

Zero sets of holomorphic functions in the bidisk

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Abstract. We characterize in geometric terms the zero sets of holomorphic functions f in the bidisk such that $\log |f| \in L^p(\mathbf{D}^2)$ for $1 < p < \infty$.

1. Introduction

In this work, we study some geometrical characterization of the analytic varieties in the bidisk $\mathbf{D}^2 = \{z \in \mathbf{C}^2 : |z_1| < 1, |z_2| < 1\}$ defined by a holomorphic function with some restriction on its growth. In a strictly pseudo-convex domain, this kind of problems are better understood and, for instance, there is a complete characterization of the zero sets of holomorphic functions in the Nevanlinna class (see [K], [S]).

In the bidisk much less is known. Nevertheless there are some cases where the zero sets have been described. For instance in the class of holomorphic functions such that $\log |f| \in L^1(\mathbf{D}^2)$ (see [C] and [A]). In this work we consider a variant of this problem, namely functions such that $\log |f| \in L^p(\mathbf{D}^2)$. We obtain a complete characterization of the zero sets of this class. This problem is closely related to one considered by Beller in one variable (see [B]), where he studied the zero sequences of functions such that $\log^+ |f| \in L^p(\mathbf{D})$. This problem in one variable has been further studied in [BO].

Acknowledgments. This work is part of my Ph. D. thesis and I want to express my sincere gratitude to my advisor Dr. Joaquim Bruna for all his help and attention.

2. Zeros of functions with $\log |f| \in L^p(\mathbf{D}^2)$

2.1. Statement of the results

In this section we will give a complete characterization of the zero sets of holomorphic functions, $f \in H(\mathbf{D}^2)$, such that $\log |f| \in L^p(\mathbf{D}^2)$. Our main tool will

⁽¹⁾ Partially supported by the DGCYT grant PB95-0956-C02-02 and grant 1996-SGR-26.

be the Poincaré–Lelong theorem [Le] that shows that this problem is related to the problem of solving the equation $i\partial\bar{\partial}u=\Theta$ with good estimates on u in terms of Θ . In order to state the theorem we need to introduce some notation.

Let Θ be a closed positive $(1,1)$ -current in the bidisk; for any $z\in\mathbf{D}$ and fixed $0<\varepsilon<1$ let D_z be the small disk

$$D_z = \left\{ \zeta \in \mathbf{D}; \frac{|\zeta - z|}{|1 - \bar{\zeta}z|} < \varepsilon \right\}.$$

If Θ is a closed positive $(1,1)$ -current then it can be expressed in coordinates as

$$\Theta(z) = i \sum_{i,j=1}^2 \theta_{ij}(z) dz_i \wedge d\bar{z}_j,$$

and $\theta_{11}(z_1, z_2)$ is a positive measure in the first variable, if we fix z_2 , because Θ can always be expressed as $i\partial\bar{\partial}u$, where u is plurisubharmonic; thus $\theta_{11} = \Delta_{z_1} u \geq 0$. Accordingly $\theta_{22}(z_1, z_2)$ is a positive measure in the second variable if we fix z_1 . Therefore we can define $\theta_{11}(D_{z_1}, z_2)$ as

$$\theta_{11}(D_{z_1}, z_2) = \int_{\zeta \in D_{z_1}} d\theta_{11}(\zeta, z_2).$$

Provided with this notation, we can state our main theorem.

Theorem 2.1. *Let Θ be a closed positive $(1,1)$ -current on the bidisk, then the equation*

$$i\partial\bar{\partial}u = \Theta$$

has a solution $u \in L^p(\mathbf{D}^2)$ for a p , $1 \leq p < \infty$, if and only if the function

$$(1) \quad f(z_1, z_2) = \theta_{11}(D_{z_1}, z_2) + \theta_{22}(z_1, D_{z_2})$$

belongs to $L^p(\mathbf{D}^2)$.

The disks D_{z_1} and D_{z_2} that appear in the statement of condition (1) depend on ε but the condition itself does not. This can be proved in a way completely analogous to the case of one variable in [L1]: Let $0 < \delta < \varepsilon < 1$, we call

$$\varkappa_\varepsilon(z) = \theta_{11}(D_{z_1}(\varepsilon), z_2).$$

Then we have the following proposition.

Proposition 2.2. (Luecking) *The function $\varkappa_\varepsilon \in L^p(\mathbf{D}^2)$ if and only if $\varkappa_\varepsilon \in L^p(\mathbf{D}^2)$.*

Theorem 2.1 has an immediate corollary for zero-varieties. Fixed $z \in \mathbf{D}^2$, we consider the cross formed by

$$C_z = \{\zeta \in \mathbf{D}^2; \zeta_1 = z_1, \zeta_2 \in D_{z_2}\} \cup \{\zeta \in \mathbf{D}^2; \zeta_2 = z_2, \zeta_1 \in D_{z_1}\}.$$

Let $n_V(z)$ be the number of points that an analytic variety V meets the cross C_z (counted with multiplicity), then we obtain the following result.

Corollary 2.3. *The analytic variety V is the zero set of a function f with $\log |f| \in L^p(\mathbf{D}^2)$ if and only if $n_V \in L^p(\mathbf{D}^2)$.*

Proof. If we take into account the Lelong–Poincaré theorem, the corollary is in fact a reformulation of Theorem 2.1. \square

Remark. Theorem 2.1 and Corollary 2.3 have direct generalizations to the case of the polydisk \mathbf{D}^n with $n > 2$. In this setting, the geometric condition that appears in the theorem is $f \in L^p(\mathbf{D}^n)$, where

$$f(z) = \sum_{i=1}^n \theta_{ii}(z_1, \dots, D_{z_i}, \dots, z_n).$$

For the sake of simplicity in the computations we will give the proof in the case $n=2$, although the same proof can be carried out in higher dimensions. Moreover we will not consider the case $p=1$. This case is already known, it has been studied by Charpentier [C, p. 58] and Andersson [A, Thm. 1]. The theorem that they prove is the following.

Theorem 2.4. (Charpentier) *If we have a closed positive $(1, 1)$ -current Θ in the bidisk, there is a solution $u \in L^1(\mathbf{D}^2)$ to the equation*

$$i\partial\bar{\partial}u = \Theta$$

if and only if

$$\int_{\mathbf{D}^2} (1 - |z_1|^2)^2 \theta_{11}(z) + (1 - |z_2|^2)^2 \theta_{22}(z) < +\infty.$$

This weighted Blaschke condition is equivalent to condition (1) when $p=1$. Indeed, if we apply Fubini’s theorem,

$$\begin{aligned} \int_{\mathbf{D}^2} \theta_{11}(D_{z_1}, z_2) + \theta_{22}(z_1, D_{z_2}) \, dm(z) &= \int_{\mathbf{D}^2} |D_{z_1}| \theta_{11}(z_1, z_2) + |D_{z_2}| \theta_{22}(z_1, z_2) \\ &\simeq \int_{\mathbf{D}^2} (1 - |z_1|^2)^2 \theta_{11}(z) + (1 - |z_2|^2)^2 \theta_{22}(z). \end{aligned}$$

We will divide the proof of Theorem 2.1 into two parts. In the first one we will show the necessity of condition (1); in the second one, which is slightly more technical since we need estimates of some integral kernels, we will show the sufficiency.

2.2. Proof of the necessity of (1)

The scheme of the proof is the following: We start by a Riesz-type decomposition of the plurisubharmonic function u . We evaluate it at the origin, and get a new decomposition by composing u with the automorphisms of the bidisk. This new decomposition has better properties than the original for our interests. This technique has been used in one variable, at least by Pascuas in [P], Ahern and Čučković in [AČ] and Luecking in [L2]. The necessity of the condition follows immediately from the new decomposition.

Let $u \in \text{PSH}(\mathbf{D}^2) \cap L^p(\mathbf{D}^2)$, $1 < p < \infty$. We consider the decomposition of u into

$$(2) \quad u = \Pi[u] + L[\partial\bar{\partial}u],$$

where $\Pi[u]$ is the orthogonal projection of u onto the pluriharmonic functions with the natural scalar product in $L^2(\mathbf{D}^2)$, i.e.

$$\Pi[u](z) = \frac{1}{\pi} \int_{\mathbf{D}^2} \pi(\zeta, z) u(\zeta) dm(\zeta),$$

where the kernel is

$$\pi(\zeta, z) = \frac{1}{(1 - \bar{\zeta}_1 z_1)^2 (1 - \bar{\zeta}_2 z_2)^2} + \frac{1}{(1 - \bar{z}_1 \zeta_1)^2 (1 - \bar{z}_2 \zeta_2)^2} - 1.$$

A priori, this decomposition is valid only in the case $p=2$, but, since the kernel defines a bounded operator in $L^p(\mathbf{D}^2)$ for $1 < p < \infty$, we can extend the decomposition to all L^p spaces.

The other term in the decomposition (2) has an integral expression of the type

$$L[\partial\bar{\partial}u](z) = \int_{\zeta \in \mathbf{D}^2} l(\zeta, z) \wedge \partial\bar{\partial}u(\zeta),$$

and the function $L[\Theta]$ is the minimal solution in $L^2(\mathbf{D}^2)$ of the equation $i\partial\bar{\partial}v = \Theta$. Note that in view of (2), the operator L is just determined on closed $(1, 1)$ -currents Θ .

The computation of the kernel $l(\zeta, z)$ and the estimates of its size were carried out by Andersson in [A]. The expression of the kernel $l(\zeta, z)$ is not unique. There are other kernels that give the same solution. Instead of writing down $l(\zeta, z)$ explicitly, we exhibit L as a linear combination of compositions of explicit operators. In the statement that follows $A_1 B_2$ denotes the integral operator

$$A_1 B_2[\Theta](z) = \int_{\zeta \in \mathbf{D}^2} a(\zeta_1, z_1) \wedge b(\zeta_2, z_2) \wedge \Theta(\zeta),$$

and the kernels appearing are

$$\begin{aligned}
 m(\zeta, z) &= \frac{1}{2\pi} \left[\log \left| \frac{\zeta - z}{1 - \bar{\zeta}z} \right|^2 + \frac{(1 - |\zeta|^2)(1 - |z|^2)}{|1 - z\bar{\zeta}|^2} + |z|^2 \left(\frac{1 - |\zeta|^2}{|1 - \bar{\zeta}z|} \right)^2 \right], \\
 \pi^0(\zeta, z) &= \frac{i}{2\pi} d\zeta \wedge d\bar{\zeta}, \\
 t(\zeta, z) &= \frac{i}{2\pi} \left[\frac{\bar{z}(1 - |\zeta|^2)}{1 - \zeta\bar{z}} + \frac{\bar{z}(1 - |\zeta|^2)}{(1 - \zeta\bar{z})^2} \right] d\zeta, \\
 k(\zeta, z) &= \frac{i}{2\pi} \frac{(1 - |\zeta|^2)}{(1 - \bar{\zeta}z)(\zeta - z)} d\zeta, \\
 i(\zeta, z) &= \frac{i}{2} \delta(\zeta - z) d\zeta \wedge d\bar{\zeta}, \\
 p(\zeta, z) &= \frac{i}{2\pi} \frac{d\zeta \wedge d\bar{\zeta}}{(1 - \bar{\zeta}z)^2}.
 \end{aligned}
 \tag{3}$$

Theorem 2.5. (Andersson) *Let us call*

$$L = \Pi_1^0 M_2 + M_1 \Pi_2^0 + \bar{T}_1 K_2 + \bar{K}_1 T_2 - \bar{T}_1 T_2 + M_1 I_2 - M_1 \bar{P}_2 + T_1 \bar{K}_2.
 \tag{4}$$

Then

$$L[\partial\bar{\partial}u](z) = \int_{\zeta \in \mathbf{D}^2} l(\zeta, z) \wedge \partial\bar{\partial}u(\zeta)$$

gives the second term of the decomposition (2).

From the explicit decomposition (2) we can already draw some conclusions. Since $u \in L^p$ by hypothesis, then $\Pi[u] \in L^p$. Therefore, because of decomposition (2), we conclude that $L[\partial\bar{\partial}u] \in L^p(\mathbf{D}^2)$. This is a necessary condition on $\partial\bar{\partial}u$ if $u \in L^p$. In fact, since the operator L solves the $\partial\bar{\partial}$ -equation, we can say that $L[\Theta] \in L^p(\mathbf{D}^2)$ is a necessary and sufficient condition for the existence of an L^p solution to the $i\partial\bar{\partial}u = \Theta$ equation. This condition is poorly handled, since the kernel that defines L is not of constant sign and therefore there are cancelations in $L[\Theta]$ that do not allow to obtain geometric conditions in the variety associated to Θ .

We will find another decomposition of u composing with the automorphisms of the bidisk. In order to avoid some technical difficulties, we will assume that $u \in C^\infty(\bar{\mathbf{D}}^2)$. Afterwards, using an appropriate regularizing process, we will get the general case.

Let us call

$$\tau_z(\zeta) = \left(\frac{\zeta_1 + z_1}{1 + \zeta_1 \bar{z}_1}, \frac{\zeta_2 + z_2}{1 + \zeta_2 \bar{z}_2} \right), \quad \zeta, z \in \mathbf{D}^2.$$

We define $u_z(\zeta) = u \circ \tau_z(\zeta)$. We have $u_z(0) = u(z)$ and applying the decomposition (2) to the function u_z at the origin we get

$$(5) \quad u(z) = \int_{\zeta \in \mathbf{D}^2} \pi(\zeta, 0) u_z(\zeta) dm(\zeta) + \int_{\zeta \in \mathbf{D}^2} l(\zeta, 0) \wedge \partial \bar{\partial} u_z(\zeta).$$

We take the first integral on the right hand side of (5), and we make the change of variables $\eta = \tau_z(\zeta)$,

$$\begin{aligned} \int_{\zeta \in \mathbf{D}^2} \pi(\zeta, 0) u_z(\zeta) dm(\zeta) &= \int_{\zeta \in \mathbf{D}^2} u_z(\tau_z(\zeta)) dm(\zeta) \\ &= \int_{\eta \in \mathbf{D}^2} u(\eta) \left(\frac{1 - |z_1|^2}{|1 - \bar{z}_1 \eta_1|^2} \frac{1 - |z_2|^2}{|1 - \bar{z}_2 \eta_2|^2} \right)^2 dm(\eta). \end{aligned}$$

Now we will prove that the operator R defined by

$$R[u](z) = \int_{\eta \in \mathbf{D}^2} u(\eta) \left(\frac{1 - |z_1|^2}{|1 - \bar{z}_1 \eta_1|^2} \frac{1 - |z_2|^2}{|1 - \bar{z}_2 \eta_2|^2} \right)^2 dm(\eta),$$

is bounded in $L^p(\mathbf{D}^2)$, i.e. $\|R[u]\|_p \lesssim \|u\|_p$. In order to do so, we will use Schur's lemma.

Lemma 2.6. (Schur) *Assume that (X, μ) is a measure space and K a measurable non-negative function defined in $X \times X$. Let T be the integral operator defined by K ,*

$$Tf(x) = \int_X K(x, y) f(y) d\mu(y).$$

Let p be such that $1 < p < \infty$ and q be such that $1/p + 1/q = 1$. If there is a constant $C > 0$ and a positive measurable function h such that

$$\int_X K(x, y) h(y)^q d\mu(y) \leq Ch(x)^q \quad \text{for } \mu\text{-a.e. } x \in X$$

and

$$\int_X K(x, y) h(x)^p d\mu(y) \leq Ch(y)^p \quad \text{for } \mu\text{-a.e. } y \in X,$$

then T is bounded in $L^p(X, d\mu)$ with norm less than or equal to C .

In our case we take $h(w) = (1 - |w_1|^2)^\beta (1 - |w_2|^2)^\beta$, with a properly chosen β such that $-1 < \beta < 0$. The estimates needed in the hypothesis of Schur's lemma come from the inequality ([R, Prop. 1.4.10])

$$\int_{\mathbf{D}} \frac{(1 - |z|^2)^\alpha}{|1 - w\bar{z}|^\beta} dm(z) \lesssim (1 - |w|^2)^{\alpha - \beta + 2}, \quad \text{if } \alpha > -1 \text{ and } \alpha - \beta + 2 < 0.$$

We have proved that if $u \in L^p(\mathbf{D}^2)$, then

$$\int_{\zeta \in \mathbf{D}^2} l(\zeta, 0) \wedge \partial \bar{\partial} u_z(\zeta) \in L^p(\mathbf{D}^2)$$

with L^p -norm controlled by that of u . Now, considering the expression (4) of $l(\zeta, z)$ and taking into account that $t(\zeta, 0) = 0$, $p(\zeta, 0) = \overline{p(\zeta, 0)} = \pi^0(\zeta, 0)$, we obtain

$$(6) \quad \int_{\zeta \in \mathbf{D}^2} l(\zeta, 0) \wedge \partial \bar{\partial} u_z(\zeta) = \int_{\zeta \in \mathbf{D}^2} \frac{i}{2\pi} (m(\zeta_2, 0) d\zeta_1 \wedge d\bar{\zeta}_1 + m(\zeta_1, 0) \delta_0(\zeta_2) d\zeta_2 \wedge d\bar{\zeta}_2) \wedge \partial \bar{\partial} u_z(\zeta).$$

Let us consider the first term in the right-hand side of (6). For bidegree reasons it is enough to consider

$$\partial_2 \bar{\partial}_2 u_z(\zeta) = \theta_{22}(\tau_z(\zeta)) \frac{(1 - |z_2|^2)^2}{|1 + \zeta_2 \bar{z}_2|^4} d\zeta_2 \wedge d\bar{\zeta}_2,$$

and therefore

$$\int_{\zeta \in \mathbf{D}^2} m(\zeta_2, 0) d\zeta_1 \wedge d\bar{\zeta}_1 \wedge \partial \bar{\partial} u_z(\zeta) = \int_{\zeta \in \mathbf{D}^2} \left(\log \left| \frac{z_2 - \zeta_2}{1 - \bar{\zeta}_2 z_2} \right|^2 + \frac{(1 - |\zeta_2|^2)(1 - |z_2|^2)}{|1 - \bar{\zeta}_2 z_2|^2} \right) \left(\frac{1 - |z_1|^2}{|1 - \zeta_1 \bar{z}_1|^2} \right)^2 \theta_{22}(\zeta_1, \zeta_2).$$

From the second term in the right hand side of (6) we only have to consider

$$\partial_1 \bar{\partial}_1 u_z(\zeta) = \theta_{11}(\tau_z(\zeta)) \frac{(1 - |z_1|^2)^2}{|1 + \zeta_1 \bar{z}_1|^4} d\zeta_1 \wedge d\bar{\zeta}_1,$$

and therefore

$$\int_{\zeta \in \mathbf{D}^2} m(\zeta_1, 0) \delta_0(\zeta_2) d\zeta_2 \wedge d\bar{\zeta}_2 \wedge \partial \bar{\partial} u_z(\zeta) = \int_{\zeta_1 \in \mathbf{D}} \left(\log \left| \frac{z_1 - \zeta_1}{1 - \bar{\zeta}_1 z_1} \right|^2 + \frac{(1 - |\zeta_1|^2)(1 - |z_1|^2)}{|1 - \bar{\zeta}_1 z_1|^2} \right) \theta_{11}(\zeta_1, z_2).$$

Both terms of (6) are therefore negative and moreover

$$\log \left| \frac{z - \zeta}{1 - \bar{\zeta} z} \right|^2 + \frac{(1 - |\zeta|^2)(1 - |z|^2)}{|1 - \bar{\zeta} z|^2} \leq -\frac{1}{2} \frac{(1 - |\zeta|^2)^2(1 - |z|^2)^2}{|1 - \bar{\zeta} z|^4}.$$

In consequence,

$$(7) \quad \int_{\zeta_1 \in \mathbf{D}} \frac{(1-|\zeta_1|^2)^2(1-|z_1|^2)^2}{|1-\bar{\zeta}_1 z_1|^4} \theta_{11}(\zeta_1, z_2) \in L^p(\mathbf{D}^2)$$

with norm controlled by the norm of u . Obviously, the same happens if we permute the indices, since the solution $L[\Theta]$ is symmetric in both variables.

Finally, note that for $\zeta_1 \in D_{z_1}$,

$$|1-\bar{\zeta}_1 z_1| \simeq 1-|z_1|^2 \simeq 1-|\zeta_1|^2.$$

From (7) we can conclude

$$\|\theta_{11}(D_{z_1}, z_2)\|_p \lesssim \|u\|_p.$$

From this inequality, we can obtain the general case (recall that we have only proved the necessity when $u \in \mathcal{C}^\infty(\bar{\mathbf{D}}^2)$). For an arbitrary $u \in L^p(\mathbf{D}^2)$ we pick a sequence of $u_n \in \mathcal{C}^\infty(\bar{\mathbf{D}}^2)$ such that $u_n \rightarrow u$ in L^p . Since the convergence in L^p implies the convergence in the distribution sense, $\Theta^n = \partial\bar{\partial}u_n \rightarrow \partial\bar{\partial}u = \Theta$, weakly.

We want to check that $\|\theta_{11}(D_{z_1}, z_2)\|_p < \infty$. But, for any function $\psi \in \mathcal{C}^\infty(\mathbf{D}^2)$ positive with compact support, and if $1/p+1/q=1$ the following holds

$$\begin{aligned} \left| \int_{\mathbf{D}^2} \psi(z) \theta_{11}(D_{z_1}, z_2) dm(z) \right| &= \left| \lim_{n \rightarrow \infty} \int_{\mathbf{D}^2} \psi(z) \theta_{11}^n(D_{z_1}, z_2) dm(z) \right| \\ &\lesssim \lim_{n \rightarrow \infty} \|u_n\|_p \|\psi\|_q \lesssim \|u\|_p \|\psi\|_q, \end{aligned}$$

thus we have obtained the desired result. \square

2.3. Proof of the sufficiency of (1)

We will show that condition (1) is sufficient in order to obtain a solution $u \in L^p(\mathbf{D}^2)$ to $i\partial\bar{\partial}u = \Theta$. We assume initially that Θ is a closed positive (1,1)-form and that $\Theta \in \mathcal{C}^\infty(\bar{\mathbf{D}}^2)$. We will show that under this hypothesis the solution $L[\Theta] \in L^p(\mathbf{D}^2)$ with controlled norm. Later on we will drop the regularity hypothesis. In order to show this we will make use of the expression of $l(\zeta, z)$ that we have given in (4), i.e.

$$(8) \quad l(\zeta, z) = \pi_1^0 \wedge m_2 + m_1 \wedge \pi_2^0 + \bar{t}_1 \wedge k_2 + \bar{k}_1 \wedge t_2 - \bar{t}_1 \wedge t_2 + m_1 \wedge i_2 - m_1 \wedge \bar{p}_2 + t_1 \wedge \bar{k}_2.$$

Recall that the definition of the kernels that appear in (8) are given explicitly in (3).

We need the following bounds of the moduli of the respective kernels that can be found in [A],

$$(9) \quad \begin{aligned} |m(\zeta, z)| &\lesssim \frac{(1-|\zeta|^2)^2}{|1-\bar{\zeta}z|^2} \left[1 + \log \left| \frac{1-\bar{\zeta}z}{\zeta-z} \right| \right], \\ |t(\zeta, z)| &\lesssim \frac{1-|\zeta|^2}{|1-\bar{\zeta}z|^2}, \end{aligned}$$

where $\zeta, z \in \mathbf{D}$. The terms of the type $\bar{T}_1 K_2$ are

$$\int_{\zeta \in \mathbf{D}^2} \overline{t(\zeta_1, z_1)} d\bar{\zeta}_1 \wedge \frac{1-|\zeta_2|^2}{(1-\bar{\zeta}_2 z_2)(\zeta_2 - z_2)} d\zeta_2 \wedge \Theta(\zeta) = D(z_1, z_2).$$

We are going to rewrite $D(z_1, z_2)$ in a more convenient way. Since $t(\zeta_1, z_1) = 0$ when $|\zeta_1| = 1$, by Stokes' theorem

$$\begin{aligned} 0 &= \int_{\zeta \in \partial \mathbf{D}^2} \overline{t(\zeta_1, z_1)} d\bar{\zeta}_1 \wedge \frac{(1-|\zeta_2|^2)(1-\zeta_2 \bar{z}_2)}{(1-\bar{\zeta}_2 z_2)(1-|z_2|^2)} \log \left| \frac{1-\bar{\zeta}_2 z_2}{\zeta_2 - z_2} \right|^2 \wedge \Theta(\zeta) \\ &= \int_{\zeta \in \mathbf{D}^2} \partial_{\zeta_1} \overline{t(\zeta_1, z_1)} \frac{(1-|\zeta_2|^2)(1-\zeta_2 \bar{z}_2)}{(1-\bar{\zeta}_2 z_2)(1-|z_2|^2)} \log \left| \frac{1-\bar{\zeta}_2 z_2}{\zeta_2 - z_2} \right|^2 \theta_{22}(\zeta) dm(\zeta) \\ &\quad + \int_{\zeta \in \mathbf{D}^2} \overline{t(\zeta_1, z_1)} \partial_{\zeta_2} \left[\frac{(1-|\zeta_2|^2)(1-\zeta_2 \bar{z}_2)}{(1-\bar{\zeta}_2 z_2)(1-|z_2|^2)} \right] \log \left| \frac{1-\bar{\zeta}_2 z_2}{\zeta_2 - z_2} \right|^2 \theta_{12}(\zeta) dm(\zeta) + D(z_1, z_2). \end{aligned}$$

An immediate computation yields the following estimates

$$\begin{aligned} |\partial_{\zeta_1} \overline{t(\zeta_1, z_1)}| &\lesssim \frac{1}{|1-\bar{z}_1 \zeta_1|^2}, \\ \left| \partial_{\zeta_2} \frac{(1-|\zeta_2|^2)(1-\zeta_2 \bar{z}_2)}{(1-\bar{\zeta}_2 z_2)(1-|z_2|^2)} \right| &\lesssim \frac{1}{1-|z_2|^2}. \end{aligned}$$

Hence,

$$\begin{aligned} |D(z_1, z_2)| &\lesssim \int_{\zeta \in \mathbf{D}^2} \frac{1}{|1-\bar{\zeta}_1 z_1|^2} \frac{1-|\zeta_2|^2}{1-|z_2|^2} \log \left| \frac{1-\bar{\zeta}_2 z_2}{\zeta_2 - z_2} \right|^2 \theta_{22}(\zeta) dm(\zeta) \\ &\quad + \int_{\zeta \in \mathbf{D}^2} \frac{1-|\zeta_1|^2}{|1-\bar{\zeta}_1 z_1|^2} \frac{1}{1-|z_2|^2} \log \left| \frac{1-\bar{\zeta}_2 z_2}{\zeta_2 - z_2} \right|^2 |\theta_{12}(\zeta)| dm(\zeta). \end{aligned}$$

Finally, as

$$\log \left| \frac{1-\bar{\zeta}_2 z_2}{\zeta_2 - z_2} \right|^2 \simeq \frac{(1-|\zeta_2|^2)(1-|z_2|^2)}{|1-\bar{\zeta}_2 z_2|^2}, \quad \text{if } \left| \frac{\zeta_2 - z_2}{1-\bar{\zeta}_2 z_2} \right| \geq \frac{1}{2},$$

we can estimate $D(z_1, z_2)$ as follows,

$$(10) \quad |D(z_1, z_2)| \lesssim \int_{\zeta \in \mathbf{D}^2} \frac{1}{|1 - \bar{\zeta}_1 z_1|^2} \frac{(1 - |\zeta_2|^2)^2}{|1 - \bar{\zeta}_2 z_2|^2} \left(1 + \log \left| \frac{1 - \bar{\zeta}_2 z_2}{\zeta_2 - z_2} \right|^2\right) \theta_{22}(\zeta) dm(\zeta) \\ + \int_{\zeta \in \mathbf{D}^2} \frac{1 - |\zeta_1|^2}{|1 - \bar{\zeta}_1 z_1|^2} \frac{1 - |\zeta_2|^2}{|1 - \bar{\zeta}_2 z_2|^2} \left(1 + \log \left| \frac{1 - \bar{\zeta}_2 z_2}{\zeta_2 - z_2} \right|^2\right) |\theta_{12}|(\zeta) dm(\zeta).$$

We claim that $|L[\Theta](z)|$ is controlled by a sum of terms of the type

$$(11) \quad \int_{\zeta \in \mathbf{D}^2} \frac{1}{|1 - \bar{\zeta}_1 z_1|^2} \frac{(1 - |\zeta_2|^2)^2}{|1 - \bar{\zeta}_2 z_2|^2} \left(1 + \log \left| \frac{1 - \bar{\zeta}_2 z_2}{\zeta_2 - z_2} \right|^2\right) \theta_{22}(\zeta) dm(\zeta),$$

$$(12) \quad \int_{\zeta \in \mathbf{D}^2} \frac{1 - |\zeta_1|^2}{|1 - \bar{\zeta}_1 z_1|^2} \frac{1 - |\zeta_2|^2}{|1 - \bar{\zeta}_2 z_2|^2} \left(1 + \log \left| \frac{1 - \bar{\zeta}_2 z_2}{\zeta_2 - z_2} \right|^2\right) |\theta_{12}|(\zeta) dm(\zeta),$$

$$(13) \quad \int_{\zeta_1 \in \mathbf{D}} \frac{(1 - |\zeta_1|^2)^2}{|1 - \bar{\zeta}_1 z_1|^2} \left(1 + \log \left| \frac{1 - \bar{\zeta}_1 z_1}{\zeta_1 - z_1} \right|^2\right) \theta_{11}(\zeta_1, z_2) dm(\zeta_1).$$

Indeed, in (10) we have already seen that $\bar{T}_1 K_2$ is bounded by (11)+(12). The term in $\bar{T}_1 T_2$ is smaller than (12) and moreover (9) implies that $M_1 I_2$ is bounded by (13) and $\bar{P}_1 M_2, \Pi_1^Q M_2$ by (11). Similar estimates apply to the kernels we obtain by permuting indices in the previous ones, like $\Pi_2^Q M_1, \bar{K}_1 T_2$ and $M_1 \bar{P}_2$. We will show that under the hypothesis (1) all of them belong to $L^p(\mathbf{D}^2)$. The terms of type (11) and (12) will be considered jointly by means of the following two lemmas.

Lemma 2.7. *The condition (1) implies*

$$\frac{(1 - |\zeta_1|^2)^2 \theta_{11}(D_z) + (1 - |\zeta_2|^2)^2 \theta_{22}(D_z)}{|D_z|} \in L^p(\mathbf{D}^2), \\ \frac{(1 - |\zeta_1|^2)(1 - |\zeta_2|^2) |\theta_{12}|(D_z)}{|D_z|} \in L^p(\mathbf{D}^2),$$

where $D_z = D_{z_1} \times D_{z_2}$ and $|D_z|$ is the Lebesgue measure of D_z .

Proof. Indeed, consider the operator ψ defined as

$$\psi[f](w) = \frac{1}{|D_w|} \int_{\zeta \in D_w} |f(\zeta)| dm(\zeta), \quad w \in \mathbf{D}.$$

It is bounded from $L^p(\mathbf{D})$ to $L^p(\mathbf{D})$. If we apply this operator to the function $f(z_1, z_2) = \theta_{11}(D_{z_1}, z_2)$ (as a function of z_2) and to $g(z_1, z_2) = \theta_{22}(z_1, D_{z_2})$ (as a function of z_1), we have proved the first part of the statement of the claim. The second

is an immediate consequence of the first. In fact, by the positivity of the current Θ , one has

$$\frac{(1-|\zeta_1|^2)(1-|\zeta_2|^2)|\theta_{12}|(D_z)}{|D_z|} \leq \frac{(1-|\zeta_1|^2)^2\theta_{11}(D_z)}{|D_z|} + \frac{(1-|\zeta_2|^2)^2\theta_{22}(D_z)}{|D_z|}.$$

And this function belongs to $L^p(\mathbf{D}^2)$. \square

Lemma 2.8. *If μ is a positive measure in the bidisk such that*

$$\frac{\mu(D_z)}{|D_z|} \in L^p(\mathbf{D}^2),$$

then

$$\int_{\zeta \in \mathbf{D}^2} \frac{d\mu(\zeta)}{|1-\bar{\zeta}_1 z_1|^2 |1-\bar{\zeta}_2 z_2|^2} \left(1 + \log \left| \frac{1-\bar{\zeta}_2 z_2}{\zeta_2 - z_2} \right|^2 \right) \in L^p(\mathbf{D}^2).$$

Proof. We will split the integral that we want to estimate into two parts

$$(14) \quad \int_{\mathbf{D}^2} \frac{d\mu(\zeta)}{|1-\bar{\zeta}_1 z_1|^2 |1-\bar{\zeta}_2 z_2|^2} + \int_{\zeta \in \mathbf{D} \times D_{z_2}} \frac{d\mu(\zeta)}{|1-\bar{\zeta}_1 z_1|^2 (1-|\zeta_2|^2)^2} \log \left| \frac{1-\bar{\zeta}_2 z_2}{\zeta_2 - z_2} \right|^2.$$

As the function

$$\frac{1}{|1-\zeta_1 \bar{z}_1|^2 |1-\zeta_2 \bar{z}_2|^2}$$

is plurisubharmonic, we have the sub-mean inequality

$$\frac{1}{|1-\zeta_1 \bar{z}_1|^2 |1-\zeta_2 \bar{z}_2|^2} \leq \frac{1}{|D_\zeta|} \int_{\eta \in D_\zeta} \frac{dm(\eta)}{|1-\eta_1 \bar{z}_1|^2 |1-\eta_2 \bar{z}_2|^2}.$$

If this inequality is inserted in (14) we get that the first integral in (14) is bounded by

$$\begin{aligned} \int_{\zeta \in \mathbf{D}^2} \int_{\eta \in D_\zeta} \frac{1}{|D_\zeta|} \frac{dm(\eta) d\mu(\zeta)}{|1-\eta_1 \bar{z}_1|^2 |1-\eta_2 \bar{z}_2|^2} &= \int_{\eta \in \mathbf{D}^2} \int_{\zeta \in D_\eta} \frac{1}{|D_\zeta|} \frac{d\mu(\zeta) dm(\eta)}{|1-\eta_1 \bar{z}_1|^2 |1-\eta_2 \bar{z}_2|^2} \\ &\lesssim \int_{\eta \in \mathbf{D}^2} \frac{\mu(D_\eta)}{|D_\eta|} \frac{dm(\eta)}{|1-\eta_1 \bar{z}_1|^2 |1-\eta_2 \bar{z}_2|^2}. \end{aligned}$$

Now, Schur's lemma implies that the kernel

$$\frac{1}{|1-\eta_1 \bar{z}_1|^2 |1-\eta_2 \bar{z}_2|^2},$$

defines a bounded operator in $L^p(\mathbf{D}^2)$ and hence we can conclude that the first term in (14) is in $L^p(\mathbf{D}^2)$. The second one is a bit more involved.

Due to the subharmonicity of the function

$$\frac{1}{|1 - \bar{\zeta}_1 z_1|^2},$$

we have, as before,

$$\begin{aligned} & \int_{\zeta \in \mathbf{D} \times D_{z_2}} \frac{d\mu(\zeta)}{|1 - \bar{\zeta}_1 z_1|^2} \frac{1}{(1 - |\zeta_2|^2)^2} \log \left| \frac{1 - \bar{\zeta}_2 z_2}{\zeta_2 - z_2} \right|^2 \\ & \lesssim \int_{\zeta \in \mathbf{D} \times D_{z_2}} \int_{\eta \in D_{\zeta_1}} \frac{dm(\eta)}{(1 - |\eta|^2)^2 |1 - \bar{\eta} z_1|^2} \frac{1}{(1 - |\zeta_2|^2)^2} \log \left| \frac{1 - \bar{\zeta}_2 z_2}{\zeta_2 - z_2} \right|^2 d\mu(\zeta) \\ & = \int_{\eta \in \mathbf{D}} \frac{1}{|1 - \bar{\eta} z_1|^2} \left(\int_{\substack{\zeta_1 \in D_\eta \\ \zeta_2 \in D_{z_2}}} \frac{d\mu(\zeta)}{(1 - |\eta|^2)^2 (1 - |\zeta_2|^2)^2} \log \left| \frac{1 - \bar{\zeta}_2 z_2}{\zeta_2 - z_2} \right|^2 \right) dm(\eta) \\ & = A(z_1, z_2). \end{aligned}$$

We want to prove that $A \in L^p(\mathbf{D}^2)$. Since $1/|1 - \bar{\eta} z_1|^2$ defines a bounded operator from $L^p(\mathbf{D})$ to $L^p(\mathbf{D})$,

$$\begin{aligned} & \int_{z \in \mathbf{D}^2} |A(z_1, z_2)|^p dm(z) \\ & \lesssim \int_{\eta \in \mathbf{D}} \left(\int_{\zeta \in D_\eta \times D_{z_2}} \frac{d\mu(\zeta)}{(1 - |\eta|^2)^2 (1 - |\zeta_2|^2)^2} \log \left| \frac{1 - \bar{\zeta}_2 z_2}{\zeta_2 - z_2} \right|^2 \right)^p dm(\eta) dm(z_2). \end{aligned}$$

Now we apply Jensen's inequality to estimate this by

$$\int_{\substack{z_2 \in \mathbf{D} \\ \eta \in \mathbf{D}}} \int_{\substack{\zeta_1 \in D_\eta \\ \zeta_2 \in D_{z_2}}} \frac{\mu(D_\eta \times D_{z_2})^{p-1}}{(1 - |\zeta_1|^2)^{2p} (1 - |\zeta_2|^2)^{2p}} \log^p \left| \frac{1 - \bar{\zeta}_2 z_2}{\zeta_2 - z_2} \right|^2 d\mu(\zeta) dm(\eta) dm(z_2).$$

If $z_2 \in D_{\zeta_2}(\varepsilon)$, then $D_{z_2} \subset D_{\zeta_2}(\delta)$ for some $\delta > \varepsilon$. Since the hypothesis of the lemma is independent of ε , we may assume that $D_{z_2} \subset D_{\zeta_2}$ and $D_\eta \subset D_{\zeta_1}$. If we apply Fubini's theorem we get the estimate

$$\int_{\zeta \in \mathbf{D}^2} \int_{\substack{\eta \in D_{\zeta_1} \\ z_2 \in D_{\zeta_2}}} \frac{\mu(D_\zeta)}{(1 - |\zeta_1|^2)^{2p} (1 - |\zeta_2|^2)^{2p}} \log^p \left| \frac{1 - \bar{\zeta}_2 z_2}{\zeta_2 - z_2} \right|^2 dm(\eta) dm(z_2) d\mu(\zeta).$$

Since

$$\int_{\substack{\eta \in D_{\zeta_1} \\ z_2 \in D_{\zeta_2}}} \log^p \left| \frac{1 - \bar{\zeta}_2 z_2}{\zeta_2 - z_2} \right|^2 dm(z_2) dm(\eta) \lesssim (1 - |\zeta_2|^2)^2 (1 - |\zeta_1|^2)^2,$$

we get as a final bound

$$\lesssim \int_{\zeta \in \mathbf{D}^2} \frac{\mu(D_\zeta)^{p-1}}{(1-|\zeta_1|^2)^{2p-2}(1-|\zeta_2|^2)^{2p-2}} d\mu(\zeta).$$

It is then enough to show that this last integral is bounded by

$$\int_{z \in \mathbf{D}^2} \frac{\mu(D_z)^p}{(1-|z_1|^2)^{2p}(1-|z_2|^2)^{2p}} dm(z).$$

In order to check this

$$\begin{aligned} & \int_{z \in \mathbf{D}^2} \frac{\mu(D_z)^p}{(1-|z_1|^2)^{2p}(1-|z_2|^2)^{2p}} dm(z) \\ (15) \quad &= \int_{z \in \mathbf{D}^2} \frac{\mu(D_z)^{p-1}}{(1-|z_1|^2)^{2p}(1-|z_2|^2)^{2p}} \int_{\zeta \in D_z} d\mu(\zeta) dm(z) \\ &\gtrsim \int_{\zeta \in \mathbf{D}^2} \int_{z \in \mathbf{D}_\zeta} \frac{\mu(D_\zeta^2)^{p-1}}{(1-|\zeta_1|^2)^{2p}(1-|\zeta_2|^2)^{2p}} dm(z) d\mu(\zeta) \\ &= \int_{\zeta \in \mathbf{D}^2} \frac{\mu(D_\zeta^2)^{p-1}}{(1-|\zeta_1|^2)^{2p-2}(1-|\zeta_2|^2)^{2p-2}} d\mu(\zeta). \end{aligned}$$

With this, we have proved the lemma. \square

Now, combining Lemmas 2.7 and 2.8, taking

$$d\mu(\zeta) = (1-|\zeta_1|^2)(1-|\zeta_2|^2)|\theta_{12}(\zeta)| dm(\zeta)$$

and

$$d\mu(\zeta) = (1-|\zeta_2|^2)^2 \theta_{22}(\zeta) dm(\zeta)$$

respectively we see that the terms (11) and (12) belong to $L^p(\mathbf{D}^2)$.

In order to finish the proof of the theorem it remains to estimate the terms of type (13). In order to control it we split it into two parts, according to the contribution of $D_{z_1}^c$ and D_{z_1} in the integral. The contribution of $D_{z_1}^c$ is bounded by

$$(16) \quad \int_{\zeta_1 \in \mathbf{D}} \frac{(1-|\zeta_1|^2)^2 \theta_{11}(\zeta_1, z_2)}{|1-\bar{\zeta}_1 z_1|^2} dm(\zeta_1).$$

Due to the subharmonicity of

$$c(\zeta_1, z_1) = \frac{1}{|1-\bar{\zeta}_1 z_1|^2},$$

we have

$$\int_{\zeta_1 \in \mathbf{D}} \frac{(1-|\zeta_1|^2)^2 \theta_{11}(\zeta_1, z_2)}{|1-\bar{\zeta}_1 z_1|^2} dm(\zeta_1) \lesssim \int_{\zeta_1 \in \mathbf{D}} \frac{\theta_{11}(D_{\zeta_1}, z_2)}{|1-\bar{\zeta}_1 z_1|^2} dm(\zeta_1).$$

Since $c(\zeta_1, z_1)$ is a kernel that defines an operator bounded from $L^p(\mathbf{D})$ to $L^p(\mathbf{D})$ and, by hypothesis (1), $\theta_{11}(D_{z_1}, z_2) \in L^p(\mathbf{D}^2)$, it follows that (16) belongs to $L^p(\mathbf{D}^2)$.

The contribution of D_{z_1} is bounded by

$$\int_{\zeta_1 \in D_{z_1}} \log \left| \frac{1-\bar{\zeta}_1 z_1}{\zeta_1 - z_1} \right| \theta_{11}(\zeta_1, z_2) dm(\zeta_1).$$

We want to prove that

$$\int_{z \in \mathbf{D}^2} \left(\int_{\zeta_1 \in D_{z_1}} \log \left| \frac{1-\bar{\zeta}_1 z_1}{\zeta_1 - z_1} \right| \theta_{11}(\zeta_1, z_2) dm(\zeta_1) \right)^p dm(z) < +\infty.$$

In order to check this, we apply Jensen's inequality and estimate the integral by

$$\int_{z \in \mathbf{D}^2} \int_{\zeta_1 \in D_{z_1}} \theta_{11}(D_{z_1}, z_2)^{p-1} \log^p \left| \frac{1-\bar{\zeta}_1 z_1}{\zeta_1 - z_1} \right| \theta_{11}(\zeta_1, z_2) dm(\zeta_1) dm(z).$$

We apply Fubini's theorem, integrating first in z_1 , and obtain the estimate

$$\int_{\zeta_1 \in \mathbf{D}} \theta_{11}(D_{\zeta_1}, z_2)^{p-1} (1-|\zeta_1|^2)^2 \theta_{11}(\zeta_1, z_2) dm(\zeta_1).$$

If we now argue as in (15), this is in turn bounded by

$$\int_{z \in \mathbf{D}^2} \theta_{11}(D_{z_1}, z_2)^p dm(z).$$

Therefore we have already proved that there exists a solution u of $i\partial\bar{\partial}u = \Theta$ with

$$\|u\|_p \lesssim \|\theta_{11}(D_{z_1}, z_2)\|_p + \|\theta_{22}(z_1, D_{z_2})\|_p,$$

if $\Theta \in \mathcal{C}^\infty(\bar{\mathbf{D}}^2)$. In the general case, by a standard regularizing process (see for instance [C]), we can find a sequence $\Theta^n \rightarrow \Theta$ in a distributional sense and with $\Theta^n \in \mathcal{C}^\infty(\bar{\mathbf{D}}^2)$, in such a way that

$$\|\theta_{11}^n(D_{z_1}, z_2)\|_p + \|\theta_{22}^n(z_1, D_{z_2})\|_p \lesssim \|\theta_{11}(D_{z_1}, z_2)\|_p + \|\theta_{22}(z_1, D_{z_2})\|_p.$$

We have already proved that there exists a sequence u_n with $i\partial\bar{\partial}u_n = \Theta^n$ and such that

$$\|u_n\|_p \lesssim \|\theta_{11}(D_{z_1}, z_2)\|_p + \|\theta_{22}(z_1, D_{z_2})\|_p.$$

There is a subsequence such that $u_n \rightarrow u$ with $i\partial\bar{\partial}u = \Theta$ and

$$\|u\|_p \lesssim \|\theta_{11}(D_{z_1}, z_2)\|_p + \|\theta_{22}(z_1, D_{z_2})\|_p. \quad \square$$

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Received November 29, 1995

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