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On polarized 3-folds (X,L) with $g(L)=q(X)+1$ and $h^0(L)\geq 4$

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Abstract. Let (X,L) be a polarized 3-fold over the complex number field. In [Fk3], we proved that $g(L)\geq q(X)$ if $h^0(L)\geq 2$ and moreover we classified (X,L) with $h^0(L)\geq 3$ and $g(L)=q(X)$, where $g(L)$ is the sectional genus of (X,L) and $q(X)=\dim H^1(\mathcal{O}_X)$ the irregularity of X. In this paper we will classify polarized 3-folds (X,L) with $h^0(L)\geq 4$ and $g(L)=q(X)+1$ by the method of [Fk3].

0. Introduction

Let X be a smooth projective variety over the complex number field $\mathbf{C}$ with $\dim X=n$ and L a Cartier divisor on X. Then we call (X,L) a polarized (resp. quasi-polarized) manifold if L is ample (resp. nef-big). Then the sectional genus $g(L)$ of (X,L) is defined by

$$g(L)=1+\tfrac{1}{2}(K_X+(n-1)L)L^{n-1},$$

where K_X is the canonical divisor of X.

Then there exists the following conjecture which is interesting but difficult.

Conjecture. *Let (X,L) be a quasi-polarized manifold. Then $g(L)\geq q(X)$, where $q(X)=\dim H^1(\mathcal{O}_X)$ is the irregularity of X.*

In [Fk3], we proved that $g(L)\geq q(X)$ if (X,L) is a quasi-polarized 3-fold with $h^0(L)\geq 2$, and we classified polarized 3-folds (X,L) with $g(L)=q(X)$ and $h^0(L)\geq 3$. The method of [Fk3] enables us to classify polarized 3-folds (X,L) for small values of $g(L)-q(X)$.

In this paper, we will classify polarized 3-folds (X,L) with $g(L)=q(X)+1$ and $h^0(L)\geq 4$. In particular we prove the following theorem.

Theorem 2.1. *Let (X,L) be a polarized 3-fold with $g(L)=q(X)+1$. Assume that $h^0(L)\geq 4$. Then (X,L) is a Del Pezzo manifold.*

We use the customary notation in algebraic geometry.

1. Preliminaries

Definition 1.1. Let X be a smooth projective variety with $\dim X > \dim Y \geq 1$. Then a morphism $f: X \to Y$ is a fiber space if f is surjective with connected fibers. Let L be a Cartier divisor on X. Then (f, X, Y, L) is called a quasi-polarized (resp. polarized) fiber space if $f: X \to Y$ is a fiber space and L is nef and big (resp. ample).

Definition 1.2. Let X be a smooth projective variety with $\dim X = n$ and let L be a line bundle on X. Then we say that (X, L) is a scroll over Y if there exists a fiber space $\pi: X \to Y$ such that any fiber of π is isomorphic to $\mathbf{P}^{n-m}$ and $L|_F = \mathcal{O}_{\mathbf{P}^{n-m}}(1)$, where $1 \leq m = \dim Y < \dim X$.

Definition 1.3. Let (X, L) be a polarized manifold with $\dim X = n$. Then (X, L) is called a Del Pezzo manifold if $g(L) = 1$ and $\Delta(L) = 1$, where $\Delta(L) = n + L^n - h^0(L)$. (We remark that the classification of Del Pezzo manifolds is complete. See Chapter I, §8 in [Fj9].)

Theorem 1.4. *Let (X, L) be a polarized manifold with* $\dim X = n$. *If $K_X + (n-1)L$ is not nef, then (X, L) is one of the following types.*

(1) $\Delta(L) = 0$. (*See* [Fj9].)

(2) *(X, L) is a scroll over a curve.*

Proof. See [Fj4] or [I].

Theorem 1.5. *Let (X, L) be a quasi-polarized manifold with $n = \dim X \geq 2$. Then $g(L) \geq 0$ if L is ample, or if L is nef-big and $n \leq 3$.*

Proof. See [Fj4] and [Fj6].

Theorem 1.6. *Let (X, L) be a polarized manifold with* $\dim X = n \geq 2$. *Then the following are true.*

(1) *$g(L) = 0$ if and only if $\Delta(L) = 0$.*

(2) *If $g(L) = 1$, then (X, L) is a scroll over an elliptic curve or a Del Pezzo manifold.*

Proof. See [Fj4] or [I].

Definition 1.7.

(1) Let (X, L) and (X', L') be polarized manifolds and $\mu: X \to X'$ a birational morphism. Then μ is called a simple blowing up if μ is a blowing up at one point on X' and $L = \mu^* L' - E$, where E is the μ-exceptional effective reduced divisor.

(2) Let (X, L) be a polarized manifold. Then (X, L) is called a minimal reduction model if (X, L) is not obtained by a finite number of simple blowing ups of another polarized manifold. If (X, L) is not a minimal reduction model, then there

exist a smooth projective variety Y, an ample divisor A on Y, and a finite number of simple blowing ups $\mu: X\to Y$ such that (Y, A) is a minimal reduction model. We call (Y, A) a minimal reduction of (X, L).

Remark 1.8. If a polarized manifold (X, L) is obtained by a finite number of simple blowing ups of another polarized manifold (Y, A), then $g(L)=g(A)$ and $q(X)=q(Y)$.

Theorem 1.9. *Let (X, L) be a polarized manifold with* $\dim X=n\geq 3$. *Assume that $K_X+(n-1)L$ is nef. If $K_X+(n-2)L$ is not nef, then (X, L) is one of the following types.*

(a) *(X, L) is obtained by a simple blowing up of another polarized manifold.*

(b0) *(X, L) is a Del Pezzo manifold with $b_2(X)=1$, or $(\mathbf{P}^3, \mathcal{O}(j))$ with $j=2$ or 3, $(\mathbf{P}^4, \mathcal{O}(2))$, or a hyperquadric in $\mathbf{P}^4$ with $L=\mathcal{O}(2)$.*

(b1) *There is a fibration $\Phi: X\to W$ over a curve W with one of the following properties*:

(b1-v) *$(F, L_F)\cong(\mathbf{P}^2, \mathcal{O}(2))$ for any fiber F of Φ.*

(b1-q) *Every fiber F of Φ is an irreducible hyperquadric in $\mathbf{P}^n$ having only isolated singularities.*

(b2) *(X, L) is a scroll over a smooth surface W.*

Proof. See [Fj4] or [I].

Theorem 1.10. (Fujita) *Let (X, L) be a polarized manifold with* $\dim X=n\geq 3$ *and $g(L)=2$. Then (X, L) is one of the following types.*

(1) *$K_X\equiv(3-n)L$, $d=L^n=1$, and $q(X)=0$, where $\equiv$ denotes the numerical equivalence.*

(2) *X is a double covering of $\mathbf{P}^n$ with branch locus being a smooth hypersurface of degree* 6 *and L is the pullback of $\mathcal{O}_{\mathbf{P}^n}(1)$.*

(2′) *X is the blowing up at a point of another polarized manifold (X', L') of type* (2). *$L=L'_X-E$, where L'_X is the pullback of L and E is the exceptional divisor.*

(3) *(X, L) is a scroll over a smooth surface.*

(4) *There exists a fiber space $r: X\to T$ such that a general fiber F of r is hyperquadric in $\mathbf{P}^n$ with $L_F=\mathcal{O}_F(1)$, where T is a smooth curve.*

(5) *(X, L) is a scroll over a smooth curve of genus two.*

Proof. See [Fj5].

Notation 1.11. Let (X, L) be a quasi-polarized manifold with $h^0(L)\geq 2$. Let $\Lambda\subset|L|$ be a linear pencil such that $\Lambda=\Lambda_M+Z$, where Λ_M is the movable part of Λ and Z is the fixed part of $|L|$. Then there is the rational map $\varphi_{\Lambda_M}: X \dashrightarrow \mathbf{P}^1$ defined by Λ_M. Let $\theta: X_1\to X$ be an elimination of indeterminacy of φ_{Λ_M} and let $t: X_1\to\mathbf{P}^1$

be its morphism. By taking Stein factorization, there exist a smooth curve C, a finite morphism $\delta: C \to \mathbf{P}^1$, and a fiber space $f_1: X_1 \to C$ such that $t = \delta \circ f_1$. Let $a = \deg \delta$, F_1 a general fiber of f_1, and $L' = \theta^* L$.

Theorem 1.12. *Let (X, L) be a polarized 3-fold with $h^0(L) \geq 2$. We use Notation 1.11. Assume that $K_X + 2L$ is nef. Then the following are true.*

(1) $g(L) \geq ag(L'_{F_1}) \geq aq(X)$ *if* $g(C) = 0$.

(2) $g(L) \geq g(C) + ag(L'_{F_1}) \geq q(X) + (a-1)g(L'_{F_1})$ *if* $g(C) \geq 1$.

Proof. See the proof of Theorem 2.8 in [Fk3].

Lemma 1.13. *Let X be a smooth surface and let C be a smooth curve. Let $f: X \to C$ be a surjective morphism (not necessary a fiber space). Then $g(L) \geq g(C)$ for any nef-big divisor L on X.*

Furthermore if $g(L) = g(C)$, then $\varkappa(X) = -\infty$.

Proof. By taking Stein factorization, there exist a smooth curve B, a fiber space $f': X \to B$, and a finite morphism $\delta: B \to C$ such that $f = \delta \circ f'$. By Theorem 2.1 and Theorem 5.5 in [Fk1], $g(L) \geq g(B)$. On the other hand, $g(B) \geq g(C)$. Hence $g(L) \geq g(C)$.

If $g(L) = g(C)$, then $g(F) \leq 1$ by Theorem 5.5 in [Fk1], where F is a general fiber of f'. If $g(F) = 1$, then $K_X L \geq 2g(B) - 2$ by the canonical bundle formula. Hence $g(L) \geq g(B) + 1 \geq g(C) + 1$. So this is a contradiction. Hence $g(F) = 0$ and $\varkappa(X) = -\infty$. □

Definition 1.14.

(1) Let (X, L) be a quasi-polarized surface. Then (X, L) is L-minimal if $LE > 0$ for any (-1)-curve E on X.

(2) Let (X, L) be a quasi-polarized surface. Then there exist a quasi-polarized surface (X', L') and a birational morphism $\pi: X \to X'$ such that (X', L') is L'-minimal and $L = \pi^* L'$. Then we say that (X', L') is an L-minimalization of (X, L).

Lemma 1.15. *Let (X, L) be a quasi-polarized surface with $\varkappa(X) = -\infty$. If $g(L) = q(X)$, then $\varkappa(K_X + L) = -\infty$.*

Proof. Let (X', L') be an L-minimalization of (X, L). Since $g(L) = q(X)$ and $\varkappa(X) = -\infty$, then $(X', L') = (\mathbf{P}^2, \mathcal{O}(r))$ $(r = 1, 2)$ or (X', L') is a scroll over a smooth curve by Theorem 3.1 in [Fk1]. Hence we obtain $\varkappa(K_{X'} + L') = -\infty$. On the other hand $h^0(m(K_X + L)) = h^0(m(K_{X'} + L'))$ for any $m > 0$. Hence $\varkappa(K_X + L) = -\infty$. □

Lemma 1.16. *Let (X, L) be a quasi-polarized surface with $\varkappa(X) = -\infty$, and (X', L') an L'-minimalization of (X, L). If (X', L') is not a scroll over a surface, then $g(L) \geq 2q(X)$.*

Proof. If $q(X)=0$, then this is true. Hence we may assume that $q(X)>0$. Then if (X', L') is not a scroll over a curve, then $K_{X'}+L'$ is nef by Mori theory (see [Fk1]). We remark that $K_{X'}^2\leq 8(1-q(X'))$ if $q(X)=q(X')\geq 1$. On the other hand,

$$\begin{aligned}(K_{X'}+L')^2 &= K_{X'}^2+2(K_{X'}+L')L'-(L')^2\\ &\leq 8(1-q(X'))+4(g(L')-1)-(L')^2=4(g(L')-2q(X')+1)-(L')^2.\end{aligned}$$

If $K_{X'}+L'$ is nef, then $(K_{X'}+L')^2\geq 0$. So we have $g(L')\geq 2q(X')$. Since $g(L)=g(L')$ and $q(X)=q(X')$, we obtain that $g(L)\geq 2q(X)$. □

Lemma 1.17. (Biancofiore–Livorni) *Let C be a smooth projective curve with genus g and $\mathcal{E}$ a normalized vector bundle of rank 2 on C. Let C_0 be the minimal section of $f\colon \mathbf{P}_C(\mathcal{E})\to C$ and F be a fiber of f. We put $e=-C_0^2$. Let $D\in\mathrm{Pic}(\mathbf{P}_C(\mathcal{E}))$ such that $D\equiv aC_0+bF$ and $a\geq 1$, where $\equiv$ denotes the numerical equivalence. Then $h^1(D)=0$ if one of the following conditions is satisfied.*

(1) *$b>ae+2g-2$, $a=1$ and any e.*
(2) *$b>ae+2g-2$, $a\geq 2$ and $e\geq 0$.*
(3) *$b>\frac{1}{2}ae+2g-2$, $a\geq 2$ and $e<0$.*

Proof. See [BL].

Lemma 1.18. *Let $\mathcal{E}$ be an indecomposable vector bundle on an elliptic curve and $d=c_1(\mathcal{E})$.*

(1) *If $d>0$, then $h^0(\mathcal{E})=d$ and $h^1(\mathcal{E})=0$.*
(2) *If $d<0$, then $h^0(\mathcal{E})=0$ and $h^1(\mathcal{E})=-d$.*

Proof. See [H].

Lemma 1.19. *Let (f, X, Y, L) be a quasi-polarized fiber space. Assume that $K_{X/Y}+tL$ is f-nef, where t is a positive integer. Then $(K_{X/Y}+tL)L^{n-1}\geq 0$.*
Moreover if $\dim Y=1$, then $K_{X/Y}+tL$ is nef.

Proof. See Lemma 0.2 in [Fk2].

Definition 1.20. Let X be a projective variety. Then the Kodaira dimension $\varkappa(X)$ of X is defined by $\varkappa(X)=\varkappa(\widetilde{X})$, where $\widetilde{X}$ is a resolution of X. (We remark that $\varkappa(X)$ is independent of the choice of resolutions.)

Lemma 1.21. *Let (X, L) be a polarized manifold with $\dim X\geq 3$ such that (X, L) is a scroll over a smooth surface S and $g(L)\neq q(X)$, and let $\pi\colon X\to S$ be the natural projection. Let $\mathcal{E}$ be an ample vector bundle on S such that $X=\mathbf{P}_S(\mathcal{E})$ and $L=\mathcal{O}_{\mathbf{P}_S(\mathcal{E})}(1)$, where $\mathcal{O}_{\mathbf{P}_S(\mathcal{E})}(1)$ is the tautological line bundle.*

We put $m=g(L)-q(X)$ *and* $n=\dim X$. *If* L *is spanned,* $h^0(L)\geq n+m$, $q(X)\geq 1$, *and* S *is a* $\mathbf{P}^1$*-bundle over a smooth curve* C, *then*

$$q(X)\leq 1+\frac{4m-3n+3}{2n^2-6n+8}.$$

Proof. Let $\mathcal{F}$ be a vector bundle of rank 2 on C such that $\mathcal{F}$ is normalized, and $S=\mathbf{P}_C(\mathcal{F})$. Let $\theta\colon S\to C$ be the natural projection. Let C_0 be a minimal section of θ and let F_θ be a fiber of θ. We put $e=-C_0^2$ and $\det\mathcal{E}=A\equiv aC_0+bF_\theta$. Then $AF_\theta=a\geq\operatorname{rank}(\mathcal{E})=n-1$ because $\mathcal{E}$ is an ample vector bundle and $F_\theta\cong\mathbf{P}^1$. Since $K_S\equiv-2C_0+(2g(C)-2-e)F_\theta$, we obtain

$$K_SA=2g(C)-4+(a-1)(2g(C)-2)+ae-2b+2.$$

We remark that $g(L)=g(A)$ and $1\leq q(X)=q(S)=g(C)$. Hence $g(A)=q(S)+m$.

(A) *The case in which* $2b-ae\leq(a-1)(2g(C)-2)+2$. Then $K_SA\geq 2g(C)-4=2q(S)-4$ and $A^2\leq 2m+2$. On the other hand, $A^2=L^n+c_2(\mathcal{E})$. Since $\mathcal{E}$ is ample, $c_2(\mathcal{E})\geq 1$.

If $c_2(\mathcal{E})=1$, then $S\cong\mathbf{P}^2$ by [LS] because L is spanned. But this is impossible because $q(S)=q(X)\geq 1$. Therefore $c_2(\mathcal{E})\geq 2$ and $L^n=A^2-c_2(\mathcal{E})\leq 2m$. Let $L^n=2m-t$, where t is a non negative integer. Then $\Delta(L)\leq m-t$ since $h^0(L)\geq m+n$ by hypothesis. Therefore $L^n\geq 2\Delta(L)+t$ and $g(L)\geq q(X)+\Delta(L)+t$.

If $t\geq 1$, then $q(X)=0$ by Chapter I (3.5) in [Fj9] since L is spanned. If $t=0$ and $g(L)>\Delta(L)$, then $q(X)=0$ by Theorems 1.4 and 6.1 in [Fj2] because $g(L)\neq q(X)$.

If $t=0$ and $g(L)=\Delta(L)$, then $q(X)=0$ because $g(L)\geq q(X)+\Delta(L)+t$, $t\geq 0$, and $q(X)\geq 0$.

Therefore $q(X)=0$ if $2b-ae\leq(a-1)(2g(C)-2)+2$. But this is impossible since $q(X)\geq 1$.

(B) *The case in which* $2b-ae\geq(a-1)(2g(C)-2)+3$. Then

$$A^2=2ab-a^2e\geq a(a-1)(2g(C)-2)+3a.$$

On the other hand we obtain

$$\begin{aligned}(K_S+A)^2&=K_S^2+2(K_S+A)A-A^2=8(1-q(S))+4(g(A)-1)-A^2\\&=4(g(A)-2q(S)+1)-A^2=4(m-q(S)+1)-A^2.\end{aligned}$$

Since $AF_\theta=a\geq n-1\geq 2$, K_S+A is nef and $(K_S+A)^2\geq 0$. Hence $A^2\leq 4m-4q(S)+4$.

Therefore since $AF_\theta=a\geq n-1\geq 2$ and $g(C)=q(S)=q(X)\geq 1$, we have

$$(n-1)(n-2)(2q(X)-2)+3(n-1)\leq 4m-4q(X)+4.$$

So we obtain

$$q(X)\leq 1+\frac{4m-3n+3}{2n^2-6n+8}. \quad \square$$

2. The main result

Theorem 2.1. *Let (X, L) be a polarized 3-fold with $g(L)=q(X)+1$ and $h^0(L)\geq 4$. Then (X, L) is a Del Pezzo manifold.*

Proof. By Theorem 1.4, K_X+2L is nef. We use Notation 1.11.

(1) *The case in which $g(C)=0$ and $a\geq 2$.* Then by Theorem 1.12, $q(X)+1=g(L)\geq 2q(X)$. Hence $q(X)\leq 1$ and $g(L)\leq 2$.

(2) *The case in which $g(C)\geq 1$.* We remark that $\theta=\mathrm{id}$ and $a\geq 2$ in this case.

Then by Theorem 1.12, $q(X)+1=g(L)\geq q(X)+g(L'_{F_1})$. Therefore $g(L'_{F_1})\leq 1$ and $\varkappa(F_1)=-\infty$. Since $g(L'_{F_1})\geq q(F_1)$, we have the following three types:

(2-1) $(g(L'_{F_1}), q(F_1))=(1,1)$;

(2-2) $(g(L'_{F_1}), q(F_1))=(1,0)$;

(2-3) $(g(L'_{F_1}), q(F_1))=(0,0)$.

We remark that (F_1, L'_{F_1}) is a polarized surface because of $\theta=\mathrm{id}$.

Claim 2.1.1. *The case (2-2) is impossible.*

Proof. If $g(L'_{F_1})=1$ and $q(F_1)=0$, then $q(X)=g(C)$. Hence by Theorem 1.12,

$$g(C)+1=q(X)+1=g(L)\geq g(C)+ag(L'_{F_1})\geq g(C)+2.$$

This is a contradiction. This completes the proof of this claim.

Therefore $g(L'_{F_1})=q(F_1)$. Since $\varkappa(F_1)=-\infty$, we obtain that $\varkappa(K_{F_1}+L'_{F_1})=-\infty$ by Lemma 1.15. Hence $h^0(m(K_X+L)_{F_1})=0$ for any $m\in\mathbf{N}$. Hence K_X+L is not nef.

(3) *The case in which $a=1$.* Then Theorem 1.12 gives $q(F_1)+1\geq q(X)+1=g(L)\geq g(L'_{F_1})$. On the other hand $h^0(L'_{F_1})\geq 3$ by hypothesis.

(3-1) *The case in which $\varkappa(F_1)\geq 0$.*

Claim 2.1.2. *$p_g(F_1)=0$ and $q(F_1)\leq 1$.*

Proof. By the Riemann–Roch theorem and the vanishing theorem, we obtain

$$h^0(K_{F_1}+L'_{F_1})-h^0(K_{F_1})=g(L'_{F_1})-q(F_1).$$

If $p_g(F_1)>0$, then $h^0(K_{F_1}+L'_{F_1})-h^0(K_{F_1})\geq 2$ because $h^0(L'_{F_1})\geq 3$. But this is impossible because $g(L'_{F_1})\leq q(F_1)+1$. Hence $p_g(F_1)=0$. Since $\varkappa(F_1)\geq 0$, we obtain $q(F_1)\leq 1$. This completes the proof of this claim.

By Claim 2.1.2, $q(X)\leq 1$ and $g(L)=q(X)+1\leq 2$.

(3-2) *The case in which $\varkappa(F_1)=-\infty$.*

(3-2-1) *The case in which an L'_{F_1}-minimalization of (F_1, L'_{F_1}) is not a scroll over a smooth curve.* Then by Theorem 1.12 and Lemma 1.16, $q(F_1)+1 \geq q(X)+1 = g(L) \geq g(L'_{F_1}) \geq 2q(F_1)$. Hence $q(F_1) \leq 1$ and $g(L) \leq q(X)+1 \leq q(F_1)+1 \leq 2$.

(3-2-2) *The case in which an L'_{F_1}-minimalization of (F_1, L'_{F_1}) is a scroll over a smooth curve.* Then $\varkappa(K_{F_1}+L'_{F_1}) = -\infty$ by Lemma 1.15. So we obtain that

$$0 = h^0(m(K_{F_1}+L'_{F_1})) = h^0(m(K_{X_1}+L')_{F_1}) = h^0(m(\theta^*(K_X+L)+E_\theta)_{F_1})$$

for any positive integer m, where E_θ is an effective θ-exceptional divisor. If K_X+L is nef, then by the base point free theorem (see [KMM]) Bs $|m(K_X+L)| = \phi$ for some $m \gg 0$. Therefore $h^0(m(\theta^*(K_X+L)+E_\theta)_{F_1}) > 0$. Therefore K_X+L is not nef.

By the above argument, it is sufficient to study (X, L) which satisfies one of the following two conditions.

(A) The case in which K_X+L is not nef.

(B) The case in which $g(L) \leq 2$.

(A) *The case in which K_X+L is not nef.*

(A-1) *The case in which (X, L) is a minimal reduction model.* We study (X, L) by Theorem 1.9. We remark that $\dim X = 3$ and $g(L) = q(X)+1$.

(A-1-1) *The case in which (X, L) is the type* (b0) *in Theorem* 1.9. By calculation, (X, L) is a Del Pezzo manifold with $b_2(X)=1$ or $(X, L) = (\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(2))$. Then in both cases $g(L)=1$ and $q(X)=0$. In particular, (X, L) is a Del Pezzo manifold.

(A-1-2) *The case in which (X, L) is the type* (b1) *in Theorem* 1.9. We use the notation of Theorem 1.9. Let F be a general fiber of Φ.

(A-1-2-1) *The case in which $(F, L_F) = (\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(2))$.* If $g(W) \leq 1$, then $q(X) \leq 1$ and $g(L) = q(X)+1 \leq 2$. So this case is reduced to the case (B) below.

If $g(W) \geq 2$, then by Lemma 1.19

$$g(L) = g(W) + \tfrac{1}{2}(K_{X/W}+2L)L^2 + (L^2F-1)(g(W)-1) \geq g(W)+3 = q(X)+3$$

since $K_{X/W}+2L$ is Φ-nef and $L^2F=4$, where $K_{X/W} = K_X - \Phi^*K_W$.

But this is a contradiction.

(A-1-2-2) *The case in which (F, L_F) is hyperquadric and $L_F = \mathcal{O}_F(1)$.* If $g(W) \leq 1$, then $g(L) = q(X)+1 = g(W)+1 \leq 2$. So this case is reduced to the case (B) below.

If $g(W) \geq 2$, then $(L^2F-1)(g(W)-1) \geq 1$ since $L^2F=2$. On the other hand, $h^0(K_F+2L_F)=1$. Therefore $(K_{X/W}+2L)L^2 > 0$ by Theorem 2.4 and Corollary 2.5 in [EV].

Hence

$$g(L) = g(W) + \tfrac{1}{2}(K_{X/W}+2L)L^2 + (L^2F-1)(g(W)-1) \geq g(W) + \tfrac{1}{2} + 1 = q(X) + \tfrac{3}{2}.$$

So we obtain $g(L)\geq q(X)+2$ because $g(L)\in\mathbf{Z}$. But this is a contradiction.

(A-1-3) *The case in which* (X, L) *is the type* (b2) *in Theorem* 1.9. If $g(L)\leq 2$, then this case is reduced to the case (B) below. So we assume $g(L)\geq 3$. We use the notation of Theorem 1.9. Let $\Phi: X\to W$ be the natural projection. First we prove the following claim.

Claim 2.1.3. $\varkappa(W)=-\infty$.

Proof. We use Notation 1.11. Let $Z=\sum_{i=1}^{m} a_i Z_i$ be the prime decomposition of Z. Let $\theta_1: X_2\to X_1$ be a birational morphism such that $Z_{i,2}$ is smooth for each i, where $Z_{i,2}$ is the strict transform of $Z_{i,1}$ by θ_1 and $Z_{i,1}$ is the strict transform of Z_i by θ. Let $\pi=\theta\circ\theta_1$ and $F=\theta(F_1)$.

(a) *The case in which* $g(C)=0$. If $a\geq 2$, then $g(L)\leq 2$ by the case (1). If $a=1$ and $\varkappa(F_1)\geq 0$, then $g(L)\leq 2$ by the case (3-1).

So these cases are impossible because we assume $g(L)\geq 3$. Hence $\varkappa(F_1)=-\infty$ and $a=1$.

We remark that $|L|\ni D=F+\sum_{i=1}^{m} a_i Z_i$.

By the proof of Theorem 1.12, we can prove $g(L)\geq g(L'_{F_1})+\sum_{i=1}^{m} g((\pi^*L)_{Z_{i,2}})$. Since $q(X)+1=g(L)$ and $g(L'_{F_1})\geq q(F_1)\geq q(X)$, we obtain that $g((\pi^*L)_{Z_{i,2}})\leq 1$ for each i. Therefore $\varkappa(Z_i)=-\infty$ for each i.

On the other hand, one of the irreducible components $F, Z_1, \dots, Z_m$ is surjective to W by Φ because L is ample and $F+\sum_{i=1}^{m} a_i Z_i\in|L|$. Hence $\varkappa(W)=-\infty$.

(b) *The case in which* $g(C)\geq 1$. We remark that $\theta=\mathrm{id}$ and $\varkappa(F_1)=-\infty$ in this case.

If $\Phi(F_1)=W$, then $\varkappa(W)=-\infty$ since $\varkappa(F_1)=-\infty$. So we may assume that $\Phi(F_1)\neq W$ for any general fiber F_1 of f_1. Since L is ample, there is a $Z_{i,2}$ such that $\pi|_{Z_{i,2}}: Z_{i,2}\to C$ is surjective. Hence $g((\pi^*L)_{Z_{i,2}})\geq g(C)$ by Lemma 1.13 and $g((\pi^*L)_{Z_{j,2}})\geq 0$ for any $j\neq i$ by Theorem 1.5.

So by the proof of Theorem 1.12 we obtain that

$$q(X)+1=g(L)\geq\sum_{i=1}^{m} g((\pi^*L)_{Z_{i,2}})+ag(L'_{F_1})\geq\sum_{i=1}^{m} g((\pi^*L)_{Z_{i,2}})+q(F_1)+g(L'_{F_1}).$$

(b-1) *The case in which* $g((\pi^*L)_{Z_{i,2}})\geq g(C)+1$. Then $g(L'_{F_1})=0$ and $q(F_1)=0$ by the above inequalities and $q(F_1)+g(C)\geq q(X)$. Since $g(C)\geq 1$, there exists a morphism $\alpha: W\to C$ such that $f_1=\alpha\circ\Phi$. Then a general fiber of α is $\mathbf{P}^1$ because $q(F_1)=0$. Therefore $\varkappa(W)=-\infty$.

(b-2) *The case in which* $g((\pi^*L)_{Z_{i,2}})=g(C)$. By Lemma 1.13 $\varkappa(Z_{i,2})=-\infty$. On the other hand $g((\pi^*L)_{Z_{j,2}})\leq 1$ for any $j\neq i$ by the above inequalities and $q(F_1)+g(C)\geq q(X)$. Hence $\varkappa(Z_{j,2})=-\infty$ for any $j\neq i$. Therefore $\varkappa(Z_{i,2})=-\infty$ for any i.

Since θ=id, L is ample. Hence $h|_{Z_i}: Z_i \to W$ is surjective for some i. Therefore $\varkappa(W) = -\infty$. This completes the proof of Claim 2.1.3.

If $q(W)=0$, then $q(X)=0$ and $g(L)=1$. Then (X, L) is a Del Pezzo manifold by Theorem 1.6.

So we may assume that $q(W) \geq 1$. Let $\beta: W \to B$ be the Albanese map of W. Let $X = \mathbf{P}_W(\mathcal{E})$, $L = \mathcal{O}_X(1)$, and $A = \det \mathcal{E}$, where $\mathcal{E}$ is an ample vector bundle on W and $\mathcal{O}_X(1)$ is the tautological line bundle. Then (W, A) is a polarized surface with $g(A) = g(L)$ and $q(W) = q(X)$. Hence $g(A) = q(W) + 1$. Therefore (W, A) is not a scroll over a smooth curve. By Lemma 1.16, $2q(W) \leq g(A) = q(W) + 1$. Hence $q(W) \leq 1$. Therefore $q(X) \leq 1$ and $g(L) \leq 2$. So this case is impossible because we assume $g(L) \geq 3$.

(A-2) *The case in which* (X, L) *is not a minimal reduction model.* Let (Y, A) be a minimal reduction of (X, L). In this case, $g(L) = g(A)$, $q(X) = q(Y)$, and $h^0(A) \geq 4$. Hence $g(A) = q(Y) + 1$ and (Y, A) is a Del Pezzo manifold or $g(A) \leq 2$ by the above argument.

If (Y, A) is a Del Pezzo manifold, then (X, L) is also a Del Pezzo manifold because $1 = g(A) = g(L)$ and $0 = q(Y) = q(X)$. Hence (X, L) is a Del Pezzo manifold or $g(L) \leq 2$.

Therefore in the case (A) we obtain that (X, L) is a Del Pezzo manifold or $g(L) \leq 2$.

(B) *The case in which* $g(L) \leq 2$.

(B-1) *The case in which* $g(L) = 2$. By Theorem 1.10, we check each type of Theorem 1.10.

If (X, L) is the type (1), (2), or (2') of Theorem 1.10, then $q(X) = 0$. So this is impossible. If (X, L) is the type (5) of Theorem 1.10, then this is also impossible because $g(L) = q(X)$ in this case.

So it is sufficient to check the type (3) and (4) of Theorem 1.10.

(B-1-1) *The case in which* (X, L) *is the type* (3) *of Theorem* 1.10. Let S be a smooth surface and $\mathcal{E}$ an ample vector bundle on S such that $X = \mathbf{P}_S(\mathcal{E})$ and $L = \mathcal{O}_{\mathbf{P}_S(\mathcal{E})}(1)$. Let $\psi: X \to S$ be the natural projection. We put $A = \det \mathcal{E}$. Then $g(L) = g(A)$ and $q(X) = q(S)$. Hence $g(A) = q(S) + 1$. So by Theorem 2.25 in [Fj7], the following cases can occur.

(α) $S \cong \mathbf{P}(\mathcal{F})$ for some stable vector bundle $\mathcal{F}$ of rank 2 on an elliptic curve W_1 with $c_1(\mathcal{F}) = 1$, $A^2 = 3$, and $L^3 = 1, 2$.

(β) $S \cong \mathbf{P}(\mathcal{F})$, $\mathcal{E} \cong \varrho^* \mathcal{G} \otimes H(\mathcal{F})$ for some semistable vector bundles $\mathcal{F}$ and $\mathcal{G}$ of rank 2 on an elliptic curve W_2, where $\varrho: S \to W_2$ is the natural projection. Moreover $(c_1(\mathcal{F}), c_1(\mathcal{G})) = (1, 0), (0, 1)$, $A^2 = 4$ and $L^3 = 3$.

(B-1-1-1) *The case in which* (S, A) *satisfies the case* (α). But in this case this is impossible. If $L^3 = 1$, then $\Delta(L) = 0$ since $h^0(L) \geq 4$. Hence $g(L) = 0$ and this cannot occur. If $L^3=2$, then $\Delta(L)\leq 1$ since $h^0(L)\geq 4$. If $\Delta(L)=0$, then $g(L)=0$ and this case cannot occur. If $\Delta(L)=1$, then $q(X)=0$ by Fujita's classification of $\Delta(L)$ (see [Fj1]). So this case cannot occur.

(B-1-1-2) *The case in which* (S,A) *satisfies the case* (β). Since $L^3\leq 3$ and $h^0(L)\geq 4$, we obtain that $\Delta(L)\leq 2$.

If $\Delta(L)=0$, then $g(L)=0$ by Theorem 1.6. If $\Delta(L)=1$, then $2\leq L^3\leq 3$ and $q(X)=0$ by Fujita's classification ([Fj1]). Therefore these cases are impossible.

So we assume $\Delta(L)=2$. Hence $L^3=3$ and $h^0(L)=4$. Since $\Delta(L)>\dim \mathrm{Bs}\,|L|$, we obtain $\dim \mathrm{Bs}\,|L|\leq 1$.

If $\dim \mathrm{Bs}\,|L|=1$, then $q(X)=0$ by Theorem 1.14(5), Theorems 2.4, 4.2, and Proposition 4.6 in [Fj3]. But this is a contradiction because $q(X)=1$ in the case (β).

If $\dim \mathrm{Bs}\,|L|=0$, then since $3=L^3=2\Delta(L)-1$ and $g(L)=2$, we obtain $q(X)=0$ by (2.17), (3.15), and (4.15) in [Fj8]. But this is impossible because $q(X)=1$ in the case (β).

So we assume that $\mathrm{Bs}\,|L|=\phi$. Since $g(L)=q(X)+1$, we obtain $q(X)=0$ by Lemma 1.21. But this is also impossible.

Therefore case (β) cannot occur.

(B-1-2) *The case in which* (X,L) *is the type* (4) *of Theorem* 1.10. We use the notation of Theorem 1.10. Then there exist a vector bundle $\mathcal{A}$ of rank 4 on T and X is a member of $|2H(\mathcal{A})+\gamma^*B|$, where $\gamma\colon \mathbf{P}(\mathcal{A})\to T$ is the natural projection and $B\in \mathrm{Pic}(T)$.

Since $2=g(L)=q(X)+1$, we have $q(X)=1$. By the argument from (3.1) to (3.7) in [Fj5], we obtain $(b,e,d)=(1,0,1)$, $(0,1,2)$, and $(-1,2,3)$, where $b=\deg B$, $e=c_1(\mathcal{A})$, and $d=L^3$.

(B-1-2-1) *The case in which* $(b,e,d)=(1,0,1)$. This is impossible because $\Delta(L)=0$ in this case and so $q(X)=0$.

(B-1-2-2) *The case in which* $(b,e,d)=(0,1,2)$. This is also impossible because $\Delta(L)=1$ and so $q(X)=0$ by Fujita's classification ([Fj1]).

(B-1-2-3) *The case in which* $(b,e,d)=(-1,2,3)$. This is also impossible by the same argument as the case (B-1-1-2).

(B-2) *The case in which* $g(L)=1$. By Theorem 1.6, (X,L) is a Del Pezzo manifold.

Hence we obtain that (X,L) is a Del Pezzo manifold if $g(L)\leq 2$.

Therefore (X,L) is a Del Pezzo manifold. This completes the proof of Theorem 2.1. □

Theorem 2.2. *Let* (X,L) *be a polarized manifold with* $\dim X=n\geq 3$. *If* L *is spanned and* $g(L)=q(X)+1$, *then* (X,L) *is a Del Pezzo manifold.*

Proof. If $\dim X=3$, then this theorem is true by Theorem 2.1 because the spannedness of L implies $h^0(L)\geq 4$. So we assume that $\dim X=n\geq 4$. By hypothesis, there exist $(n-3)$ general elements $D_1, \dots, D_{n-3}$ of $|L|$ such that $V=D_1\cap\dots\cap D_{n-3}$ is a smooth projective 3-fold. Since $g(L)=g(L_V)$ and $q(X)=q(V)$, we have $g(L_V)=q(V)+1$ and $\mathrm{Bs}\,|L_V|=\phi$. By Theorem 2.1, $g(L_V)=1$ and $q(V)=0$. Hence $g(L)=1$ and $q(X)=0$. Therefore we obtain that (X,L) is a Del Pezzo manifold by Theorem 1.6. □

By the above results, we conjecture the following.

Conjecture 2.3. *Let (X,L) be a polarized manifold with* $\dim X=n\geq 4$, *$g(L)=q(X)+1$, and $h^0(L)\geq n+1$. Then (X,L) is a Del Pezzo manifold.*

Remark 2.4. We remark that if $\dim X=2$, $g(L)=q(X)+1$, and $h^0(L)\geq 3$, then there exists an example of (X,L) which is not a Del Pezzo surface: Let C be an elliptic curve and $\mathcal{E}$ an indecomposable vector bundle of rank 2 on C with $c_1(\mathcal{E})=1$. Then $\mathcal{E}$ is normalized. Let $X=\mathbf{P}_C(\mathcal{E})$ and H be the tautological line bundle $\mathcal{O}_{\mathbf{P}_C(\mathcal{E})}(1)$. We put $L=2H$. Then $g(L)=2$, $q(X)=1$, and $h^0(L)=3$.

References

[BLP] Beltrametti, M., Lanteri, A. and Palleschi, M., Algebraic surfaces containing an ample divisor of arithmetic genus two, *Ark. Mat.* **25** (1987), 189–210.

[BL] Biancofiore, A. and Livorni, E. L., On the genus of a hyperplane section of a geometrically ruled surface, *Ann. Mat. Pura Appl.* (4) **147** (1987), 173–185.

[EV] Esnault, H. and Viehweg, E., Effective bounds for semi positive sheaves and the height of points on curves over complex function fields, *Compositio Math.* **76** (1990), 69–85.

[Fj1] Fujita, T., On the structure of polarized manifolds with total deficiency one I, *J. Math. Soc. Japan* **32** (1980), 709–725. II, *J. Math. Soc. Japan* **33** (1981), 415–434. III, *J. Math. Soc. Japan* **36** (1984), 75–89.

[Fj2] Fujita, T., On hyperelliptic polarized varieties, *Tôhoku Math. J.* (2) **35** (1983), 1–44.

[Fj3] Fujita, T., On polarized manifolds of Δ-genus two. I, *J. Math. Soc. Japan* **36** (1984), 709–730.

[Fj4] Fujita, T., On polarized manifolds whose adjoint bundles are not semipositive, in *Algebraic Geometry, Sendai, 1985* (Oda, T., ed.), Adv. Stud. Pure Math. **10**, pp. 167–178, North-Holland, Amsterdam–New York, 1987.

[Fj5] Fujita, T., Classification of polarized manifolds of sectional genus two, in *Algebraic Geometry and Commutative Algebra* (Hijikata, H., Hironaka, H., Maruyama, M., Matsumura, H., Miyanishi, M., Oda, T. and Ueno, K., eds.), vol. 1, pp. 73–98, Kinokuniya, Tokyo, 1988.

[Fj6] Fujita, T., Remarks on quasi-polarized varieties, *Nagoya Math. J.* **115** (1989), 105–123.

[Fj7] FUJITA, T., Ample vector bundles of small c_1-sectional genera, *J. Math. Kyoto Univ.* **29** (1989), 1–16.

[Fj8] FUJITA, T., Polarized manifolds of degree three and Δ-genus two, *J. Math. Soc. Japan* **41** (1989), 311–331.

[Fj9] FUJITA, T., *Classification Theories of Polarized Varieties,* London Math. Soc. Lecture Note Ser. **155**, Cambridge Univ. Press, Cambridge, 1990.

[Fk1] FUKUMA, Y., A lower bound for the sectional genus of quasi-polarized surfaces, *Geom. Dedicata* **64** (1997), 229–251.

[Fk2] FUKUMA, Y., A lower bound for sectional genus of quasi-polarized manifolds II, Preprint, 1994.

[Fk3] FUKUMA, Y., On sectional genus of quasi-polarized 3-folds, *Preprint,* 1995.

[H] HARTSHORNE, R., Ample vector bundles on curves, *Nagoya Math. J.* **43** (1971), 73–89.

[I] IONESCU, P., Generalized adjunction and applications, *Math. Proc. Cambridge Philos. Soc.* **99** (1986), 457–472.

[KMM] KAWAMATA, Y., MATSUDA, K. and MATSUKI, K., Introduction to the minimal model problem, in *Algebraic Geometry, Sendai, 1985* (Oda, T., ed.), Adv. Stud. Pure Math. **10**, pp. 283–360, North-Holland, Amsterdam–New York, 1987.

[LS] LANTERI, A. and SOMMESE, A. J., A vector bundle characterization of $\mathbf{P}^n$, *Abh. Math. Sem. Univ. Hamburg* **58** (1988), 89–96.

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