mu(\phi) := \int_G \phi(\delta_z \exp(-p_k(z)))\, d\lambda(z), \quad \phi \in C(M),$$

where λ denotes the Lebesgue measure. By the choice of k, there is $C > 0$ such that by the subaveraging property of $|f|$, for each $f \in A_n$ and $w \in G$

$$\begin{aligned} |f(w)| \exp(-p_n(w)) &\le \lambda(B(w, d(w)/2))^{-1} \int_{B(w,d(w)/2)} |f|\, d\lambda \exp(-p_n(w)) \\ &\le C \int_{B(w,d(w)/2)} |f| \exp(-p_k)\, d\lambda \le C \int_G |f| \exp(-p_k)\, d\lambda. \end{aligned}$$

Hence $\|f\|_n \le C \int_M |\langle f, a \rangle|\, d\mu(a)$, which proves that the inclusion map $A_k \to A_n$ is absolutely summing (Pietsch [26, Theorem 2.3.3]). By Pietsch [26, Theorem 3.3.5], the projective spectrum $(A_k)_{k \in \mathbf{N}}$ is even nuclear and so is $A_{\mathbf{P}}^0$.

1.2. Lemma. *Let Ψ be as in 1.1. For $k\in\mathbf{N}$ and $r\in\mathbf{R}_+$ we put*

$$\omega_k(r):=\inf_{x>0}(\psi_k(1/x)+rx).$$

Then $\Omega:=(\omega_k)_{k\in\mathbf{N}}$ is a sequence of continuous nonnegative nondecreasing unbounded concave functions on $\mathbf{R}_+$, such that $\omega_k(e^x)$ is a convex function of $x\in\mathbf{R}$ for each $k\in\mathbf{N}$. Moreover for each $n\in\mathbf{N}$ there are $k\in\mathbf{N}$ and $r_0\geq 0$ such that for all $r\geq r_0$

(1) $\omega_{n+1}(r)\leq\omega_n(r)$

(2) $\log(1+r)=o(\omega_n(r))$

(3) $\omega_n(r)=o(r)$

(4) $2\omega_k(r)\leq\omega_n(r)$

(5) $\omega_k(2r)\leq\omega_n(r)$.

Proof. Obviously for each $k\in\mathbf{N}$, the function ω_k is concave (as an infimum of affine functions), nonnegative and nondecreasing. Since ψ_k is unbounded, so is ω_k. By Napalkov [25, Lemma 5], the function $x\mapsto\omega_k(e^x)$ is convex. (3) follows directly from the definition of ω_n. Next note that for each $r>0$ there is $x(r)>0$ with $\omega_k(r)=\psi_k(1/x(r))+rx(r)$; because of (3) we have $\lim_{r\to\infty}x(r)=0$. This shows that 1.1 (α) implies (1), (δ) implies (2), (β) implies (4), and (γ) implies (5).

Remark. If vice versa a system $\Omega=(\omega_k)_{k\in\mathbf{N}}$ with the properties stated in 1.2 is given, then defining $\psi_k(x):=\sup_{t\geq 0}(\omega_k(t)-t/x)$, $x>0$, $k\in\mathbf{N}$, we get a system $\Psi=(\psi_k)_{k\in\mathbf{N}}$ which is equivalent to one (see 1.4) which satisfies the conditions of 1.1. This correspondence is one to one (up to equivalence, see [17, 1.9, 6.9, 6.10]).

Convention. In the sequel let $\mathbf{P}$ and Ω be as in 1.1 and 1.2, respectively.

1.3. *Definition.* We extend the definition of ω_k to the whole of $\mathbf{C}^N$ by $\omega_k(z):=\omega_k(|z|)$, $z\in\mathbf{C}^N$, $k\in\mathbf{N}$. We define

$$A_{H-\Omega}:=\{\,f\in A(\mathbf{C}^N)\mid |f|_k=\sup_{z\in\mathbf{C}^N}|f(z)|e^{-H(z)+\omega_k(z)}<\infty \text{ for some } k\in\mathbf{N}\,\}$$

and denote by A^0_Ω the algebra of all entire functions $F\in A(\mathbf{C}^N)$ such that for each $k\in\mathbf{N}$ there is some $C>0$ with

$$|F(z)|\leq C\exp\omega_k(z),\quad z\in\mathbf{C}^N.$$

We endow $A_{H-\Omega}$ with the inductive limit topology induced by $(|\ |_k)_{k\in\mathbf{N}}$. Obviously we have $A^0_\Omega\cdot A_{H-\Omega}\subset A_{H-\Omega}$.

Remark. By a similar reasoning as after Definition 1.1, one can show that $A_{H-\Omega}$ is the dual of a nuclear Fréchet space. (This will also follow from Proposition 1.6.)

1.4. *Remark.* There is $K\geq 1$ such that for each $k\in\mathbf{N}$ there exists $C_k>0$ such that

$$H(z)-\omega_k(Kz)-C_k\leq \sup_{\zeta\in G}(\mathrm{Re}\langle z,\zeta\rangle-p_k(\zeta))\leq H(z)-\omega_k(z)+C_k,\quad z\in\mathbf{C}^N,$$

(Napalkov [25, formula (25)], although only stated for $N=1$ there). In particular there is a system $(u_k)_{k\in\mathbf{N}}$ consisting of plurisubharmonic functions on $\mathbf{C}^N$ which is equivalent to $(H-\omega_k)_{k\in\mathbf{N}}$.

Here and in the sequel two systems $(v_k)_{k\in\mathbf{N}}$ and $(\tilde{v}_k)_{k\in\mathbf{N}}$ are called *equivalent* if for each $n\in\mathbf{N}$ there are $k\in\mathbf{N}$ and $C>0$ such that $v_k\leq\tilde{v}_n+C$ and $\tilde{v}_k\leq v_n+C$.

1.5. Proposition. *For each $F\in A^0_\Omega$, $F(z)=\sum_{\alpha\in\mathbf{N}_0^N}a_\alpha z^\alpha$, a continuous linear operator is given by*

$$L_F\colon A^0_{\mathbf{P}}\to A^0_{\mathbf{P}},\quad L_F(f):=\sum_{\alpha\in\mathbf{N}_0^N}a_\alpha f^{(\alpha)}.$$

Proof. Let $k\in\mathbf{N}$. By Cauchy's integral formula, there is $C_k>0$ with

$$|a_\alpha|\leq C_k\inf_{r>0}r^{-|\alpha|}\exp\omega_k(r),\quad \alpha\in\mathbf{N}_0^N.$$

By the same reason, we get for $f\in A^0_{\mathbf{P}}$

$$|f^{(\alpha)}(z)|\leq\|f\|_k\alpha!(d(z)/2)^{-|\alpha|}\exp\psi_k(2/d(z)),\quad z\in G,\ \alpha\in\mathbf{N}_0^N.$$

Let $z\in G$ and $\alpha\in\mathbf{N}_0^N$. We apply the estimate for $|a_\alpha|$ with $r=4|\alpha|/d(z)$. By the definition of ω_k, we get

$$\begin{aligned}|a_\alpha||f^{(\alpha)}(z)|&\leq C_k\|f\|_k(2|\alpha|)^{-|\alpha|}\alpha!\exp\{\omega_k(4|\alpha|/d(z))+\psi_k(2/d(z))\}\\&\leq C_k\|f\|_k(2|\alpha|)^{-|\alpha|}\alpha!\exp\{\psi_k(4/d(z))\\&\quad+(4|\alpha|/d(z))(d(z)/4)+\psi_k(2/d(z))\}\\&\leq C_k\|f\|_k(2|\alpha|/e)^{-|\alpha|}|\alpha|!\exp\{\psi_k(4/d(z))+\psi_k(2/d(z))\}.\end{aligned}$$

Hence by Stirling's formula, there is $C_k'>0$ not depending on f with

$$\sum_\alpha|a_\alpha||f^{(\alpha)}(z)|\leq C_k'\|f\|_k\exp\{\psi_k(4/d(z))+\psi_k(2/d(z))\},\quad z\in G.$$

By 1.1(β) and (γ), for each $n\in\mathbf{N}$ there are $k\in\mathbf{N}$ and $C_k''>0$ with

$$\sum_\alpha|a_\alpha||f^{(\alpha)}(z)|\exp(-p_n(z))\leq C_k''\|f\|_k,\quad z\in G,\ f\in A^0_{\mathbf{P}}.$$

This proves the assertion.

1.6. Proposition. *The Fourier–Borel transform $\mathcal{F}\colon A^0_{\mathbf{P}}{}' \to A_{H-\Omega}$, $\mathcal{F}(\mu)(z) := \mu(e^{\langle z,\cdot\rangle})$, $z\in\mathbf{C}^N$, is an isomorphism. If we identify the strong dual space $A^0_{\mathbf{P}}{}'$ with $A_{H-\Omega}$ by this isomorphism, then for each $F\in A^0_\Omega$ the transposed map L^t_F (see 1.5) is the operator $M_F\colon A_{H-\Omega}\to A_{H-\Omega}$ of multiplication by F. Hence by duality theory, L_F is surjective if and only if $F\not\equiv 0$ and $FA_{H-\Omega}$ is a closed subspace of $A_{H-\Omega}$.*

Proof. The assertion about the Fourier–Borel transform (although only stated for $N=1$ there) can be found in Napalkov [25]. We have

$$\mathcal{F}(L^t_F(\mu))(z) = (\mu\circ L_F)(e^{\langle z,\cdot\rangle}) = F(z)\mathcal{F}(\mu)(z), \quad \mu\in A^0_{\mathbf{P}}{}',\ z\in\mathbf{C}^N.$$

1.7. *Examples.* The following systems $\Psi=(\psi_k)_{k\in\mathbf{N}}$ satisfy the conditions of 1.1. The associated functions $\Omega=(\omega_k)_{k\in\mathbf{N}}$ (see 1.2) are calculated up to equivalence (see 1.4):

(1) $\psi_k(x)=x^\beta/k$, $\omega_k(r)=r^{\beta'}/k$, where $\beta>0$ and $\beta':=\beta/(\beta+1)$.

(2) $\psi_k(x)=x^{\beta_k}$, $\omega_k(r)=r^{\beta'_k}$, where $0<\beta_k$ strictly decreases with k and $\beta'_k:=\beta_k/(\beta_k+1)$.

(3) $\psi_k(x)=(\log(1+x))^s/k$, $\omega_k(r)=(\log(1+r))^s/k$, where $s>1$.

To get (3) we note that direct computation gives for each $r>0$ that $\omega_k(r)=(1/k)(\log(1+1/x))^s+rx$, for some $x>0$ with $rx=(s/k)(\log(1+1/x))^{s-1}/(x+1)$. For large r the value of x is small. This shows that the dominating part in the representation of $\omega_k(r)$ is the first one. The equation for x implies the rough bounds $(s/(2k))(1/x)\le r\le (s/k)(1/x)^s$ for large $r>0$. Inserting this, we obtain the desired estimates.

2. Solution of the division problem

In view of 1.6, looking for a characterization of surjective operators L_F, we have to characterize when $FA_{H-\Omega}$ is closed in $A_{H-\Omega}$. We will apply a procedure which is roughly that one which has been introduced by Ehrenpreis [4]. In order to estimate more carefully we use the following notation.

2.1. *Definition.* For $a\in\mathbf{C}^N$ and $R>0$ we consider the real-valued function $h=h(H;a,R)$ which equals H on $\mathbf{C}^N\setminus U(a,R)$ and with $h(z)$ for $z\in(a,R)$ given by

$$\sup\{u(z)\mid u \text{ psh. on } U(a,R),\ \limsup_{w\to\zeta} u(w)\le H(\zeta) \text{ for } \zeta\in\partial U(a,R)\}.$$

Then h is continuous and plurisubharmonic (psh.) (see [19, Lemma 2]). We put

$$\Delta(H;a,R) := \sup_{z\in U(a,R)} (h(H;a,R)(z)-H(z)).$$

Then $\Delta(H;a,R)\geq 0$ because H is plurisubharmonic. Since H is positively homogeneous, so is $\Delta(H;\cdot,\cdot)$, i.e. $\Delta(H;\lambda a,\lambda R)=\lambda\Delta(H;a,R)$ for all $\lambda>0$.

By S_H^* we denote the set of all $a\in S$, such that $\Delta(H;a,R)>0$ for all $R>0$. It is easy to see that S_H^* is compact. We consider the cone $\Gamma_H:=\{\lambda a \mid \lambda\geq 0, a\in S_H^*\}$.

Remark. By Bedford and Taylor [1, §9], Γ_H is the support of the Monge–Ampère measure $(dd^cH)^N$.

2.2. Proposition. *For each $a\in\Gamma_H$ the function $\Delta(H;a,\cdot)\colon]0,\infty[\to]0,\infty[$, $R\mapsto\Delta(H;a,R)$ is continuous, strictly increasing and surjective. By $\Delta^{-1}(H;a,\cdot)$ we denote the inverse function. The function $\Delta^{-1}(H;\cdot,\cdot)$ is positively homogeneous.*

Proof. For all $a,\tilde a\in\mathbf{C}^N$ and $R,\widetilde R>0$ we have

$$\begin{aligned}\Delta(H;a,R)&=\sup_{z\in B(0,1)}\left(h(H(a+R\cdot);0,1)(z)-H(a+Rz)\right)\\&\leq\sup_{z\in B(0,1)}\left(h(\widetilde H(\tilde a+\widetilde R\cdot);0,1)(z)-\widetilde H(\tilde a+\widetilde Rz)\right)\\&\quad+2\sup_{z\in B(0,1)}|H(a+Rz)-\widetilde H(\tilde a+\widetilde Rz)|\\&=\Delta(\widetilde H;\tilde a,\widetilde R)+2\sup_{z\in B(0,1)}|H(a+Rz)-\widetilde H(\tilde a+\widetilde Rz)|\end{aligned}$$

and thus

$$|\Delta(H;a,R)-\Delta(\widetilde H;\tilde a,\widetilde R)|\leq 2\sup_{z\in B(0,1)}|H(a+Rz)-\widetilde H(\tilde a+\widetilde Rz)|.$$

In particular, $\Delta(H;a,R)$ is a continuous function of $(a,R)\in\mathbf{C}^N\times]0,\infty[$.

If $a\in\mathbf{C}^N$ and $0<R<\widetilde R$ we put $\lambda:=(|a|+\widetilde R)/(|a|+R)$. We then get $U(\lambda a,\lambda R)\subset U(a,\widetilde R)$ and $\lambda\Delta(H;a,R)\leq\Delta(H;a,\widetilde R)$. If $a\in\Gamma_H$, this shows that $\Delta(H;a,\cdot)$ is strictly increasing and unbounded. Since $\lim_{R\to 0}\Delta(H;a,R)=0$, $\Delta(H;a,\cdot)$ is onto.

Our theorem is prepared by the following lemmas of which the first one contains the essential idea.

2.3. Lemma. *Let $F\in A_\Omega^0\setminus\{0\}$. Then* (i) $\Rightarrow$ (ii):

(i) *For each $k\in\mathbf{N}$ there are $n\in\mathbf{N}$ and $C>0$ such that for all $f\in A_{H-\Omega}$ we have $|f|_n\leq C|Ff|_k$.*

(ii) *For each $k\in\mathbf{N}$ there is $R>0$ such that for each $z\in\Gamma_H$ with $|z|\geq R$ there exists $w\in B(z,\Delta^{-1}(H;z,\omega_k(z)))$ such that $|F(w)|\geq\exp(-\omega_k(w))$.*

Proof. For $k\in\mathbf{N}$ we put

$$\sigma_k(z):=\Delta^{-1}(H;z,\omega_k(z))=|z|\Delta^{-1}(H;z/|z|,\omega_k(z)/|z|),\quad z\in\Gamma_H\setminus\{0\}.$$

Since $\lim_{R\to 0}\Delta^{-1}(H;a,R)=0$ uniformly for $a\in S_H^*$, by 1.2, we may assume that $\sigma_k(z)\le|z|/2$, $z\in\Gamma_H\backslash\{0\}$, $k\in\mathbf{N}$.

Assume that (ii) does not hold. Then there are $\tilde{k}\in\mathbf{N}$ and a sequence $(z^j)_{j\in\mathbf{N}}$ in Γ_H with $\lim_{j\to\infty}|z^j|=\infty$ such that

$$(1)\qquad |F(w)|<\exp(-\omega_{\tilde{k}}(w))\quad \text{for all}\quad w\in B(z^j,\sigma_{\tilde{k}}(z^j))\quad\text{and}\quad j\in\mathbf{N}.$$

Since H is Lipschitz continuous, by 2.1, there is $L>0$ such that for each $z\in\Gamma_H$ we have $\Delta(H;z,R+1)\le\Delta(H;z,R)+L$ for $R>0$. Thus we obtain $\Delta^{-1}(H;z,x)+1\le\Delta^{-1}(H;z,x+L)$, $x>0$. Hence by 1.2, we can choose $m\ge\tilde{k}$ and $t_0>0$ such that

$$(2)\qquad \omega_m(2z)\le\omega_{\tilde{k}}(z),\ z\in\mathbf{C}^N,\quad\text{and}\quad \sigma_m(z)+1\le\sigma_{\tilde{k}}(z),\ z\in\Gamma_H,$$

if $|z|\ge t_0$. We are going to derive a contradiction to (i). Fix $j\in\mathbf{N}$ with $|z^j|\ge 2t_0$. Put $h_j:=h(H;z^j,\sigma_m(z^j))$ (see 2.1). Choose $x^j\in B(z^j,\sigma_j(z^j))$ with

$$(3)\qquad h_j(x^j)=H(x^j)+\Delta(H;z^j,\sigma_m(z^j))=H(x^j)+\omega_m(z^j).$$

According to 1.2 and 1.4, we choose a psh. function u with

$$(4)\qquad H(z)-\omega_m\left(\tfrac{2}{3}z\right)\le u(z)\le H(z)-\omega_{\tilde{n}}(z)+C,\quad z\in\mathbf{C}^N,$$

for some $\tilde{n}\in\mathbf{N}$ and $C>0$. We consider the psh. function $\phi_j:=(h_j+u)/2$. By Theorem 4.4.4 of Hörmander [8] and standard arguments, we get $f_j\in A(\mathbf{C}^N)$ such that

$$(5)\qquad f_j(x^j)=\exp\phi_j(x^j)$$

and

$$(6)\qquad |f_j(z)|\le C(N)\exp\Big\{\sup_{w\in B(z,1)}\phi_j(w)+C(N)\log(1+|z|^2)\Big\},\quad z\in\mathbf{C}^N,$$

where $C(N)>0$ only depends on N. Since H is Lipschitz, applying (3) and (4) we get

(7)
$$\sup_{w\in B(z,1)}\phi_j(w)\le\begin{cases} H(z)+L'-\omega_{\tilde{n}}(|z|-1)/2 & \text{for } z\notin B(z^j,\sigma_m(z^j)+1)\\ H(z)+L'-\omega_{\tilde{n}}(|z|-1)/2+\omega_m(z^j) & \text{for } z\in B(z^j,\sigma_m(z^j)+1)\end{cases}$$

with some $L'>0$ not depending on j. In particular $f_j\in A_{H-\Omega}$.

We now estimate Ff_j. If $z\in B(z^j,\sigma_m(z^j)+1)$, then $|z-z^j|\le|z^j|/2$ and $|z|\ge|z^j|/2\ge t_0$, and by (1), (2), (6), and (7), we obtain

$$\begin{aligned}|F(z)f_j(z)|&\le C(N)e^{L'}\exp\{H(z)+C(N)\log(1+|z|^2)-\omega_{\tilde{n}}(|z|-1)/2\\&\qquad+\omega_m(2z)-\omega_{\tilde{k}}(z)\}\\&\le C(N)e^{L'}\exp\{H(z)+C(N)\log(1+|z|^2)-\omega_{\tilde{n}}(|z|-1)/2\}.\end{aligned}$$

Hence by (6), (7), and 1.2, there is some $k\in\mathbf{N}$ with

$$\sup_{j\in\mathbf{N}}\sup_{z\in\mathbf{C}^N}|F(z)f_j(z)|\exp(-H(z)+\omega_k(z))<\infty. \tag{8}$$

Since $|x^j-z^j|\le|z^j|/2$ and $|z^j|\ge\frac{2}{3}|x^j|$, on the other hand, we obtain from (5), (3) and (4) that

$$|f_j(x^j)|\ge\exp\{H(x^j)+\omega_m(z^j)/2-\omega_m\left(\tfrac{2}{3}x^j\right)/2\}\ge\exp(H(x^j)).$$

Since $|x^j|\ge|z^j|/2$, we have $\lim_{j\to\infty}|x^j|=\infty$. Hence by 1.2, we conclude that for each $n\in\mathbf{N}$

$$\lim_{j\to\infty}|f_j(x^j)|\exp\{-H(x^j)+\omega_n(x^j)\}=\infty.$$

Together with (8) this is a contradiction to (i).

2.4. Lemma. *For $0<\lambda<1$ and $z\in\mathbf{C}^N\setminus\{0\}$ let $0<\tau(z)<\sigma(z)<|z|$ with $\sigma(z)=\tau(z)^\lambda|z|^{1-\lambda}$. Let F be analytic in $B(z,|z|)$. Let $\omega,\widetilde{\omega}\colon B(z,|z|)\to\mathbf{R}_+$ with*

$$\frac{1}{\lambda}\sup_{w\in B(z,\sigma(z))}\omega(w)+\frac{1-\lambda}{\lambda}\log\sup_{w\in B(z,|z|)}|F(w)|\le\inf_{w\in B(z,\tau(z))}\widetilde{\omega}(w). \tag{9}$$

If there is $w\in B(z,\sigma(z))$ with $|F(w)|\ge\exp(-\omega(w))$ then there is $w\in B(z,\tau(z))$ with $|F(w)|\ge\exp(-\widetilde{\omega}(w))$.

Proof. We prove the contraposition. Assume that for all $w\in B(z,\tau(z))$ we have $|F(w)|<\exp(-\widetilde{\omega}(w))$. We fix $w\in B(z,\sigma(z))\setminus\{z\}$, put $h:=w-z$, and

$$M(r):=\max\{\,|F(z+\zeta h/|h|)|\mid\zeta\in\mathbf{C},|\zeta|=r\,\},\quad r>0.$$

Put $r_1:=\tau(z)$, $r_2:=\sigma(z)$, and $r_3:=|z|$. For $|\zeta|\le r_1$ and $x:=z+\zeta h/|h|$, we have $|x-z|\le\tau(z)$. For $|\zeta|\le r_3$ and $x:=z+\zeta h/|h|$, we have $|x-z|\le|z|$. We apply Hadamard's three-circles-theorem to r_1, r_2, and r_3, and we obtain by (9)

$$\begin{aligned}\log|F(w)|&\le\log M(r_2)\\&\le\frac{\log(r_3/r_2)}{\log(r_3/r_1)}\log M(r_1)+\frac{\log(r_2/r_1)}{\log(r_3/r_1)}\log M(r_3)\\&=\lambda\log M(r_1)+(1-\lambda)\log M(r_3)\\&<-\lambda\inf_{x\in B(z,\tau(z))}\widetilde{\omega}(x)+(1-\lambda)\log\sup_{x\in B(z,|z|)}|F(x)|\\&\le-\sup_{x\in B(z,\sigma(z))}\omega(x)\le-\omega(w).\end{aligned}$$

We will apply the following lemma from Hörmander [7, Lemma 3.2.]

2.5 Lemma. *Let g, F and g/F be entire. Then we have for all $r>0$ and $z\in\mathbf{C}^N$*

$$|g(z)/F(z)| \le \sup_{|z-\zeta|<4r} |g(\zeta)| \sup_{|z-\zeta|<4r} |F(\zeta)|/(\sup_{|z-\zeta|<r} |F(\zeta)|)^2.$$

We are now ready to state our main result.

2.6. Theorem. *Let $\Omega=(\omega_k)_{k\in\mathbf{N}}$ be as in 1.2. Assume in addition that H and Ω satisfy the following condition: There is some $\lambda>0$ such that for each $n\in\mathbf{N}$ there are $k\in\mathbf{N}$ and $r_0>0$ with*

$$\omega_k(z)/|z| \le \Delta(H; z/|z|, (\omega_n(z)/|z|)^\lambda), \quad z\in\Gamma_H, \ |z|\ge r_0.$$

Then for each $F\in A^0_\Omega\setminus\{0\}$ the following are equivalent:

(i) *Whenever $g\in A_{H-\Omega}$ and $f:=g/F\in A(\mathbf{C}^N)$, then $f\in A_{H-\Omega}$.*

(ii) *$FA_{H-\Omega}$ is a closed subspace of $A_{H-\Omega}$.*

(iii) *For each $k\in\mathbf{N}$ there are $n\in\mathbf{N}$ and $C>0$ such that for all $f\in A_{H-\Omega}$ we have $|f|_n\le C|Ff|_k$.*

(iv) *For each $k\in\mathbf{N}$ there is $R>0$ such that for each $z\in\Gamma_H$ with $|z|\ge R$ there exists $w\in B(z,\Delta^{-1}(H;z,\omega_k(z)))$ such that $|F(w)|\ge\exp(-\omega_k(w))$.*

(v) *For each $k\in\mathbf{N}$ there is $R>0$ such that for each $z\in\Gamma_H$ with $|z|\ge R$ there exists $w\in B(z,\omega_k(z))$ such that $|F(w)|\ge\exp(-\omega_k(w))$.*

Without the additional condition for H and Ω, we still have (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) *and* (v) $\Rightarrow$ (i).

Proof. (i) $\Rightarrow$ (ii): $A_{H-\Omega}$ is continuously embedded in $A(\mathbf{C}^N)$.

(ii) $\Rightarrow$ (iii): Since $FA_{H-\Omega}$ is closed, $\mathrm{ind}_{k\to\infty}(A_{H-\omega_k}\cap FA_{H-\Omega})$ is an (LF)-space, where $A_{H-\omega_k}:=\{f\in A_{H-\Omega} \mid |f|_k<\infty\}$. Hence we can apply the open mapping theorem for (LF)-spaces to the continuous and bijective map $M_F: A_{H-\Omega}\to \mathrm{ind}_{k\to\infty}(A_{H-\omega_k}\cap FA_{H-\Omega})$, $M_F(f)=Ff$, which shows that M_F is a linear topological isomorphism onto $FA_{H-\Omega}$ endowed with the topology of $\mathrm{ind}_{k\to\infty}(A_{H-\omega_k}\cap FA_{H-\Omega})$. Thus (iii) follows from Grothendieck's factorization theorem (see e.g., Köthe [10, § 19, 5.(4)]).

(iii) $\Rightarrow$ (iv): Lemma 2.3.

(iv) $\Rightarrow$ (v): We may assume that $0<\lambda<1$. Let $\tilde{k}\in\mathbf{N}$. Since $\Delta(H;\cdot,\cdot)$ is homogeneous, by the extra hypothesis on H and Ω, there are $k\in\mathbf{N}$ and $R>0$ with $\Delta^{-1}(H;z,\omega_k(z))\le\omega_{\tilde{k}}(z)^\lambda|z|^{1-\lambda}$, $z\in\Gamma_H$, $|z|\ge R$. We may assume that R is so large that (iv) holds. We put $\tau(z):=\omega_{\tilde{k}}(z)$. By 1.2 we may furthermore assume that k and R are chosen so large that for all $|z|\ge R$ we have $\sigma(z):=\tau(z)^\lambda|z|^{1-\lambda}<|z|/2$ and that

$$\frac{1}{\lambda}\omega_k(2|z|)+\frac{1-\lambda}{\lambda}\log \sup_{w\in B(z,|z|)} |F(w)| \le \omega_{\tilde{k}}(|z|/2).$$

Applying 2.4 with $\omega=\omega_k$ and $\widetilde{\omega}=\omega_{\tilde{k}}$, we get (v) from (iv).

(v) $\Rightarrow$ (i): Let $g\in A_{H-\Omega}$ with $f:=g/F\in A(\mathbf{C}^N)$. Let $n\in\mathbf{N}$ with $|g|_n<\infty$. For $k\geq n$ (which will be determined later) we choose $R>0$ according to (v). We may assume $|F(\zeta)|\leq\exp\omega_k(\zeta)$, $\zeta\in\mathbf{C}^N$, and that $0<\omega_k(z)\leq|z|/8$, $|z|\geq R$. Fix $z\in\Gamma_H$ with $|z|\geq R$. We choose $w\in B(z,\omega_k(z))$ according to (v) and apply Lemma 2.5 with $r=\omega_k(z)$ to obtain

$$|f(z)|\leq|g|_n \sup_{|z-\zeta|<4r}\exp\{H(\zeta)-\omega_n(\zeta)\}\sup_{|z-\zeta|<4r}\exp(\omega_k(\zeta))/(\exp(-\omega_k(w))^2.$$

If $|z-\zeta|\leq 4r\leq|z|/2$ then $|z|/2\leq|\zeta|\leq\frac{3}{2}|z|$. Since H is Lipschitz continuous, there is a constant $L>0$ depending on H such that $|H(z)-H(\zeta)|\leq L|z-\zeta|$ for all $\zeta\in\mathbf{C}^N$. Hence we obtain

$$|f(z)|\leq|g|_n\exp\big\{H(z)+4L\omega_k(z)-\omega_n(z/2)+\omega_k\big(\tfrac{3}{2}z\big)+2\omega_k\big(\tfrac{3}{2}z\big)\big\}.$$

By 1.2 we may choose k so large that there is $C>0$ with

$$\omega_k(2s)+3\omega_k(3s)\leq\omega_n(s)+C\quad\text{for all } s\in\mathbf{R}_+.$$

Therefore we obtain

$$\log|f(z)|+\omega_k(z)\leq C+\log|g|_n+H(z),\quad z\in\Gamma_H.$$

By 1.2, $x\mapsto\omega_k(e^x)$ is convex. Hence the function ω_k is plurisubharmonic on $\mathbf{C}^N$. On the other hand it is well known that f is a function of exponential type. Thus we can apply the Phragmén–Lindelöf theorem [20, Thm. 7], to the plurisubharmonic function $\log|f|+\omega_k-C-\log|g|_n$ to obtain the preceding estimate on the whole of $\mathbf{C}^N$. So we are done.

2.7. Corollary. *For $N=1$ let $\Omega=(\omega_k)_{k\in\mathbf{N}}$ be as in* 1.2. *Then for each $F\in A^0_\Omega\setminus\{0\}$ the assertions* (i), (ii), (iii) *and* (iv) *of Theorem* 2.6 *are equivalent.*

Proof. By Theorem 2.6, we have only to prove that (iv) implies (i). Let $g\in A_{H-\Omega}$ with $f:=g/F\in A(\mathbf{C})$. Let $n\in\mathbf{N}$ with $|g|_n<\infty$. For $k\geq n$ (which will be determined later) we choose $R>0$ according to (iv). We may assume $|F(\zeta)|\leq\exp\omega_k(\zeta)$, $\zeta\in\mathbf{C}$, and furthermore that $\Delta^{-1}(H;z,\omega_k(z))\leq|z|/(2(4e+1))$, $z\in\Gamma_H$, $|z|\geq R$, because $\Delta^{-1}(H;\cdot,\cdot)$ is homogeneous. Fix $z\in\Gamma_H$ with $|z|\geq R$. Choose $w\in\mathbf{C}$ according to (iv). Note that for $|\zeta-w|\leq(4e/(2(4e+1)))|z|$ we have

$$|z|\leq|z-w|+|w-\zeta|+|\zeta|\leq|z|/(2(4e+1))+(4e/(2(4e+1)))|z|+|\zeta|\leq|z|/2+|\zeta|,$$

hence $|z|\le 2|\zeta|$; on the other hand $|\zeta|\le|z|+|z-w|+|w-\zeta|\le\frac{3}{2}|z|$.

We may assume that $w\neq z$. We apply the minimum modulus theorem from Levin [12, Thm. 11 of Chapter I], with $\eta=\left(\frac{5}{4}-1\right)/\left(\frac{5}{4}\cdot 16\right)$ and $R=\frac{5}{4}|w-z|$ and get $|w-z|<r_1<\frac{5}{4}|w-z|$ and absolute constants $a, A>0$ such that

$$|F(\zeta)|\ge|F(w)|^{a+1}\Big(\max_{|x-w|\le 2eR}|F(x)|\Big)^{-a}\ge\exp(-A\omega_k(A\zeta)),\quad |\zeta-w|=r_1.$$

We choose $\zeta_0\in\mathbf{C}$ with $|\zeta_0-w|=r_1$ and $|\zeta_0-z|<|w-z|/4$. We apply [12, Thm. 11], again, with $\eta=\frac{1}{32}$ and $R=2|w-z|/4$ and get $|w-z|/4<\tilde r_1<|w-z|/2$ and an absolute constant $\widetilde A>0$ such that

$$|F(\zeta)|\ge\exp(-\widetilde A\omega_k(\widetilde A\zeta)),\quad |\zeta-\zeta_0|=\tilde r_1$$

and

$$|f(\zeta)|=|g(\zeta)/F(\zeta)|\le|g|_n\exp\{H(\zeta)-\omega_n(\zeta)+\widetilde A\omega_k(\widetilde A\zeta)\},\quad |\zeta-\zeta_0|=\tilde r_1.$$

Since $z\in B(\zeta_0,\tilde r_1)\subset B(z,\Delta^{-1}(H;z,\omega_k(z)))$, we get from 2.1

$$\begin{aligned}\log|f(z)|&\le\log|g|_n+h(H;z,\Delta^{-1}(H;z,\omega_k(z)))(z)-\omega_n(z/3)+\widetilde A\omega_k(3\widetilde Az)\\&\le\log|g|_n+H(z)+\omega_k(z)-\omega_n(z/3)+\widetilde A\omega_k(3\widetilde Az).\end{aligned}$$

If k is chosen sufficiently large, by 1.2, there is $C>0$ such that

$$\log|f(z)|+\omega_k(z)\le C+\log|g|_n+H(z),\quad z\in\Gamma_H.$$

Arguing as in 2.6, from the classical Phragmén–Lindelöf theorem for cones, we get that the preceding estimate holds on the whole of $\mathbf{C}$.

2.8. *Remark.* If G is the Cartesian product of N bounded convex domains in the plane (see 3.4), then in 2.6(iv) and (v) we can achieve that in addition $w\in\Gamma_H$. Since (with the notation of 3.4) in this case $\Gamma_H=\prod_{l=1}^N\Gamma_{H_l}$, by standard arguments applying the minimum modulus theorem N-times (see the proof of 2.7) one gets $\tilde w\in\Gamma_H$ with $|\tilde w-z|\le|w-z|$ and $|F(\tilde w)|\ge\exp(-A\omega_k(A\tilde w))$, where $A>0$ is a constant which depends only on N.

3. Surjective differential operators

Before we rephrase the assertion of Theorem 2.6 for differential operators L_F, we will discuss the extra hypothesis on H and Ω which is given in terms of the function $\Delta(H;\cdot,\cdot)$ of 2.1.

3.1. Lemma. *If G is a bounded open convex polyhedron in $\mathbf{C}^N$, then there exists $\varepsilon>0$ with*

$$\Delta(H;a,R)\geq\varepsilon R,\quad R>0,\ a\in\Gamma_H.$$

Proof. We will modify the proof of [20, Lemma 4]. With G also its polar set $\Omega:=\{z\in\mathbf{C}^N\,|\,H(z)\leq 1\}$ is a polyhedron. Hence by [20, Prop. 9], we have $\Gamma_H=\bigcup\{\lambda a\,|\,\lambda\geq 0, a\in F\}$ where the (finite) union is taken over all real exposed faces $F\subset\partial\Omega$ of Ω. We only need to know that this implies that for each $a\in\Gamma_H$ there is a face F with $a\in\{\lambda a\,|\,\lambda\geq 0, a\in F\}$ and there is the support function $\widetilde{H}$ of a suitable affine transformation of G such that

$$\Delta(H;a,R)=\Delta(\widetilde{H};a,R),\quad a\in\{a\in\mathbf{C}^N\mid\widetilde{H}(a)=0\}\subset\mathbf{R}^N,$$

and $\widetilde{H}\geq 0$. Since $\widetilde{H}$ is the maximum of finitely many linear functions, this shows that there is some $\varepsilon=\varepsilon(F)>0$ with $\widetilde{H}(z)\geq\varepsilon|\operatorname{Im}z|$ for all $z\in\mathbf{C}^N$. We define $v:=h(|\operatorname{Im}\cdot|;a,R)$. According to 2.1, v is psh. on $\mathbf{C}^N$ with $v(z)=|\operatorname{Im}z|$ for all $z\in\partial U(a,R)$. In [19, the proof of Lemma 3], a plurisubharmonic function u has been constructed (denoted by v there), with $u\leq|\operatorname{Im}\cdot|$ on $\partial U(a,R)$ and $u(a)=2R/(\pi\sqrt{N})$. That is why we obtain $v(a)\geq 2R/(\pi\sqrt{N})$. Thus

$$\Delta(H;a,R)=\Delta(\widetilde{H};a,R)\geq h(\varepsilon|\operatorname{Im}\cdot|;a,R)(a)-\widetilde{H}(a)\geq\frac{2\varepsilon}{\pi\sqrt{N}}R.$$

Taking the minimum over all F we finish the proof.

3.2. Lemma. *If G is the ball $U(0,1)$ then $H(z)=|z|$ and there are $R_0,\varepsilon>0$ with*

$$h(H;a,R)(z)-H(z)\geq\varepsilon R^2,\quad 0<R\leq R_0,\ a\in S,\ z\in B(a,R/2).$$

Proof. Fix $a\in S$ and $0<R\leq R_0<1$, where $R_0>0$ will be chosen later. Fix $z\in B(a,R)$ and put $t:=z-a$. By the Taylor series expansion, we have

$$\begin{aligned}H(z)=H(a)&+\sum_{j=1}^{N}\frac{\partial H(a)}{\partial z_j}t_j+\sum_{j=1}^{N}\frac{\partial H(a)}{\partial\bar{z}_j}\bar{t}_j\\&+\operatorname{Re}\left(\sum_{j,k=1}^{N}\frac{\partial^2H(a)}{\partial z_j\partial z_k}t_jt_k\right)+\sum_{j,k=1}^{N}\frac{\partial^2H(a)}{\partial z_j\partial\bar{z}_k}t_j\bar{t}_k+R_a(t),\end{aligned}$$

where $|R_a(t)|\leq O(|t|^3)$ (see Hörmander [8, page 51]). We abbreviate the four leading terms of this expansion by $P(z)$ and note that P is a pluriharmonic function of $z\in\mathbf{C}^N$. Hence

$$h(H;a,R)=P+h(H-P;a,R)=H+h(H-P;a,R)-(H-P).$$

We now estimate $H-P$. Note that

$$Q_a(t):=\sum_{j,k=1}^{N}\frac{\partial^2 H(a)}{\partial z_j\partial\bar z_k}t_j\bar t_k=-\tfrac14\left|\sum_{j=1}^{N}a_j\bar t_j\right|^2+\tfrac12|t|^2\ge\tfrac14|t|^2.$$

We now choose $0<R_0<1$ only depending on H such that $|R_a(t)|\le|t|^2/10$ if $0<R\le R_0$. Thus we get

$$H(z)-P(z)=Q_a(t)+R_a(t)\le\left(\tfrac12+\tfrac1{10}\right)|t|^2 \tag{10}$$

and

$$H(z)-P(z)\ge Q_a(t)-|R_a(t)|\ge\left(\tfrac14-\tfrac1{10}\right)|t|^2. \tag{11}$$

By the definition of $h(H-P;a,R)$, (11) implies that $h(H-P;a,R)(z)\ge\left(\frac14-\frac1{10}\right)R^2$ for all $z\in B(a,R)$. This and (10) give for each $z\in B(a,R/2)$

$$h(H-P;a,R)(z)-(H-P)(z)\ge\left(\tfrac14-\tfrac1{10}-\left(\tfrac12+\tfrac1{10}\right)\left(\tfrac12\right)^2\right)R^2=\varepsilon R^2.$$

3.3. Lemma. *If the bounded convex domain G has a boundary of class C^1, then $\Gamma_H=\mathbf{C}^N$. If the modulus of continuity of the Gauß map $\mathcal{N}:\partial G\to S$ is bounded by the strictly increasing function $D:\mathbf{R}_+\to\mathbf{R}_+$, then there are $R_0,\varepsilon>0$ such that*

$$\Delta(H;a,R)\ge\varepsilon R^2D^{-1}(R/2),\quad 0<R\le R_0,\ a\in S.$$

Proof. The idea of proof that $\Gamma_H=\mathbf{C}^N$ if ∂G is of class C^1 is essentially contained in the proof of Thm. 7 of Morzhakov [22].

If ∂G is of class C^1 there are $L,l>0$ such that for each $w\in\partial G$ there is a motion A_w of $\mathbf{R}^{2N}$ which maps $\mathbf{R}^{2N-1}\times\{0\}$ onto the supporting hyperplane of ∂G at w, which maps 0 to w and $(0,1)$ to the inner normal $\mathcal{N}(w)$ of ∂G at w and such that $A_wB\subset G$, where

$$B=\{\,x\in\mathbf{R}^{2N}\mid L|(x_1,\dots,x_{2N-1})|<x_{2N}<l\,\}.$$

By the hypothesis, we have

$$|\mathcal{N}(w)-\mathcal{N}(z)|\le D(|w-z|),\quad w,z\in\partial G,$$

where $\mathcal{N}:\partial G\to S$ is the Gauß map. Fix $a\in S$ and $0<R\le 2D(l)$. Put $w:=\mathcal{N}^{-1}(a)$ and $r:=D^{-1}(R/2)\le l$. To simplify the notation we assume $A_w=id$. $B\cap U(0,r)$ contains

a ball U of radius $\varkappa \geq r/(2+L)$ which is symmetric to the axis through $(0,1)$. This shows that there is a ball $U(\widetilde{w},\varkappa)\subset U(w,r)$ symmetric to the axis through $\mathcal{N}(a)$ which touches ∂G at some $w_0 \in \partial G \cap B(w,r)$. Put $z_0 := \mathcal{N}(w_0)$. We get

$$|z_0 - a| = |\mathcal{N}(w_0) - \mathcal{N}(w)| \leq D(r) = R/2.$$

Since $z \mapsto \varkappa|z| + \mathrm{Re}\langle \widetilde{w}, z\rangle$ is the support function of $U(\widetilde{w},\varkappa)$, we get

$$\varkappa|z| + \mathrm{Re}\langle \widetilde{w}, z\rangle \leq H(z) \text{ on } \mathbf{C}^N, \qquad \text{and} \qquad \varkappa|z_0| + \mathrm{Re}\langle \widetilde{w}, z_0\rangle = H(z_0).$$

Choose $\varepsilon > 0$ and $0 < R_0 \leq 2D(l)$ according to 3.2. Let $0 < R \leq R_0$. We obtain

$$h(H; a, R)(z) \geq \varkappa h(|\cdot|; a, R)(z) + \mathrm{Re}\langle \widetilde{w}, z\rangle, \quad z \in U(a,R),$$

and by 3.2

$$\Delta(H; a, R) \geq h(H; a, R)(z_0) - H(z_0) \geq \varkappa(h(|\cdot|; a, R)(z_0) - |z_0|) \geq \frac{D^{-1}(R/2)}{2+L}\varepsilon R^2.$$

Remark. The same arguments show that $\Gamma_H \neq \emptyset$ for each bounded convex domain G.

The following lemma shows how to deal with the Cartesian product of sets as considered in Lemmas 3.1, 3.2, 3.3.

3.4. Lemma. *Let the open bounded convex set be of the form $G = \prod_{l=1}^n G_l$, where $G_l \subset \mathbf{C}^{N_l}$, $\sum_{l=1}^n N_l = N$. Then $H(z) = \sum_{l=1}^n H_l(z_l)$, $z \in \mathbf{C}^N = \prod_{l=1}^n \mathbf{C}^{N_l}$, where H_l is the support function of G_l, $l = 1, \ldots, n$. Then $\Gamma_H = \prod_{l=1}^n \Gamma_{H_l}$ and the following holds*

$$\min_{1\leq l\leq n} \Delta(H_l; a_l, R/\sqrt{n}) \leq \Delta(H; a, R) \leq \min_{1\leq l\leq n} \Delta(H_l; a_l, R), \quad a \in \mathbf{C}^N, \; R > 0.$$

Proof. Fix $a \in \mathbf{C}^N$ and $R > 0$ and choose $0 < \eta \leq 1$ with (here we put $0/0 := 0$)

$$\frac{1-\eta}{\eta} = \min_{1\leq l_0\leq n}\left(\Delta(H_{l_0}; a_{l_0}, R/\sqrt{n}) \Big/ \sum_{l\neq l_0}^{n} \Delta(H_l; a_l, R/\sqrt{n})\right) =: m.$$

Put $d(H_l; \cdot, \cdot) := h(H_l; \cdot, \cdot) - H_l$, $l = 1, \ldots, n$, and consider the psh. function

$$u(z) := H(z) + \sum_{l=1}^{n} (d(H_l; a_l, R/\sqrt{n})(z_l) - \eta\Delta(H_l; a_l, R/\sqrt{n})), \quad z \in \mathbf{C}^N.$$

If $|\zeta-a|=R$, there is some $1\le l_0\le n$ with $|\zeta_{l_0}-a_{l_0}|\ge R/\sqrt{n}$. Thus we obtain $d(H_{l_0};a_{l_0},R/\sqrt{n})(\zeta_{l_0})=0$ and

$$u(\zeta)\le H(\zeta)+(1-\eta)\sum_{l\ne l_0}\Delta(H_l;a_l,R/\sqrt{n})-\eta\Delta(H_{l_0};a_{l_0},R/\sqrt{n})\le H(\zeta),$$

by the choice of η. By the definition of $\Delta(H;a,R)$ and since $1-\eta=m/(m+1)$, we get

$$\begin{aligned}\Delta(H;a,R)&\ge\sup_{z\in U(a,R)}(u(z)-H(z))=(1-\eta)\sum_{l=1}^{n}\Delta(H_l;a_l,R/\sqrt{n})\\&\ge\min_{1\le l\le n}\Delta(H_l;a_l,R/\sqrt{n}).\end{aligned}$$

To prove the upper estimate, fix $0<q<1$ and choose $x\in U(a,R)$ so that the inequality $d(H;a,R)(x)\ge q\Delta(H;a,R)$ holds. Fix $1\le l_0\le n$ and consider the psh. function

$$u(z_{l_0}):=h(H;a,R)(x_1,\dots,z_{l_0},\dots,x_n)-\sum_{l\ne l_0}H_l(x_l),\quad z_{l_0}\in\mathbf{C}^{N_{l_0}}.$$

If $|\zeta_{l_0}-a_{l_0}|\ge R$ also $|(x_1,\dots,\zeta_{l_0},\dots,x_n)-a|\ge R$ and

$$h(H;a,R)(x_1,\dots,\zeta_{l_0},\dots,x_n)=H(x_1,\dots,\zeta_{l_0},\dots,x_n),$$

which gives $u(\zeta_{l_0})=H_{l_0}(\zeta_{l_0})$. By the definition of $\Delta(H_{l_0};a_{l_0},R)$, we get

$$q\Delta(H;a,R)\le h(H;a,R)(x)-H(x)=u(x_{l_0})-H_{l_0}(x_{l_0})\le\Delta(H_{l_0};a_{l_0},R).$$

We are now ready to state our main result concerning surjective partial differential operators of infinite order.

3.5. Theorem. *Let* $\mathbf{P}$ *be as in* 1.1 *and* $\Omega=(\omega_k)_{k\in\mathbf{N}}$ *as defined in* 1.2. *Let* G *be a bounded convex domain which is a polyhedron or smooth of Hölder class* $C^{1,\lambda}$ *for some* $0<\lambda\le1$ *or which is a Cartesian product* $G=\prod_{l=1}^n G_l$ *of such domains. Then for each* $F\in A^0_{\mathbf{P}}\setminus\{0\}$ *the following assertions are equivalent for the operator* L_F *defined in* 1.5:

(i) $L_F\colon A^0_{\mathbf{P}}\to A^0_{\mathbf{P}}$ *is surjective.*

(ii) *Whenever* $g\in A_{H-\Omega}$ *and* g/F *is entire, then* $g/F\in A_{H-\Omega}$.

(iii) *For each* $k\in\mathbf{N}$ *there is* $R>0$ *such that for each* $z\in\Gamma_H$ *with* $|z|\ge R$ *there exists* $w\in\mathbf{C}^N$ *with* $|w-z|\le\omega_k(z)$ *and* $|F(w)|\ge\exp(-\omega_k(w))$.

Proof. If G is a polyhedron or smooth of Hölder class $C^{1,\lambda}$, then the assertion follows directly from 1.6, 2.6 and from 3.1 and 3.3, respectively. If G is a Cartesian product, we apply 3.4 to claim that there are $R_0, C, \varepsilon > 0$ such that

$$\Delta(H; a, R) \geq \varepsilon R^C, \quad 0 < R \leq R_0, \ a \in \Gamma_H \cap S = S_H^*.$$

For $R \geq 2$ and $a_l \in S_{H_l}^*$, $l = 1, ..., n$, we have $U(a_l, R) \supset U(0, R-1)$ and

$$\Delta(H_l; a_l, R) \geq \Delta(H_l; 0, R-1) = R\left(1 - \frac{1}{R}\right)\Delta(H_l; 0, 1) \geq \Delta(H_l; 0, 1)R/2.$$

Hence by 3.1 and 3.3 (we may assume $R_0 = 2$), there are $\varepsilon > 0$ and $C \geq 1$ with

$$\Delta(H_l; a_l, R) \geq \varepsilon \min\{R, R^C\}, \quad R > 0, \ a_l \in S_{H_l}^*, \ l = 1, ..., n.$$

Let $a \in S_H^*$ and $0 < R \leq 1$. Since $\Delta(H_l; \cdot, \cdot)$ is positively homogeneous, if $a_l \neq 0$ we have

$$\Delta(H_l; a_l, R) = |a_l| \Delta(H_l; a_l/|a_l|, R/|a_l|) \geq \varepsilon \min\left\{R, \frac{R^C}{|a_l|^{C-1}}\right\}$$
$$\geq \varepsilon \min\left\{R, \frac{R^C}{|a|^{C-1}}\right\} = \varepsilon R^C.$$

If $a_l = 0$, we have

$$\Delta(H_l; a_l, R) = R\Delta(H_l; 0, 1) \geq \varepsilon R^C.$$

Hence we get from 3.4

$$\Delta(H; a, R) \geq \min_{1 \leq l \leq n} \Delta(H_l; a_l, R/\sqrt{n}) \geq \varepsilon n^{-C/2} R^C.$$

Thus the assertion follows as above.

4. Discussion of the Ehrenpreis condition

In this last section, assuming that the extra hypothesis on H and Ω of 3.5 holds, we investigate under which restriction on Ω there are non surjective partial differential operators of infinite order. First of all we note the following:

4.1. *Remark.* Let $\mathbf{P}$ be as in 1.1. If P is a nonzero polynomial on $\mathbf{C}^N$, it is obvious that $P \in A_\Omega^0$ and it is well known that there is $\varepsilon > 0$ such that

$$\sup_{|w-z| \leq 1} |P(w)| \geq \varepsilon, \quad z \in \mathbf{C}^N.$$

Hence by 3.5 "(iii) $\Rightarrow$ (i)" (without any extra assumption on H and Ω) the partial differential operator $L_P: A_{\mathbf{P}}^0 \to A_{\mathbf{P}}^0$ is surjective.

We recall the minimum modulus theorem from Levin [12] in a somewhat more general form.

4.2. Proposition. *Let F be analytic in a neighborhood of $\{z\in\mathbf{C} \mid |z|\le 2S\}$ $(0<S)$, $F(0)=1$. For each $0<R<S$ and $0<\eta\le\frac{3}{2}e$ there is a set $\mathcal{S}\subset\mathbf{C}$ which is a union of finitely many discs with sum of radii less or equal $4\eta R$ such that for all $z\in\mathbf{C}$ with $|z|\le R$ and $z\notin\mathcal{S}$ we have*

$$\log|F(z)|\ge -2M(2R)-\log(3e/(2\eta))\frac{\log M(2S)}{\log(S/R)}.$$

Here $M(t):=\max_{|z|\le t}|F(z)|$, $t>0$.

Proof. Although Levin [12, Theorem 11], proves the assertion only for $S=eR$, his proof gives the desired result if we apply Jensen's formula to estimate the number n of zeros of F in the disc $|z|\le 2R$ by $n\le\log M(2S)/\log(S/R)$.

We will apply the following consequence of a theorem of Rubel and Taylor [27].

4.3. Proposition. *Let $\Omega=(\omega_k)_{k\in\mathbf{N}}$ be as in 1.2 (possibly without 1.2(3) and the convexity assumption). Let $(a_j)_{j\in\mathbf{N}}$ be a sequence in $\mathbf{C}\backslash\{0\}$ with $\lim_{j\to\infty}|a_j|=\infty$. Put $n(t):=\sum_{|a_j|\le t}1$ for $t\ge 0$. Assume that for each $k\in\mathbf{N}$ there is $r_0>0$ with*

$$N(r):=\int_0^r\frac{n(t)}{t}\,dt\le\omega_k(r),\quad r\ge r_0.$$

Then there exists a nonzero entire function $F\in A^0_\Omega$ of one variable which has at least the zeros $(a_j)_{j\in\mathbf{N}}$ (with respect to multiplicities).

Proof. Note that by the concavity of ω_k, there is $R=R(k)>0$ with $\omega_k(2r)\le C\omega_k(r)$ for all $r\ge R$, where $C\ge 2$ does not depend on k. Consider the function $p\colon\mathbf{R}_+\to\mathbf{R}_+$ defined by

$$p(r):=\sum_{l=0}^{\infty}(2C)^{-l}N(2^l r),\quad r\ge 0.$$

Then it is trivial that $N(r)\le p(r)$ and $p(2r)\le 2Cp(r)$ for all $r\ge 0$. By the hypothesis on $N(r)$, we claim that for each $k\in\mathbf{N}$ there is $R>0$ such that

$$p(r)\le\sum_{l=0}^{\infty}(2C)^{-l}\omega_k(2^l r)\le\sum_{l=0}^{\infty}(2C)^{-l}C^l\omega_k(r)=2\omega_k(r),\quad r\ge R. \tag{12}$$

By Rubel and Taylor [27, Thm. 5.2 and Prop. 3.5], there exists a nonzero entire function F on $\mathbf{C}$ which has at least the zeros $(a_j)_{j\in\mathbf{N}}$ and satisfies an estimate

$$\log|F(z)|\le Bp(B|z|),\quad z\in\mathbf{C},$$

with some $B>0$. By (12) and 1.2, F belongs to A^0_Ω.

4.4. Lemma. *Let $\Omega=(\omega_k)_{k\in\mathbf{N}}$ be as in* 1.2 *(possibly without* 1.2(3) *and the concavity and convexity assumptions). Let $(\varrho_k)_{k\in\mathbf{N}}$ satisfy the conditions of* 1.2 *(possibly without* 1.2(2), (5), *and the concavity and convexity assumptions). Assume that for each $k\in\mathbf{N}$ there are $m\in\mathbf{N}$ and $R_0>0$ with*

$$\inf_{S>R}\frac{\omega_m(S)}{\log(S/R)}\leq\frac{\omega_k(R)}{\log(R/\varrho_k(R))},\quad R\geq R_0.$$

Then for each $F\in A^0_\Omega\setminus\{0\}$ the following condition holds: For each $k\in\mathbf{N}$ there is $R_0>0$ such that for each $z\in\mathbf{C}^N$ with $|z|\geq R_0$ there exists $w\in\mathbf{C}^N$ with $|w-z|\leq\varrho_k(z)$ and $|F(w)|\geq\exp(-\omega_k(w))$.

Proof. We may assume that $\omega_{k+1}(r)\leq\omega_k(r)$ and $\varrho_{k+1}(r)\leq\varrho_k(r)\leq r/2$ for all $k\in\mathbf{N}$ and $r>0$. For an arbitrary $\tilde{k}\in\mathbf{N}$, according to the properties of Ω and $(\varrho_k)_{k\in\mathbf{N}}$, we choose $k\geq\tilde{k}$ and $r_0>0$ such that

$$3\omega_k(2r)\leq\omega_{\tilde{k}}(r/2)\quad\text{and}\quad 12e\,\varrho_k(r)<\varrho_{\tilde{k}}(r),\ r\geq r_0. \tag{13}$$

By the hypothesis and 1.2(5), there are $m\geq k$ and $R_0\geq r_0$ such that

$$\inf_{S>R}\frac{\omega_m(2S)}{\log(S/R)}<\frac{\omega_k(R)}{\log(R/\varrho_k(R))},\quad R\geq R_0. \tag{14}$$

Since $F\in A^0_\Omega$, we can choose $\widetilde{R}_0\geq R_0$ such that

$$M(r):=\max_{|\zeta|\leq r}|F(\zeta)|\leq\exp\omega_m(r),\quad r\geq\widetilde{R}_0.$$

Fix $z\in\mathbf{C}^N$ with $R:=|z|\geq\widetilde{R}_0$. By (14), there exists $S>R$ with

$$\frac{\omega_m(2S)}{\log(S/R)}\leq\frac{\omega_k(R)}{\log(R/\varrho_k(R))}.$$

We put $\eta:=\frac{3}{2}e\varrho_k(R)/R\leq\frac{3}{2}e$ and apply Proposition 4.2 to the function $\zeta\mapsto F(\zeta z/|z|)$, $\zeta\in\mathbf{C}$. Then there is a union $\mathcal{S}$ of discs with sum of radii less or equal $4\eta R$ such that for all $\zeta\in\mathbf{C}$ with $|\zeta|\leq R$ and $\zeta\notin\mathcal{S}$

$$\log|F(\zeta z/|z|)|\geq-2M(2R)-\log(R/\varrho_k(R))\frac{M(2S)}{\log(S/R)}. \tag{15}$$

Since $2\cdot 4\eta R=12e\,\varrho_k(R)<\varrho_{\tilde{k}}(R)$, there exists $\zeta\in\mathbf{C}$ with $|\zeta-R|\leq\varrho_{\tilde{k}}(R)$ and $|\zeta|\leq R$ which satisfies (15). For $w:=\zeta z/|z|$ we obtain $|w-z|=|\zeta-R|\leq\varrho_{\tilde{k}}(z)$ and

$$\log|F(w)|\geq-2\omega_m(2R)-\log(R/\varrho_k(R))\frac{\omega_m(2S)}{\log(S/R)}\geq-2\omega_m(2R)-\omega_k(R).$$

Since $\varrho_{\tilde{k}}(z)\leq|z|/2$, we have $|w|\geq|z|-|w-z|\geq|z|-|z|/2=|z|/2$. By (13), we conclude

$$\log|F(w)|\geq-\omega_{\tilde{k}}(|z|/2)\geq-\omega_{\tilde{k}}(w).$$

4.5. Lemma. *Let Ω be as in 1.2 (possibly without 1.2(3)). Let $(\varrho_k)_{k\in\mathbf{N}}$ be as in 4.4. Assume that there is some $a\in S$ such that for each function $F\in A^0_\Omega\setminus\{0\}$ and each $k\in\mathbf{N}$ there is $R_0>0$ such that for each $z\in\{\lambda a|\lambda\geq 0\}$ with $|z|\geq R_0$ there exists $w\in\mathbf{C}^N$ with $|w-z|\leq\varrho_k(z)$ and $|F(w)|\geq\exp(-\omega_k(w))$. Then for each $k\in\mathbf{N}$ there are $m\in\mathbf{N}$ and $R_0>0$ with*

$$\inf_{S>R}\frac{\omega_m(S)}{\log(S/R)}\leq\frac{\omega_k(R)}{\log(R/\varrho_k(R))},\quad R\geq R_0.$$

Proof. We may assume that $a=(1,0,...,0)$, that $\omega_{k+1}(r)\leq\omega_k(r)$ and $\varrho_{k+1}(r)\leq\varrho_k(r)\leq r/2$ for all $k\in\mathbf{N}$ and $r\geq 0$. In particular by 1.2(5), we may replace "$|F(w)|\geq\exp(-\omega_k(w))$" by "$|F(w)|\geq\exp(-\omega_k(z))$" in the hypothesis. Thus we obtain that for $N=1$ each function $F\in A^0_\Omega\setminus\{0\}$ satisfies the hypothesis with $a=1$ (consider $\widetilde{F}(z):=F(z_1)$, $z\in\mathbf{C}^N$).

Assume that there is $k\in\mathbf{N}$ such that for each $m\in\mathbf{N}$ and $R_0>0$ there is $a\geq R_0$ with

$$\inf_{r>a}\frac{\omega_m(r)}{\log(r/a)}>\frac{\omega_k(a)}{\log(a/\varrho_k(a))}.$$

We will derive a contradiction to the hypothesis. Inductively we can choose a strictly increasing unbounded sequence $(a_j)_{j\in\mathbf{N}}$ of positive real numbers such that

$$m_j:=\min_{r>a_j}\frac{\omega_j(r)}{\log(r/a_j)}>\frac{\omega_k(a_j)}{\log(a_j/\varrho_k(a_j))}\tag{16}$$

for all $j\in\mathbf{N}$. To simplify the notation, we assume that $m_j\in\mathbf{N}$ for all $j\in\mathbf{N}$. Put $n(t):=\sum_{a_j\leq t}m_j$, $t>0$. Since $\log t=o(\omega_j(t))$, moreover, we may assume that

$$n(a_{j-1})\log(a_j/a_1)\leq\omega_j(a_j)\quad\text{ and }\quad n(a_{j-1})\leq m_j,\ j\in\mathbf{N}.$$

For $j\in\mathbf{N}$ and $a_j<r\leq a_{j+1}$, we obtain from the definition of m_j

$$\begin{aligned}\int_0^r\frac{n(t)}{t}\,dt&=\int_0^{a_j}\frac{n(t)}{t}\,dt+\int_{a_j}^r\frac{n(t)}{t}\,dt\\&\leq n(a_{j-1})\log(a_j/a_1)+(n(a_{j-1})+m_j)\log(r/a_j)\\&\leq\omega_j(a_j)+2m_j\log(r/a_j)\leq\omega_j(r)+2\omega_j(r)\leq 3\omega_j(r).\end{aligned}$$

By 4.3, there is a function $F\in A^0_\Omega\setminus\{0\}$ of one variable which has at least the zeros a_j with multiplicities m_j, $j\in\mathbf{N}$. To show that F does not satisfy the condition in the hypothesis of the lemma let $R_0>0$ be given. Choose $\tilde{k}\geq k$ and $\widetilde{R}\geq R_0$ with

$$\max_{|x|\leq 2r}\log|F(x)|+\omega_{\tilde{k}}(2r)<\omega_k(r),\quad r\geq\widetilde{R}.$$

Choose $j \geq k$ so large that $a_j \geq \widetilde{R}$. We put $z := a_j$ and $w := z + re^{i\theta}$, where $0 \leq r \leq \varrho_{\tilde{k}}(z)$ and $0 \leq \theta \leq 2\pi$. By (16) and by the Jensen–Poisson formula (with $R := z = a_j$), we obtain (each term of the sum is nonnegative):

$$\begin{aligned}\log|F(w)| &= \frac{1}{2\pi}\int_0^{2\pi} P_{r/R}(\theta-t)\log|F(z+Re^{it})|\,dt - \sum_{\substack{|z-a|\leq R\\ F(a)=0}} \log\left|\frac{R^2-(\overline{a-z})re^{i\theta}}{R(re^{i\theta}-(a-z))}\right| \\ &\leq \max_{|x|\leq 2|z|}\log|F(x)| - m_j\log(R/r) \leq \max_{|x|\leq 2|z|}\log|F(x)| - m_j\log(|z|z/\varrho_k(z)) \\ &\leq \max_{|x|\leq 2|z|}\log|F(x)| - \omega_k(z) < -\omega_{\tilde{k}}(2z) \leq -\omega_{\tilde{k}}(w).\end{aligned}$$

4.6. Theorem. *Let* **P** *be as in* 1.1 *and* Ω *as defined in* 1.2. *Assume that* G *satisfies the extra hypothesis of* 3.5. *Then the following assertions are equivalent:*

(i) *For each* $F \in A^0_\Omega \setminus \{0\}$ *the differential operator* $L_F\colon A^0_{\mathbf{P}} \to A^0_{\mathbf{P}}$ *is surjective.*

(ii) *For each* $k \in \mathbf{N}$ *there are* $m \in \mathbf{N}$ *and* $R_0 > 0$ *with*

$$\inf_{S>R} \frac{\omega_m(S)}{\log(S/R)} \leq \frac{\omega_k(R)}{\log(R/\omega_k(R))}, \qquad R \geq R_0.$$

Proof. Combine 3.5, 4.4, 4.5.

4.7. *Examples.* (1) If $\Omega = ((1/k)\omega)_{k\in\mathbf{N}}$ satisfies the conditions of 1.2, and there are $A > 1$ and $r_0 > 0$ with $2\omega(r) \leq \omega(Ar)$ for $r \geq r_0$, then 4.6(ii) does not hold. For in this case, there is $C > 0$ such that $\omega(R)/C \leq \inf_{S>R}\omega(S)/\log(S/R) \leq C\omega(R)$ for large $R > 0$. See Example 1.7(1).

(2) Let $\Omega = (\omega_k)_{k\in\mathbf{N}}$ be a weight system such that for each $k \in \mathbf{N}$ there are $m \in \mathbf{N}$ and $r_0 > 0$ with $\omega_m(r)\log r \leq \omega_k(r)$ for all $r \geq r_0$ (this condition on Ω has been communicated by S. N. Melikhov). Then, choosing $S = 2R$ in 4.6(ii), one gets that Ω satisfies the condition 4.6(ii). See Example 1.7(2).

(3) For $\Omega = (\omega_k)_{k\in\mathbf{N}}$ where $\omega_k(r) = (\log(1+r))^s/k$ $(s > 1)$, the condition 4.6(ii) holds. See Example 1.7(3).

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