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# Uniform growth of analytic curves away from real points

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## 1. Introduction

Let $V$ be a one-dimensional analytic curve in $\{(t,s)\in\mathbf{C}^2:|s|<1\}$ such that the projection map $\pi(t,s)=s$ onto the second coordinate is proper on $V$. Then there is an integer $\nu$ such that, except over a discrete subset of $|s|<1$,

$$V=\{(t_j(s),s):1\le j\le\nu,\ |s|<1\}; \tag{1}$$

that is, $V$ is the zero set of the monic pseudopolynomial with coefficients analytic for $|s|<1$,

$$F(t,s)=\prod_{j=1}^{\nu}(t-t_j(s))=t^\nu+\sum_{j=0}^{\nu-1}a_j(s)t^j.$$

Each of the branches $t_j(s)$ then has a Puiseux series expansion about $s=0$ of the form

$$t_j(s)=t_j(0)+\sum_{k=1}^{\infty}d_{j,k}s^{k/N}$$

for some integer $N\ge1$. Suppose that there is a constant $C>0$ and a rational number $r=p/q$, $0<p/q\le1$, such that

$$|\operatorname{Im} t_j(s)|\le C|s|^{p/q},\quad s \text{ real},\ |s|<1.$$

This condition implies that $t_j(0)$ is real and that the first fractional power $k/N$ that has a nonzero coefficient $d_{j,k}$ in the series must satisfy $k/N\ge p/q$. Hence,

$$|t_j(s)-t_j(0)|\le C'|s|^{p/q},\quad |s|<1,\ 1\le j\le\nu \tag{2}$$

for some constant $C'$ which, a priori, depends on the curve $V$. We are going to prove that the constant can be chosen independent of the curve $V$, in the following sense.

**1.1. Theorem.** *Let $\nu$ and $p \leq q$ be positive integers and $C$ a positive number. Then there is a constant $C' > 0$, depending only on $\nu$, $p$, $q$, and $C$, such that each pseudopolynomial whose zero set $V$ satisfies*

$$(3) \qquad \begin{aligned} |t_j(s)| &\leq C, & |s| &< 1 \\ |\operatorname{Im} t_j(s)| &\leq C|s|^{p/q}, & s \text{ real}, \ |s| &< 1, \end{aligned}$$

*also satisfies*

$$(4) \qquad \min_{\sigma} \max_{1 \leq j \leq \nu} |t_{\sigma(j)}(s) - t_j(0)| \leq C'|s|^{p/q}, \quad |s| < 1,$$

*where $\sigma$ ranges over all permutations of the index set $\{1, 2, \dots, \nu\}$.*

Thus the estimate (2) holds uniformly, provided the $t_j$ are lined up in the right way.

Theorem 1.1 will be used in Section 4 to extend a classical result about zeros of hyperbolic polynomials. Recall that a homogeneous polynomial $P_m(z)$ is the symbol of a differential operator with constant coefficients, hyperbolic with respect to the direction $N = (0, \dots, 0, 1)$, if $P_m(0, \dots, 0, 1) \neq 0$ and the $m$ zeros of $z_n \mapsto P_m(z_1, \dots, z_{n-1}, z_n)$ are all real when $(z_1, \dots, z_{n-1}) \in \mathbf{R}^{n-1}$. Perturbing $P_m$ by adding lower order terms, $P(z) = P_m(z) + Q(z)$ with $\deg Q < m$, leaves the operator hyperbolic provided the zeros of $z_n \mapsto P(z_1, \dots, z_{n-1}, z_n)$ have uniformly bounded imaginary parts when $(z_1, \dots, z_{n-1}) \in \mathbf{R}^{n-1}$. That is, the imaginary parts of the zeros of $P_m$ are perturbed by only a bounded amount. We will use the estimate of the theorem to show that in fact the zeros themselves are perturbed by only a bounded amount as long as $(z_1, \dots, z_{n-1})$ are real (see Theorem 4.1). In fact, Theorem 1.1 grew out of the attempt to derive the perturbation result of Wakabayashi [5] in the form given in Meise–Taylor–Vogt [3, Section 3], in a way similar to Hörmander's proof [2, 12.4.6], of Svensson's perturbation theorem.

The proof is done in several steps. First, the pseudopolynomial case is reduced to the case of ordinary polynomials. Second, the principle of Tarski and Seidenberg yields an analytic one parameter family of algebraic curves where estimates are worst. Theorem 1.1 is proved by desingularizing the curve in the parameter space. This will finally reduce the problem to the case of an unramified covering, where it is trivial. The desingularization is done in a rather special way, which is possible because of the inequalities (3). This last part of the proof is close to Hörmander's proof [2, 12.4.6], of Svensson's theorem.

## 2. Reduction to a semi-algebraic 1-parameter family of curves

First, we reduce the problem to finitely many parameters, i.e., we replace the pseudopolynomial by an ordinary polynomial. Then, we use the principle of Tarski and Seidenberg to show that the parameters leading to the worst estimates lie on some analytic curve. In Section 3, the claim will be shown for that curve.

For $\varrho_1, \varrho_2 > 0$ define the disk and the bidisk

$$\Delta(\varrho_1) := \{s \in \mathbf{C} : |s| < \varrho_1\}, \quad \Delta(\varrho_1, \varrho_2) := \{(t, s) \in \mathbf{C}^2 : |t| < \varrho_1,\ |s| < \varrho_2\}.$$

By $H(G)$ we denote the space of all holomorphic functions on a domain $G$.

For $m \in \mathbf{N}$ and $\varrho_2, C > 0$ let $\mathcal{V}(\varrho_2, m, C)$ denote the set of all analytic curves in $\overline{\Delta(C)} \times \Delta(\varrho_2)$ which are exactly $m$-sheeted over the $s$-plane, i.e.,

$$\mathcal{V}(\varrho_2, m, C) = \left\{ F \in H(\mathbf{C} \times \Delta(\varrho_2)) : \forall s\ \exists t_1(s), \dots, t_m(s) \in \mathbf{C} : \right.$$
$$\left. |t_j(s)| \le C,\ F(t, s) = \prod_{j=1}^{m} (t - t_j(s)) \right\}.$$

It is possible to arrange the indices in such a way that the functions $t_j : \Delta(\varrho_2) \to \overline{\Delta(C)}$ are continuous and, except on a discrete subset of $\Delta(\varrho_2)$, locally holomorphic. We define, for $s \in \Delta(\varrho_2)$ and $F \in \mathcal{V}(\varrho_2, m, C)$,

$$\tilde{f}(s, F) := \min_{\sigma} \max_{j=1,\dots,m} |t_{\sigma(j)}(s) - t_j(0)|, \tag{5}$$

where $\sigma$ ranges over all permutations of $\{1, \dots, m\}$.

**2.1. Lemma.** *Fix $r > 0$ and $m \in \mathbf{N}$, and let $M$ be the smallest nonnegative integer strictly larger than $mr - 1$. Then there is $0 < \varrho_2 < 1$ such that for each pseudopolynomial $F \in \mathcal{V}(1, m, \frac{1}{2})$ there is a polynomial $P$ of the form*

$$P(t, s) = t^m + \sum_{j=0}^{m-1} \sum_{i=0}^{M} a_{i,j} s^i t^j, \quad a_{i,j} \in \mathbf{C},$$

*such that*

$$\min_{\sigma} \max_{j=1,\dots,m} |\tau_{\sigma(j)}(s) - t_j(s)| \le \tfrac{1}{2} |s|^r \quad \text{for all } s \in \Delta(\varrho_2), \tag{6}$$

*where $\sigma$ ranges over all permutations of $\{1, \dots, m\}$ and $\tau_i$ and $t_j$ are the roots of $P$ and $F$ respectively, i.e.,*

$$F(t, s) = \prod_{j=1}^{m} (t - t_j(s)), \quad P(t, s) = \prod_{j=1}^{m} (t - \tau_j(s)).$$

*Proof.* If

$$F(t,s)=t^m+\sum_{j=0}^{m-1}\sum_{i=0}^{\infty}a_{i,j}s^it^j$$

is the Taylor series expansion of $F$, define $P$ by

$$P(t,s)=t^m+\sum_{j=0}^{m-1}\sum_{i=0}^{M}a_{i,j}s^it^j.$$

Note that $|F(t,s)|=\prod_{j=1}^m|t-t_j(s)|\le 2^m$ for $(t,s)\in\Delta(1,1)$, thus $|a_{i,j}|\le 2^m$ for all $i$, $j$. Fix $s\in\Delta(\varrho_2)$, $s\neq 0$, where $\varrho_2>0$ is a sufficiently small constant, and choose one of the roots of $F(\,\cdot\,,s)$, say $t_1(s)$. Let $\varepsilon=|s|^r(8m)^{-1}$. Then the union of the intervals $]a_j^-,a_j^+[$ with $a_j^\pm=|t_1(s)-t_j(s)|\pm\varepsilon$, $j=1,\dots,m$, does not cover $\left[0,\frac14|s|^r\right[$. Hence there is $R\in\left]0,\frac14|s|^r\right[$ with $\big|R-|t_1(s)-t_j(s)|\big|\ge\varepsilon$. For each $t$ with $|t-t_1(s)|=R$

$$|F(t,s)|=\prod_{j=1}^m|t-t_j(s)|\ge\prod_{j=1}^m\big||t-t_1(s)|-|t_1(s)-t_j(s)|\big|\ge\varepsilon^m.$$

For those $t$, the choice of $M$ implies

$$|P(t,s)-F(t,s)|\le\sum_{j=0}^{m-1}\sum_{i>M}2^m|s|^i=\frac{m2^m|s|^{M+1}}{1-|s|}<\varepsilon^m\le|F(t,s)|,$$

provided $\varrho_2$ was chosen small enough. Thus, for $s$ fixed, $P(\,\cdot\,,s)$ and $F(\,\cdot\,,s)$ have the same number of roots in $\{t:|t_1(s)-t|<R\}$. We let $\sigma$ associate the $t_j$ in this disk to the $\tau_i$ in there and restart the procedure with a $t_j$ outside until all $t_j$ are covered.

2.2. *Notation.* For $m\in\mathbf{N}$, $M\in\mathbf{N}_0$, an $M+1$ by $m$ matrix $(a_{i,j})$, and complex numbers $s$, $t$ define

$$Q(s,t,(a_{i,j}))=t^m+\sum_{j=0}^{m-1}\sum_{i=0}^{M}a_{i,j}s^it^j.$$

Note that $Q$ is a polynomial in $(M+1)m+2$ complex variables. Fix a rational number $p/q$ in $]0,1]$ and define

$$\mathcal{A}(\varrho_2,m,p/q):=\big\{(a_{i,j})\in\mathbf{C}^{(M+1)\times m}:Q(\,\cdot\,,\cdot\,,(a_{i,j}))\in\mathcal{V}(\varrho_2,m,1),\ |\operatorname{Im}t|\le|s|^{p/q}$$
$$\text{for all }(s,t)\in\Delta(\varrho_2,1),\ s\text{ real with }Q(s,t,(a_{i,j}))=0\big\}.$$

In applications of the principle of Tarski and Seidenberg, we identify the space $\mathbf{C}^{k\times l}$ of all complex $k$ by $l$ matrices with a real vector space of dimension $2kl$.

**2.3. Lemma.** *$\mathcal{A}(\varrho_2, m, p/q)$ is a compact semi-algebraic subset of $\mathbf{C}^{(M+1)\times m}$.*

*Proof.* By the continuity of the roots of a polynomial, the complement of $\{(a_{i,j}):Q(s,\cdot,(a_{i,j}))\in\mathcal{V}(\varrho_2,m,1)\}$ is open. The extra condition $|\operatorname{Im} t|\le|s|^{p/q}$ is closed. Since we have bounds on all roots of $Q(s,\cdot,(a_{i,j}))$ and this polynomial is monic, we have bounds on the coefficients $\sum_{i=0}^{M} a_{i,j}s^i$ and thus also on the $a_{i,j}$. This proves compactness. Note also that $\mathcal{A}(\varrho_2,m,p/q)$ can be written as

$$\begin{aligned}\mathcal{A}(\varrho_2,m,p/q)=\{(a_{i,j})\in\mathbf{C}^{(M+1)\times m}:\forall t,s\in\mathbf{C},\ r\in\mathbf{R}:\\ (Q(s,t,(a_{i,j}))=0,\ |s|^2<\varrho_2^2\ \Rightarrow\ |t|^2\le 1)\\ \text{and } (Q(r,t,(a_{i,j}))=0,\ r^2<\varrho_2^2\ \Rightarrow\ (\operatorname{Im} t)^{2q}\le r^{2p})\}.\end{aligned}$$

Thus the principle of Tarski and Seidenberg in the form given in Hörmander [2, remark before A.2.4], implies that $\mathcal{A}(\varrho_2,m,p/q)$ is semi-algebraic, keeping in mind that "$A\Rightarrow B$" is the same as "(not $A$) or $B$".

Recall the definition of $\tilde f$ from (5) and define $f:\Delta(\varrho_2)\times\mathcal{A}(\varrho_2,m,p/q)\to\mathbf{R}$ by

$$f(s,(a_{i,j}))=\tilde f(s,Q(s,\cdot,(a_{i,j}))).$$

**2.4. Lemma.** *The graph of $f$ is semi-algebraic.*

*Proof.* Denote by $S_j$ the $j$-th elementary symmetric polynomial in $m$ variables

$$S_j(x_1,\dots,x_m)=\sum_{i_1<\dots<i_j}\prod_{k=1}^{j} x_{i_k}.$$

The claim follows again from Hörmander [2, remark before A.2.4], and the description of the graph $\mathcal{G}$ of $f$ given below. There, $t_1,\dots,t_m$ denote the roots of $Q(s,\cdot,(a_{i,j}))$ for the given value of $s$, while $t_1^0,\dots,t_m^0$ are the roots for $s=0$. If we arrange them so that $r^2:=|t_1-t_1^0|^2$ is maximal, then $r=f(s,(a_{i,j}))$, provided no permutation of the $t_j$ and of the $t_j^0$ leads to a smaller value of the maximal distance. The condition $\prod_{j=1}^{n}(t-t_j)=Q(s,t,(a_{i,j}))$ for all $t$ is expressed by comparison of the coefficients.

$$\begin{aligned}\mathcal{G}=\Big\{&(s,(a_{i,j}),r)\in\mathbf{C}\times\mathbf{C}^{(M+1)\times m}\times\mathbf{R}: r\geq 0,\ (a_{i,j})\in\mathcal{A}(\varrho_2,m,p/q),\\ &\exists t_1,\dots,t_m,\ t_1^0,\dots,t_m^0\in\mathbf{C}\ \forall T_1,\dots,T_m,\ T_1^0,\dots,T_m^0\in\mathbf{C}:\\ &S_m(t_1,\dots,t_m)=(-1)^m\sum_{i=0}^{M}a_{i,0}s^i,\dots,S_1(t_1,\dots,t_m)=-\sum_{i=0}^{M}a_{i,m-1}s^i,\\ &S_m(t_1^0,\dots,t_m^0)=(-1)^m a_{0,0},\dots,S_1(t_1^0,\dots,t_m^0)=-a_{0,m-1},\\ &r^2=|t_1-t_1^0|^2,\ r^2\geq|t_2-t_2^0|^2,\dots,r^2\geq|t_m-t_m^0|^2,\\ &\Big(S_m(T_1,\dots,T_m)=(-1)^m\sum_{i=0}^{M}a_{i,0}s^i,\dots,S_1(T_1,\dots,T_m)=-\sum_{i=0}^{M}a_{i,m-1}s^i,\\ &\quad S_m(T_1^0,\dots,T_m^0)=(-1)^m a_{0,0},\dots,S_1(T_1^0,\dots,T_m^0)=-a_{0,m-1},\\ &\quad |T_1-T_1^0|^2\geq|T_2-T_2^0|^2,\dots,|T_1-T_1^0|^2\geq|T_m-T_m^0|^2\Big)\Rightarrow|T_1-T_1^0|^2\geq r^2\Big\}.\end{aligned}$$

**2.5. Lemma.** *There are $\varrho_1>0$, an even integer $b$, and holomorphic maps $s:\Delta(\varrho_1)\to\Delta(\varrho_2)$ and $A:\Delta(\varrho_1)\to\mathbf{C}^{(M+1)\times m}$ such that for $\lambda$ with $-\varrho_1<\lambda<\varrho_1$ we have $|s(\lambda)|=\lambda^b$ and $A(\lambda)\in\mathcal{A}(\varrho_2,m,p/q)$ as well as*

$$\sup\{f(s,(a_{i,j})):|s|=\lambda^b,\ (a_{i,j})\in\mathcal{A}(\varrho_2,m,p/q)\}=f(s(\lambda),A(\lambda)).$$

*Proof.* The graph $\mathcal{G}$ of $f$ is semi-algebraic by Lemma 2.4. Thus also the set

$$\mathcal{E}=\{(\mu,y,s,(a_{i,j}))\in\mathbf{R}\times\mathbf{R}\times\mathbf{C}\times\mathbf{C}^{(M+1)\times m}:(s,(a_{i,j}),y)\in\mathcal{G},|s|^2=\mu^{-2}\}$$

is semi-algebraic. For fixed $\mu>0$ the supremum $g(\mu):=\sup\{y:(\mu,y,s,(a_{i,j}))\in\mathcal{E}\}=\sup\{f(s,(a_{i,j})):|s|=1/\mu,\ (a_{i,j})\in\mathcal{A}(\varrho_2,m,p/q)\}$ is by Lemma 2.3 obtained and finite. Thus, by Hörmander [2, A.2.8], there are $C>0$ and semi-algebraic maps $\tilde{s}$ and $\tilde{A}$ with

$$(\mu,g(\mu),\tilde{s}(\mu),\tilde{A}(\mu))\in\mathcal{E}\quad\text{for all }\mu>C.$$

This means that $g(\mu)=f(\tilde{s}(\mu),(\tilde{a}_{i,j}(\mu)))$, $|\tilde{s}(\mu)|=1/\mu$. By Hörmander [2, A.2.8], $\tilde{s}$ and the components of $\tilde{A}$ admit Puiseux series expansions for sufficiently large $\mu$. So, for some small $\varrho_1>0$ and large even $b\in\mathbf{N}$, the maps $s$ and $A$ with $s(\lambda)=\tilde{s}(\lambda^{-b})$ and $A(\lambda)=\tilde{A}(\lambda^{-b})$ are holomorphic on $\Delta(\varrho_1)\setminus\{0\}$. Since they are bounded, they can be extended to the origin. Since $\mathcal{A}(\varrho_2,m,p/q)$ is closed, it contains $A(0)$.

So far, we have reduced the compact family of analytic curves to a compact family of algebraic curves and then to an analytic one parameter family of algebraic curves. This situation is now dealt with in our main lemma, the proof of which will be postponed to the next section.

2.6. *Notation.* For $\nu\in\mathbf{N}$, denote by $\mathcal{H}(\nu)$ the set of all functions $h$, holomorphic on the closure of $\mathbf{C}\times\Delta(\varrho_3,\varrho_4)$, for suitable $\varrho_3,\varrho_4>0$ depending on $h$, of the form

$$(7)\qquad h(t,\lambda,s)=\prod_{j=1}^{\nu}(t-t_j(\lambda,s))=t^\nu+\sum_{j=0}^{\nu-1}a_{\nu-j}(\lambda,s)t^j,\quad (\lambda,s)\in\Delta(\varrho_3,\varrho_4),$$

where $a_{\nu-j}$ is holomorphic on the closure of $\Delta(\varrho_3,\varrho_4)$.

**2.7. Main Lemma.** *For each pseudopolynomial $h\in\mathcal{H}(\nu)$ for which there are $C>0$ and $p,q\in\mathbf{N}$, $p\le q$, such that*

$$(8)\qquad |\operatorname{Im} t_j(\lambda,s)|\le C|s|^{p/q},\quad (\lambda,s)\in\mathbf{R}^2\cap\Delta(\varrho_3,\varrho_4),\ j=1,\dots,\nu,$$

*there are $C',\delta'>0$ such that*

$$(9)\qquad \min_{\sigma}\max_{j=1,\dots,\nu}|t_{\sigma(j)}(\lambda,s)-t_j(\lambda,0)|\le C'|s|^{p/q},\quad (\lambda,s)\in\Delta(\delta',\varrho_4),$$

*where $\sigma$ varies over all permutations of the indices $\{1,2,\dots,\nu\}$.*

*Proof of Theorem* 1.1. Let a curve as in (1) be given, i.e., a pseudopolynomial $F\in\mathcal{V}(1,\nu,C)$ satisfying $|\operatorname{Im} t_j(s)|\le C|s|^{p/q}$ for all $s\in]-1,1[$. Replacing $t$ by $t/2C$, we may assume $C=\frac12$. For $r=p/q$, choose $M$ and the polynomial $P$ as in Lemma 2.1. This $P$ is of the form $P(t,s)=Q(s,t,(a_{i,j}))$ for some $M+1$ by $\nu$ matrix $(a_{i,j})=A$. We claim that $A\in\mathcal{A}(\varrho_2,\nu,p/q)$. To see this, denote the zeros of $P(\cdot,s)$ by $\tau_i(s)$, $i=1,\dots,\nu$. Then (6) implies $|\tau_{\sigma(j)}(s)|\le|t_j(s)|+\frac12|s|^{p/q}\le1$ for all $s\in\Delta(\varrho_2)$ and $|\operatorname{Im}\tau_{\sigma(j)}(s)|\le|s|^{p/q}$ for all real $s$. This shows that $A\in\mathcal{A}(\varrho_2,\nu,p/q)$.

For $\varrho_1$, $s(\lambda)$, and $A(\lambda)$ as in Lemma 2.5, and $Q$ as in 2.2 define

$$h(t,\lambda,s)=Q(s(\lambda),t,A(\lambda)).$$

Then $h\in\mathcal{H}(\nu)$, and $h$ satisfies the hypothesis of Lemma 2.7 with $C=1$. Hence there are $C'$ and $\varrho_3,\varrho_4>0$ with

$$\min_{\sigma}\max_{j=1,\dots,\nu}|t_{\sigma(j)}(\lambda,s)-t_j(\lambda,0)|\le C'|s|^{p/q},\quad (\lambda,s)\in\Delta(\varrho_3,\varrho_4).$$

In other words, $f(s(\lambda),A(\lambda))\le C'|s(\lambda)|^{p/q}$ for $\lambda$ sufficiently small. Keeping in mind that $|s(\lambda)|=\lambda^b$, this implies by Lemma 2.5, for each $s$ with $|s|=\lambda^b$,

$$\tilde f(s,P)\le f(s(\lambda),A(\lambda))\le C'(\lambda^b)^{p/q}\le C'|s|^{p/q}.$$

If $\sigma$ is the permutation corresponding to $\tilde f(s,P)$, and $\gamma$ the one from (6), then, for each $j$ and small $s$

$$|t_{\gamma\circ\sigma(j)}(s)-t_j(0)|\le|t_{\gamma\circ\sigma(j)}(s)-\tau_{\sigma(j)}(s)|+|\tau_{\sigma(j)}(s)-\tau_j(0)|\le\left(\tfrac12+C'\right)|s|^{p/q}.$$

Because $|t_i(s)-t_j(0)|\le2$, the estimate holds also for larger $|s|<1$ if we allow a bigger constant.

## 3. Proof of the main lemma

Our proof of the main lemma will be given in several steps. It follows that of Lemma 12.4.7 of [2], using a reduction procedure based on the form of the power series expansion of $h$ near a point. In Lemma 3.1 it is shown that the claim is equivalent to the assertion that the power series of $h$ has a certain form. This property of $h$ is obtained in Lemma 3.5 provided all the roots $t_j(\lambda, s)$ coincide when restricted to the plane $\{s=0\}$. In the other case, the proof proceeds inductively. The reduction consists in resolving the singularity of $\{(t,\lambda):h(t,\lambda,0)=0\}$ in the origin in such a way that it extends to $s\neq 0$.

Once and for all, we fix a rational number $0<p/q\leq 1$.

The first lemma gives the relationship between the magnitude of the roots $t_j(\lambda, s)$ and the form of the power series expansion.

**3.1. Lemma.** *Suppose that $h\in\mathcal{H}(\nu)$ satisfies $h(t,0,0)=t^\nu$, and let $k\in\mathbf{N}_0$ be fixed. Then the power series expansion of $h$ about the origin has the form*

$$h(t,\lambda,s)=\sum_{p\alpha+k\beta+q\gamma\geq p\nu} a_{\alpha,\beta,\gamma}t^\alpha\lambda^\beta s^\gamma \tag{10}$$

*if and only if there exists $C$ such that the zeros $t_j(\lambda,s)$ satisfy*

$$|t_j(\lambda,s)|\leq C(|\lambda|^{p/k}+|s|^{p/q}),\quad 1\leq j\leq \nu, \tag{11}$$

*in a neighborhood of the origin. If $k=0$, the term on the right hand side of* (11) *is interpreted as $C|s|^{p/q}$.*

*Proof.* We have $h(t,\lambda,s)=t^\nu+\sum_{j=0}^{\nu-1}a_{\nu-j}(\lambda,s)t^j$ as in (7). If $h$ has an expansion in the form (10), then

$$a_j(\lambda,s)=\sum_{k\beta+q\gamma\geq pj} a_{\alpha,\beta,\gamma}\lambda^\beta s^\gamma. \tag{12}$$

If $k\geq 1$, this implies

$$|a_j(\lambda,s)|\leq C_1(|\lambda|^{p/k}+|s|^{p/q})^j, \tag{13}$$

because whenever $|\lambda|\leq 1$, $|s|\leq 1$, and $k\beta+q\gamma\geq pj$, we can decrease $\beta$ and $\gamma$ to $\beta'$ and $\gamma'$, not necessarily integers, satisfying $k\beta'/pj+q\gamma'/pj=1$ and then

$$|\lambda|^\beta|s|^\gamma\leq|\lambda|^{\beta'}|s|^{\gamma'}\leq\frac{k\beta'}{pj}|\lambda|^{pj/k}+\frac{q\gamma'}{pj}|s|^{pj/q}\leq(|\lambda|^{p/k}+|s|^{p/q})^j.$$

From the well-known estimate, $|t|\leq 2\max_j |a_j|^{1/j}$, for the magnitude of the largest root of the monic polynomial $t^\nu+a_1t^{\nu-1}+\ldots+a_\nu$, we therefore conclude from (13) that (11) holds in a neighborhood of the origin. When $k=0$, the only change is that (12) holds for all $\beta$ when $q\gamma<pj$. Hence, no monomials $\lambda^\beta s^\gamma$ with $\beta>0$ and $q\gamma<pj$ appear in the expansion, so the right hand side of (13) can be replaced by $C|s|^{pj/q}$ and the rest of the proof is the same as when $k\geq 1$.

Conversely, if (11) holds, then the coefficient $a_j(\lambda,s)$, which is a sum of products of the roots $t_j(\lambda,s)$ taken $j$ at a time, satisfies the estimate (13) for some constant $C$. By Cauchy's inequalities for power series coefficients, this implies (12), which means that

$$\frac{\partial^{\alpha+\beta+\gamma}}{\partial t^\alpha\partial\lambda^\beta\partial s^\gamma}h(0,0,0)=0 \quad \text{if } p\alpha+k\beta+q\gamma<p\nu.$$

That is, (10) holds. This completes the proof.

**3.2. Corollary.** *Suppose that $h\in\mathcal{H}(\nu)$ satisfies* (8) *and $h(t,0,0)=t^\nu$. Then there is $k\in\mathbf{N}_0$ such that the power series expansion of $h$ about the origin has the form*

$$h(t,\lambda,s)=\sum_{p\alpha+k\beta+q\gamma\geq p\nu} a_{\alpha,\beta,\gamma}t^\alpha\lambda^\beta s^\gamma.$$

*Proof.* The hypothesis (8) implies that each solution curve $t_j(0,s)$ of the equation $h(t(0,s),0,s)=0$ has a Puiseux series expansion with leading term

$$t_j(0,s)=c_js^r+\ldots,\quad c_j\neq 0,$$

with $r\geq p/q$. Thus we can apply Lemma 3.1 to $(t,\lambda,s)\mapsto h(t,0,s)$ to get

$$h(t,0,s)=\sum_{p\alpha+q\gamma\geq p\nu} a_{\alpha,0,\gamma}t^\alpha s^\gamma.$$

For all but finitely many of the triples $(\alpha,\beta,\gamma)$ with $a_{\alpha,\beta,\gamma}\neq 0$ and $\beta\neq 0$ we have $p\alpha+\beta+q\gamma\geq p\nu$. Thus there are only finitely many conditions on $k$, which can all be satisfied.

The following lemma will be used in the induction step.

**3.3. Lemma.** *Let $h$ be a pseudopolynomial whose Taylor series expansion has the form* (10) *for fixed $k$ and $\nu$. Define $T(h)$ by*

$$T(h)(t,\lambda,s)=\frac{h(\lambda^pt,\lambda^k,\lambda^qs)}{\lambda^{p\nu}},\quad \lambda\neq 0,$$

*and as extended by continuity for* $\lambda=0$. *Then* $T(h)\in\mathcal{H}(\nu)$. *Furthermore,*

(1) *if the zeros of* $h$ *satisfy* (8), *then the zeros of* $T(h)$ *also satisfy* (8) *with the same constant* $C$,

(2) *if the zeros of* $T(h)$ *satisfy the conclusion* (9) *of the main lemma, so do the zeros of* $h$.

*Proof.* It is clear that $T(h)$ is a pseudopolynomial.

The range of $T$ on $\mathcal{A}(\nu)$ is contained in $\mathcal{H}(\nu)$ since the powers of $t$ in the monomials $t^\alpha\lambda^\beta s^\gamma$ are unchanged by the action of $T$. In particular, $h(t,0,0)$ is unchanged by the action of $T$. It is also easy to check that the zeros $\tau_j(\lambda,s)$ of $t\mapsto(Th)(t,\lambda,s)$ are given in terms of those of $h$, i.e., in terms of the $\{t_j(\lambda,s)\}$, by

$$\tau_j(\lambda,s)=\frac{t_j(\lambda^k,\lambda^q s)}{\lambda^p},\quad \lambda\neq 0, \tag{14}$$

and by continuity for $\lambda=0$. Therefore, if the zeros of $h$ satisfy (8), then for real $(\lambda,s)$,

$$|\operatorname{Im}\tau_j(\lambda,s)|=|\lambda^{-p}|\,|\operatorname{Im}t_j(\lambda^k,\lambda^q s)|\leq C|\lambda^{-p}||\lambda^q s|^{p/q}=C|s|^{p/q},$$

so (1) holds.

To check part (2), note first from Lemma 3.1 that the zeros of $h\in\mathcal{A}(\nu)$ always satisfy (11). Therefore, if $|\lambda|\leq C_2|s|^{k/q}$, there is nothing to prove since

$$|t_j(\lambda,s)-t_j(\lambda,0)|\leq 2C_1(|\lambda|^{p/k}+|s|^{p/q})\leq 4C_1C_2|s|^{p/q}.$$

Consequently, in proving (2) we can assume that

$$|s|\leq\delta|\lambda|^{q/k} \tag{15}$$

for some small positive constant $\delta$. Rewrite the relation (14) as

$$t_j(\lambda,s)=(\lambda^{1/k})^p\tau_j(\lambda^{1/k},s/(\lambda^{1/k})^q),$$

where the notation means that $\lambda^{1/k}$ is any fixed value of a $k$-th root of $\lambda$. By hypothesis, the zeros $\tau_j$ of $t\mapsto(Th)(t,\lambda,s)$ satisfy (9) provided $\lambda$ and $s$ are small. Therefore, provided $|\lambda^{1/k}|\leq\delta_1$ and $|s/(\lambda^{1/k})^q|\leq\delta_2$, which is exactly the condition (15) satisfied by $\lambda$ and $s$,

$$\begin{aligned}\min_\sigma\max_j|t_{\sigma(j)}-t_j(\lambda,0)|&=\min_\sigma\max_j|(\lambda^{1/k})^p(\tau_{\sigma(j)}(\lambda^{1/k},s(\lambda^{1/k})^{-q})-\tau_j(\lambda^{1/k},0))|\\&\leq C'|\lambda|^{p/k}|s(\lambda^{1/k})^{-q}|^{p/q}=C'|s|^{p/q}.\end{aligned}$$

Thus, the estimate (9) also holds for the zeros of $t\mapsto h(t,\lambda,s)$. This completes the proof.

We next want to write down the analytic graph $t=P(\lambda)$ that is "most tangent to" $h(t,\lambda,0)=0$ and an associated integer $\mu(h)$ that measures the degree of contact. Suppose that $h$ is a pseudopolynomial as in (7) with the additional property that $h(t,0,0)$ has all its zeros at the origin $t=0$. That is, assume $h(t,0,0)=t^\nu$. The roots $t_j(\lambda,0)$ of $h(t,\lambda,0)$ have Puiseux series expansions in positive fractional powers of $\lambda$. However, the fact that $t_j(\lambda,0)$ is real when $\lambda$ is real implies that no fractional powers actually occur and that the coefficients are real, as has been observed by Chaillou ([1, p. 9 and Lemma 2, p. 147]), i.e., there are $c_{j,l}\in\mathbf{R}$ with

$$t_j(\lambda,0)=\sum_l c_{j,l}\lambda^l,\quad j=1,\dots,\nu.$$

3.4. *Definition.* Let $\mu=\mu(h)$ denote the integer (or $+\infty$)

$$\mu=\sup\{l: c_{j,l}=c_{k,l} \text{ for all } 1\le j,k\le\nu\}.$$

Also, let $p_l(\lambda)=c_{j,l}\lambda^l$ denote the common value of these terms for $1\le l\le\mu$.

In other words, if we set

$$P(\lambda)=\sum_{l\le\mu}p_l(\lambda), \tag{16}$$

then

$$t_j(\lambda,0)=P(\lambda)+\sigma_j(\lambda,0),\quad 1\le j\le\nu,$$

or if $\mu(h)=+\infty$

$$t_j(\lambda,0)=P(\lambda),\quad 1\le j\le\nu.$$

**3.5. Lemma.** *Suppose $h\in\mathcal{H}(\nu)$ satisfies $t_j(\lambda,0)\equiv0$, $1\le j\le\nu$. Then the main lemma, Lemma 2.7, holds for $h$.*

*Proof.* Fix a real number $\lambda$ near 0 and consider the Puiseux series expansion of the zeros $t_j(\lambda,s)$ of the function of two variables, $(t,s)\mapsto h(t,\lambda,s)=t^\nu+\sum_{j=0}^{\nu-1}a_{\nu-j}(\lambda,s)t^j$. By hypothesis, at $s=0$ all $\nu$ of the zeros $t_j(\lambda,0)$ are equal to zero so this series expansion has the form

$$t_j(\lambda,s)=c_js^{r_j}+o(|s|^{r_j}),\quad |s|<\varepsilon(\lambda),$$

where $r_j$ is a positive rational number. If $h$ satisfies the hypothesis (8), then we must have $r_j\ge p/q$. Therefore, the coefficients $a_k(\lambda,s)$ satisfy

$$|a_k(\lambda,s)|=O(|s|^{kp/q}),\quad \lambda \text{ real},\ |\lambda|\le\delta,$$

since $a_k$ is a sum of products of the roots $t_j(\lambda, s)$ taken $k$ at a time. Therefore,

$$\frac{\partial^\gamma}{\partial s^\gamma} a_k(\lambda, s)\bigg|_{s=0} = 0 \quad \text{if } q\gamma < kp,\ \lambda \text{ real},$$

or

$$\frac{\partial^{\alpha+\gamma}}{\partial t^\alpha \partial s^\gamma} h(0, \lambda, 0) = 0 \quad \text{if } p\alpha + q\gamma < p\nu,\ \lambda \text{ real}.$$

Hence, the last equation also holds for all small complex $\lambda$ so the power series expansion of $h(t, \lambda, s)$ about the origin has the form

$$h(t, \lambda, s) = \sum_{p\alpha + q\gamma \geq p\nu} a_{\alpha,\beta,\gamma} t^\alpha \lambda^\beta s^\gamma.$$

It then follows from Lemma 3.1 that $|t_j(\lambda, s)| \leq C_1 |s|^{p/q}$, $1 \leq j \leq \nu$, which clearly implies that the conclusion (9) of Lemma 2.7 holds for $h$. This completes the proof.

**3.6. Lemma.** *Let $\sum_{ni+mj\geq c} a_{i,j} t^i s^j$ be an analytic germ such that for at least two different pairs $(i, j)$ with $ni+mj=c$ we have $a_{i,j} \neq 0$. Then there is a solution curve $t=t(s)$ of $\sum_{ni+mj\geq c} a_{i,j} t(s)^i s^j = 0$ admitting a Puiseux series expansion of the form*

$$t(s) = b_0 s^{n/m} + \sum_{l=1}^{\infty} b_l s^{r_l} \text{ with } b_0 \neq 0 \text{ and } r_l > \frac{n}{m} \text{ for } l \geq 1, \quad |s| \text{ small}.$$

*Proof.* This can be seen from the explicit construction of the Puiseux series expansions of all branches of the solution as given, e.g., in Walker [6, III §7]. Of course, there much more is proved than what we need here, so we sketch the calculation of the first term of the Puiseux series expansion.

By hypothesis, the polynomial $\sum_{ni+mj=c} a_{i,j} b^i$ has at least two terms, thus at least one non-zero root $b_0$. Consider the holomorphic function

$$F(t_1, s_1) = \sum_{ni+mj\geq c} a_{i,j} (b_0 + t_1)^i s_1^{ni+mj-c}.$$

It satisfies $F(0,0)=0$, and, by hypothesis, it has a term $a_{i,j} t_1^i$ with $a_{i,j} \neq 0$, $i \neq 0$, and $ni+mj=c$. By Whitney [7, 1.10A], there is a solution $t_1(s_1)$ of $F(t_1, s_1)=0$ satisfying $t_1(0)=0$ and admitting a Puiseux series expansion with positive exponents only. This completes the proof since

$$t(s) = (b_0 + t_1(s^{1/m})) s^{n/m}$$

is a solution of our original equation.

In the proof of the main lemma, we try to split the zeros of the solution $t(\lambda, 0)$ in $\lambda=0$ by resolving the singularity. This leads to an inductive procedure. However, in general, it might happen that this desingularization does not carry over to values of $s$ different from 0. We show that the hypothesis excludes such a behavior.

*Proof of the main lemma, Lemma* 2.7. Note first that it is enough to show (9) for $(\lambda, s)\in\Delta(\delta, \delta')$ for some $\delta'$. It will then hold for $(\lambda, s)\in\Delta(\delta, \varrho_4)$ if we replace $C'$ by the maximum of $C'$ and some bound depending on the maximum of all $|t_j(\lambda, s)|$, $(\lambda, s)\in\Delta(\delta, \varrho_4)$, which exists because of continuity.

Let $h\in\mathcal{H}(\nu)$ be given. We proceed by induction over $\nu$. If $h\in\mathcal{H}(1)$, then $h(t, \lambda, s)=t-t_1(\lambda, s)$ for a holomorphic function $t_1$. Its power series expansion gives

$$t_1(\lambda, s)-t_1(\lambda, 0)=\sum_{i=0}^{\infty}\sum_{j=1}^{\infty} b_{i,j}\lambda^i s^j=O(s).$$

This proves the claim for $\nu=1$.

Suppose now that the assertion is already shown for $\mathcal{H}(\mu)$, $\mu=1, \dots, \nu-1$. If for $h\in\mathcal{H}(\nu)$ the restriction $h(t, 0, 0)$ has several different roots $\tau_1, \dots, \tau_l$, then we can group the $t_j(\lambda, s)$ accordingly, thus write on a possibly smaller domain $\Delta(\varrho_3', \varrho_4')$

$$h(t, \lambda, s)=\prod_{j=1}^{l}\Bigl(\prod_{t_i(0,0)=\tau_j}(t-t_i(\lambda, s))\Bigr)$$

and apply the induction hypothesis to each factor.

So we only have to deal with the case that $h(t, 0, 0)$ has a $\nu$-fold root, which we may assume to be 0. For $P$ as in (16), we define

$$h_1(t, \lambda, s)=\prod_{j=1}^{\nu}(t-t_j(\lambda, s)-P(\lambda))=\prod_{j=1}^{\nu}(t-\sigma_j(\lambda, s)), \tag{17}$$

where

$$\sigma_j(\lambda, s)=t_j(\lambda, s)-P(\lambda), \quad 1\le j\le\nu,$$

are the roots of the pseudopolynomial $h_1(t, \lambda, s)=h(t-P(\lambda), \lambda, s)$.

This may lead to $\sigma_j(\lambda, 0)\equiv0$, in which case the claim follows from Lemma 3.5. Otherwise, we denote the Taylor coefficients of $h_1$ by $c_{\alpha,\beta,\gamma}$. Choose $k_1$ and $k_2$ relatively prime with

$$\frac{k_1}{k_2}=\min\{r\in\mathbf{Q}: r\ge0,\ p\alpha+r\beta+q\gamma\ge p\nu \text{ for all } \alpha, \beta, \gamma \text{ satisfying } c_{\alpha,\beta,\gamma}\neq0\}.$$

There is only a finite number of relevant conditions, so the minimum exists. If $k_1=0$, then the claim follows from Lemma 3.1. Thus we assume in the sequel $k_1/k_2>0$. Note that then there are $\alpha_1$, $\beta_1$, $\gamma_1$ with $c_{\alpha_1,\beta_1,\gamma_1}\neq 0$, $p\alpha_1+r\beta_1+q\gamma_1=p\nu$, and $\beta_1\neq 0$, since $r$ is minimal. Define

$$h_2(t,\lambda,s)=h_1(t,\lambda^{k_2},s).$$

We denote the Taylor series coefficients of $h_2$ by $a_{\alpha,\beta,\gamma}$. Obviously, $a_{\alpha,k_2\beta,\gamma}=c_{\alpha,\beta,\gamma}$, and $a_{\alpha,\beta,\gamma}=0$ if $\beta$ is not a multiple of $k_2$. Thus

$$p\alpha+k_1\beta+q\gamma\geq p\nu \quad \text{provided } a_{\alpha,\beta,\gamma}\neq 0,$$

and

$$p\alpha_1+k_1\beta_1+q\gamma_1=p\nu, \quad \text{with } \beta_1\neq 0 \text{ and } a_{\alpha_1,\beta_1,\gamma_1}\neq 0.$$

Now, two cases have to be considered.

*Case* 1. There are $\alpha_0,\beta_0,\gamma_0$ with $p\alpha_0+k_1\beta_0+q\gamma_0=p\nu$ satisfying $a_{\alpha_0,\beta_0,\gamma_0}\neq 0$ and $\beta_0\gamma_0\neq 0$.

In this case, explicit calculation of the least exponent in the Puiseux series expansion of almost any real $\lambda$ gives a contradiction to the hypothesis. To carry out this plan, fix a sufficiently small $\lambda_0>0$ with

$$\sum_{\beta=0}^{\infty} a_{\alpha_0,\beta,\gamma_0}\lambda_0^{\beta}\neq 0.$$

Set

$$\mu=\min\left\{m\in\mathbf{Q}: h_2(t,\lambda_0,s)=\sum_{\alpha+m\gamma\geq\nu}\sum_{\beta=0}^{\infty} a_{\alpha,\beta,\gamma}\, t^{\alpha}\lambda_0^{\beta}s^{\gamma}\right\}.$$

Because of $\gamma_0>0$ and $\beta_0>0$, the following inequality implies $\mu>q/p$:

$$m\geq\frac{\nu-\alpha_0}{\gamma_0}=\frac{k_1\beta_0+q\gamma_0}{p\gamma_0}>\frac{q}{p}.$$

There are at least two pairs $(\alpha,\gamma)$ satisfying $\alpha+\mu\gamma=\nu$ and $\sum_{\beta=0}^{\infty} a_{\alpha,\beta,\gamma}\lambda_0^{\beta}\neq 0$, namely $(\nu,0)$ and the pair for which $\mu$ is obtained. By Lemma 3.6, there is a solution curve $t(s)$ of $h_2(t(s),\lambda_0,s)\equiv 0$ whose Puiseux series expansion starts with $t(s)=b_0t^{1/\mu}+\dots$, $b_0\neq 0$. Choose $k$ and $m$ relatively prime with $\mu=k/m$. Since $\mu>q/p\geq 1$, we have $k\geq 2$. So at least on one of the intervals $]-\varepsilon,0[$ or $]0,\varepsilon[$, $\varepsilon$ sufficiently small, there is a branch of the $k$-th root for which $b_0s^{1/\mu}$ is not real. This implies $|t(s)|=O(|\operatorname{Im} t(s)|)$ for the corresponding branch. Because of $1/\mu<p/q$, this contradicts the hypothesis.

*Case* 2. For all $\alpha$, $\beta$, and $\gamma$ with $a_{\alpha,\beta,\gamma}\neq 0$ and $p\alpha+k_1\beta+q\gamma=p\nu$ we have $\gamma=0$ or $\beta=0$.

We apply the operator $T$ of Lemma 3.3 with $k=k_1$

$$\tilde{h}(t,\lambda,s)=\frac{h_2(\lambda^p t,\lambda^{k_1},\lambda^q s)}{\lambda^{p\nu}}.$$

By that lemma, the hypothesis holds also for $\tilde{h}$, and it suffices to show the claim for $\tilde{h}$. Since $\beta_1\neq 0$, the hypothesis of the second case implies $\gamma_1=0$. Thus in the Taylor series expansion of $h_2$ there is the non-vanishing term $a_{\alpha_1,\beta_1,0}t^{\alpha_1}\lambda^{\beta_1}$ with $p\alpha_1+k_1\beta_1=p\nu$ as well as $t^\nu$. By Lemma 3.6, this implies that at least one of the roots $\sigma_j(\lambda,s)$ of $h_2(t,\lambda,s)$ has, in the $(t,\lambda)$ variables, a Puiseux series expansion with a leading term

$$\sigma_j(\lambda,0)=c_j\lambda^{p/k_1}+\ldots,\quad c_j\neq 0,$$

(necessarily a Taylor series in $\lambda$, although this fact is not needed). By Lemma 3.2, none of the $\sigma_j(\lambda,0)$ has a leading term with exponent strictly smaller than $p/k_1$. Further, from the definition of $P(\lambda)$, it follows that not all the leading terms of the $\sigma_j(\lambda,0)$ are identical. That is, $c_j\neq c_k$ for at least one pair $(j,k)$. However,

$$\tilde{h}(t,\lambda,0)=\prod_{j=1}^{\nu}\left(t-\frac{\sigma_j(\lambda^{k_1},0)}{\lambda^p}\right)=\prod_{j=1}^{\nu}(t-c_j-o(1)),$$

so $\tilde{h}(t,0,0)$ has at least two distinct zeros. Thus, as at the start of this proof, $\tilde{h}$ factors on a possibly smaller domain into pseudopolynomials to which the induction hypothesis applies. Consequently, the conclusion of the theorem holds for $\tilde{h}$ and then, by Lemma 3.3, also for $h_2$. This completes the proof.

## 4. Applications

In this section we apply Theorem 1.1 to derive a result on the zeros of hyperbolic perturbations of homogeneous hyperbolic polynomials that extends the theorems of Svensson [4], Wakabayashi [5], and Meise–Taylor–Vogt [3]. To do so, we first recall some facts on hyperbolic polynomials (see, e.g., Hörmander [2, Section 12.4]).

A polynomial $P$ in $n$ variables is said to be hyperbolic in the direction $N=(0,\ldots,0,1)$ if, with $\xi=(\xi_1,\ldots,\xi_n)=(\xi',\xi_n)$, the polynomial in one variable $\xi_n\mapsto$

$P(\xi',\xi_n)$ has degree $m{=}\deg P$ for every choice of $\xi'{\in}\mathbf{R}^{n-1}$, and the zeros of this polynomial

$$P(\xi',\xi_n)=\prod_{j=1}^{m}(\xi_n-\alpha_j(\xi'))$$

satisfy

$$|\operatorname{Im}\alpha_j(\xi')|\le C,\quad \xi'\in\mathbf{R}^{n-1}.$$

If $P_m(\xi)$ is the principal part of $P$, i.e., $P_m$ is the homogeneous polynomial of degree $m$ given by the top degree monomials in $P$,

$$P(\xi)=P_m(\xi)+Q(\xi),\quad \deg Q<m,$$

then

$$P_m(\xi',\xi_n)=\prod_{j=1}^{m}(\xi_n-\beta_j(\xi')),$$

where

$$\beta_j(\xi') \text{ is real for } \xi'\in\mathbf{R}^{n-1}.$$

Therefore, hyperbolicity of $P$ requires that the imaginary parts of the zeros of $P_m$ are perturbed by the lower order terms in $P$ by at most a bounded amount. Theorem 1.1 can be used to show that also the real parts of the zeros are perturbed only by a bounded amount. This is the case $\omega(\xi)\equiv 1$ of the following theorem.

**4.1. Theorem.** *Let $P$ be a complex polynomial of degree $m$ in $n$ variables, and let $P_m$ denote its principal part. Assume that $P_m(\xi',\xi_n)=\prod_{j=1}^{m}(\xi_n-\beta_j(\xi'))$ and that all roots $\beta_j$ are real whenever $\xi'{\in}\mathbf{R}^{n-1}$. If*

$$P(\xi)=\prod_{j=1}^{m}(\xi_n-\alpha_j(\xi')),\quad \xi=(\xi',\xi_n)\in\mathbf{C}^n,$$

*is such that*

$$|\operatorname{Im}\alpha_j(\xi')|\le C\omega(|\xi'|),\quad \xi'\in\mathbf{R}^{n-1}, \tag{18}$$

*for some positive, continuous increasing function $\omega(t)$ with $\omega(t)=o(t)$ and $\omega(0)>0$, then there is a constant $C'>0$ such that*

$$\min_{\sigma}\max_{1\le j\le m}|\alpha_{\sigma(j)}(\xi')-\beta_j(\xi')|\le C'\omega(|\xi'|),\quad \xi'\in\mathbf{R}^{n-1},$$

*where the minimum ranges over all permutations $\sigma$ of the set $\{1,2,\dots,m\}$.*

*Proof.* By the arguments used in the proof of Hörmander [2, Theorem 12.3.1], we get from (18) the existence of constants $A, A', A''>0$ and $a\in\mathbf{Q}\cap[0,1[$ such that

$$|\operatorname{Im}\alpha_j(\xi')|\le A|\xi'|^a+A' \quad \text{for all } \xi'\in\mathbf{R}^{n-1} \tag{19}$$

and

$$\max(1,t^a)\le A''\omega(t) \quad \text{for all } t\ge 0. \tag{20}$$

Next we let

$$P=P_m+Q=\sum_{j=0}^{m}P_j,$$

where $P_j$ is homogeneous of degree $j$, and we let

$$p(s,t;\xi'):=\sum_{j=0}^{m}s^{m-j}P_j(\xi',t), \quad (s,t,\xi')\in\mathbf{C}^2\times\mathbf{R}^{n-1}.$$

Note that

$$p(s,t;\xi')=s^mP\left(\frac{\xi'}{s},\frac{t}{s}\right), \quad (s,t,\xi')\in(\mathbf{C}\setminus\{0\})\times\mathbf{C}\times\mathbf{R}^{n-1}.$$

Hence, for $s,t\in\Delta(1)$, $s\in\mathbf{R}$, and $\xi'\in\mathbf{R}^{n-1}$ satisfying $p(s,t;\xi')=0$ we have

$$P\left(\frac{\xi'}{s},\frac{t}{s}\right)=0 \text{ if } s\ne 0 \quad \text{and} \quad P_m(\xi',t)=0 \text{ if } s=0.$$

If $s\ne0$ we get from (19), for some $j$ and $|\xi'|\le1$,

$$|\operatorname{Im}t|=|s|\left|\operatorname{Im}\alpha_j\left(\frac{\xi'}{s}\right)\right|\le\left(A\left|\frac{\xi'}{s}\right|^a+A'\right)|s|=A|\xi'|^a|s|^{1-a}+A'|s|\le C_1|s|^{1-a} \tag{21}$$

for an appropriate constant $C_1>0$. Since, by hypothesis, $P_m$ is hyperbolic with respect to $(0,\dots,0,1)$, the estimate (21) holds trivially when $s=0$.

Multiplying $P_m$ with a suitable constant, we may assume that

$$P_m(0,t)=t^m, \quad t\in\mathbf{C}.$$

Hence there exists $\delta_0>0$ such that for each $\xi'\in\mathbf{R}^{n-1}$ satisfying $|\xi'|\le\delta_0$ the polynomial $t\mapsto P_m(\xi',t)$ has all its zeros in $|t|<\frac{3}{4}$. Therefore we can choose $0<\varrho_2<1$ such that for $|s|<\varrho_2$, $|\xi'|\le\delta_0$, all the zeros of $t\mapsto p(s,t;\xi')$ satisfy $|t|<\frac{7}{8}$. Hence we get from (21), Theorem 1.1, and a scaling argument that there is a constant $C_2$ such that for each $\xi'\in\mathbf{R}^{n-1}$, $|\xi'|\le\delta_0$, and $s\in\Delta(\varrho_2)$ we have

$$\min_{\sigma}\max_{1\le j\le m}|t_{\sigma(j)}(s,\xi')-t_j(0,\xi')|\le C_2|s|^{1-a}. \tag{22}$$

To conclude the theorem from this, fix $\xi'\in\mathbf{R}^{n-1}$ with $|\xi'|\ge 2\delta_0/\varrho_2$ and let $s:=\delta_0/|\xi'|$, $\eta':=s\xi'$. Then note that for $1\le j\le n$ we have

$$p\Big(s,s\alpha_j\Big(\frac{\eta'}{s}\Big);\eta'\Big)=s^mP\Big(\frac{\eta'}{s},\alpha_j\Big(\frac{\eta'}{s}\Big)\Big)=s^mP(\xi',\alpha_j(\xi'))=0$$

and

$$p(0,\beta_j(\eta');\eta')=P_m(\eta',\beta_j(\eta'))=s^mP_m\Big(\frac{\eta'}{s},\beta_j\Big(\frac{\eta'}{s}\Big)\Big)=s^mP_m(\xi',\beta_j(\xi'))=0.$$

Hence (22) implies

$$\begin{aligned}|s|\min_{\sigma}\max_{1\le j\le m}|\alpha_{\sigma(j)}(\xi')-\beta_j(\xi')|&=|s|\min_{\sigma}\max_{1\le j\le m}\Big|\alpha_{\sigma(j)}\Big(\frac{\eta'}{s}\Big)-\beta_j\Big(\frac{\eta'}{s}\Big)\Big|\\&=\min_{\sigma}\max_{1\le j\le m}\Big|s\alpha_{\sigma(j)}\Big(\frac{\eta'}{s}\Big)-\beta_j(\eta')\Big|\le C_2|s|^{1-a}.\end{aligned}$$

By (20) this implies

$$\min_{\sigma}\max_{1\le j\le m}|\alpha_{\sigma(j)}(\xi')-\beta_j(\xi')|\le C_2|s|^{-a}=C_2\Big(\frac{\delta_0}{|\xi'|}\Big)^{-a}=\frac{C_2}{\delta_0^a}|\xi'|^a\le C'\omega(|\xi'|) \tag{23}$$

for some constant $C'$ and all $\xi'\in\mathbf{R}^{n-1}$ satisfying $|\xi'|\ge 2\delta_0/\varrho_2$. Since the roots $\beta_j$ and $\alpha_j$ depend continuously on $\xi'$ we see that (23) holds for all $\xi'\in\mathbf{R}^{n-1}$ with a possibly larger constant $C'$.

4.2. *Example.* Though the constant in Theorem 1.1 is a uniform one, the constant in Theorem 4.1 cannot be uniform, even if $P_m$ is fixed. This can be seen easily by considering the polynomials

$$P(\xi,a):=(\xi_1+a)^2+\xi_2^2-\xi_3^2,\quad a\in[0,\infty[,\ \xi\in\mathbf{R}^3.$$

**4.3. Corollary.** *Let $P$, $P_m$, and $\omega$ be as in Theorem* 4.1. *If the zeros of $P$ satisfy* (19), *then there exists a constant $C''>0$ such that*

$$|P(\xi',\xi_n+it)|\le C''|P_m(\xi',\xi_n+it)| \quad \text{for all } (\xi,t)\in\mathbf{R}^{n+1},\ |t|\ge\omega(\xi').$$

*Proof.* Using the notation from Theorem 4.1 we have for fixed $(\xi,t)\in\mathbf{R}^n$, $t\neq 0$,

$$\frac{P(\xi',\xi_n+it)}{P_m(\xi',\xi_n+it)}=\frac{\prod_{j=1}^m(it+\xi_n-\alpha_j(\xi'))}{\prod_{j=1}^m(it+\xi_n-\beta_j(\xi'))}=\prod_{j=1}^m\left(1+\frac{\beta_j(\xi')-\alpha_j(\xi')}{it+\xi_n-\beta_j(\xi')}\right).$$

By Theorem 4.1 we can assume that the roots $\alpha_j$, $\beta_j$ are lined up so that the choice $\sigma=\mathrm{id}$ gives the best estimate. Since all $\beta_j$ and $\xi_n$ are real, we get for $|t|\ge\omega(|\xi'|)$

$$\left|\frac{P(\xi',\xi_n+it)}{P_m(\xi',\xi_n+it)}\right|\le\prod_{j=1}^m\left(1+\frac{C'\omega(|\xi'|)}{t}\right)\le C''.$$

4.4. *Remark.* In the case $\omega\equiv 1$ the corollary implies the necessary condition of Svensson for hyperbolic perturbations of homogeneous hyperbolic polynomials, as it is indicated in Hörmander [2, Theorem 12.4.6(i)]. For general weight functions $\omega$, the corollary implies Proposition 3.6 of Meise–Taylor–Vogt [3], which was derived from the extension of Svensson's perturbation theorem to Gevrey classes by Wakabayashi [5, Remark after 1.2.5].

Note that our proof of Theorem 4.1 is a modification of Hörmander's proof of Svensson's theorem. The modification consists in replacing [2, Lemma 12.4.7], by Theorem 1.1. In [3, 3.8], it was outlined that Corollary 4.3 (resp. [3, Proposition 3.6]) can also be obtained along the lines of Hörmander's proof if one replaces [2, Lemma 12.4.7], by the following lemma, which is an easy consequence of the main lemma, Lemma 2.7.

**4.5. Lemma.** *If $h$ is a pseudopolynomial which satisfies the hypotheses of the main lemma, Lemma* 2.7, *then there exist $C''>0$ and $\delta'>0$ such that*

$$|h(it^p,\lambda,t^q)|\le C''|h(it^p,\lambda,0)| \quad \text{for all } (t,\lambda)\in\mathbf{R}^2,\ |t|<\delta',\ |\lambda|<\delta'.$$

*Proof.* Using the same notation as in 2.6 we fix $(t,\lambda)\in\mathbf{R}^2$, $t\neq 0$, so that $(\lambda,t^q)\in\Delta(\varrho_3,\varrho_4)$. Then we have

$$\frac{h(it^p,\lambda,t^q)}{h(it^p,\lambda,0)}=\frac{\prod_{j=1}^\nu(it^p-t_j(\lambda,t^q))}{\prod_{j=1}^\nu(it^p-t_j(\lambda,0))}=\prod_{j=1}^\nu\left(1+\frac{t_j(\lambda,0)-t_j(\lambda,t^q)}{it^p-t_j(\lambda,0)}\right).$$

By Lemma 2.7 we can assume that the roots $t_j(\lambda, t^q)$ and $t_j(\lambda, 0)$ are lined up so that the choice $\sigma = \mathrm{id}$ gives the best estimate. Since $\lambda$ is real, the estimate (8) implies that the roots $t_j(\lambda, 0)$ are real. Hence (9) implies

$$\left|\frac{h(it^p, \lambda, t^q)}{h(it^p, \lambda, 0)}\right| \le \prod_{j=1}^{\nu}\left(1 + \frac{C'|t|^p}{|t|^p}\right) \le C''$$

provided that $|\lambda| < \delta$, where $\delta > 0$ is chosen according to Lemma 2.7.

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