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# Invariant fundamental solutions and solvability for symmetric spaces of type $G_{\mathbf{C}}/G_{\mathbf{R}}$ with only one conjugacy class of Cartan subspaces

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## Introduction

Let $G/H$ be a reductive symmetric space and let $D: C^\infty(G/H) \to C^\infty(G/H)$ be a non-trivial $G$-invariant differential operator. An invariant fundamental solution for $D$ is a left-$H$-invariant distribution $T$ on $G/H$ solving the differential equation

$$(*) \qquad DT = \delta,$$

where $\delta$ is the Dirac measure at the origin of $G/H$.

Assume that $G/H$ is of type $G_{\mathbf{C}}/G_{\mathbf{R}}$ ($G$ complex and $H$ a real form of $G$) and that $H$, up to conjugacy, has only one Cartan subalgebra. Let $A$ denote the associated Cartan subset of $G/H$, identified with a real abelian subgroup of $G$. Using results from the theory of orbital integrals defined on $G/H$ obtained by Bouaziz, Harinck and Sano, we can then reduce $(*)$ to a differential equation on $A$,

$$\Gamma(D) T_A = \delta_A$$

for some distribution $T_A$ on $A$, where $\Gamma(D)$ is a uniquely defined differential operator with constant coefficients on $A$ and $\delta_A$ is the Dirac measure at the origin of $A$, i.e. $T_A$ is by definition a fundamental solution for $\Gamma(D)$. Our main result is the following theorem.

**Theorem 5.** *Let $D$ be as above. Then $D$ has an invariant fundamental solution on $G/H$ if $\Gamma(D)$ has a fundamental solution on $A$.*

Our result is similar to results obtained by Helgason for Riemannian symmetric spaces, see [11, Theorem 4.2], and by Rouvière for semisimple Lie groups with only one conjugacy class of Cartan subalgebras, see [15, Theorem 4.2], and our approach is very much inspired by their works.

Assume now that $D$ has an invariant fundamental solution on $G/H$. Then $D$ is solvable, in the sense that $DC^\infty(G/H)=C^\infty(G/H)$, if $G/H$ is $D$-convex, see [1, pp. 301ff.]. Van den Ban and Schlichtkrull give in [1, Theorem 2] a necessary condition on $D$ for $D$-convexity of $G/H$, and we show that this condition implies that $\Gamma(D)$ has a fundamental solution on $A$, and hence, as an application of Theorem 5, that $D$ is solvable, see Theorem 7.

## Notation

Let $G$ be a reductive complex connected Lie group with Lie algebra $\mathfrak{g}$, and let $H$ be a real form of $G$ with Lie algebra $\mathfrak{h}$. Let $\sigma$ denote the conjugation of $\mathfrak{g}$ relative to $\mathfrak{h}$ and let also $\sigma$ denote the involution of $G$ whose differential is $\sigma$, then $H$ is the open connected subgroup of $G^\sigma$, the fixpoint set of $\sigma$ in $G$. The space $G/H$ is said to be a reductive symmetric space of type $G_{\mathbf{C}}/G_{\mathbf{R}}$. Let $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{q}$ be the decomposition of $\mathfrak{g}$ into the $\pm1$-eigenspaces of $\sigma$, where $\mathfrak{q}=i\mathfrak{h}$. Let $\theta$ be a Cartan involution of $\mathfrak{g}$ commuting with $\sigma$, and let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ be the usual Cartan decomposition into the $\pm1$-eigenspaces of $\theta$. Let $K=G^\theta$ be the maximal compact subgroup of $G$ consisting of fixpoints of $\theta$, with Lie algebra $\mathfrak{k}$.

Let $p$ be the canonical projection of $G$ onto $G/H$ and let $\varphi$ be the map of $G/H$ into $G$ defined by $G/H\ni p(g)\mapsto g\sigma(g)^{-1}\in G$, $g\in G$. The image of $\varphi$ in $G$, denoted by $\mathbf{X}$, is a closed submanifold of $G$, see [14, p. 402], and $\varphi$ is seen to be a $G$-isomorphism from $G/H$ onto $\mathbf{X}$, equipped with the $G$-action $g\cdot x=gx\sigma(g)^{-1}$, $x\in\mathbf{X}$, $g\in G$. We will in the following use this realization of the symmetric space $G/H$.

Denote the space of distributions on $\mathbf{X}$ by $\mathcal{D}'(\mathbf{X})$. The group $G$ acts naturally on $\mathcal{D}'(\mathbf{X})$ via the contragradient representation (on $G/H$), and we denote the $H$-invariant distributions under this action by $\mathcal{D}'(\mathbf{X})^H$.

Let exp denote the exponential map of $\mathfrak{g}$ into $G$.

## Cartan subspaces, Cartan subsets and root systems

A Cartan subspace $\mathfrak{a}$ for $\mathbf{X}$ is defined (cf. [14, §1]) as a maximal abelian subspace of $\mathfrak{q}$ consisting of semisimple elements. We see, since $\mathfrak{h}$ is a real form of $\mathfrak{g}$, that $\mathfrak{a}$ is a Cartan subspace for $\mathbf{X}$ if and only if $i\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{h}$. The Cartan subset $A$ of $\mathbf{X}$ associated to a Cartan subspace $\mathfrak{a}$ for $\mathbf{X}$, is defined (cf. [14, §1]) as the set of elements $x\in\mathbf{X}$ centralizing $\mathfrak{a}$ in $G$ (under the adjoint action).

So let $\mathfrak{a}$ be a Cartan subspace of $\mathfrak{q}$. We denote by $\Delta=\Delta(\mathfrak{g},\mathfrak{a}_{\mathbf{C}})$ the root system of the pair $(\mathfrak{g},\mathfrak{a}_{\mathbf{C}})$, where $\mathfrak{a}_{\mathbf{C}}=\mathfrak{a}+i\mathfrak{a}$. We choose a set of positive roots denoted by $\Delta^+$. Let $W$ denote the Weyl group corresponding to the root system $\Delta$. Let $H_\alpha$, respectively $\mathfrak{g}_\alpha$, denote the coroot, respectively the root space, of the root $\alpha\in\Delta$.

We say that a root $\alpha\in\Delta$ is real, respectively imaginary or complex, if it is real-valued, respectively imaginary-valued, or neither real- nor imaginary-valued, on the Cartan subalgebra $i\mathfrak{a}$ of $\mathfrak{h}$. The set of real roots, positive real roots, imaginary roots, positive imaginary roots, complex roots and positive complex roots are denoted by $\Delta_{\mathbf{R}}$, $\Delta_{\mathbf{R}}^+$, $\Delta_I$, $\Delta_I^+$, $\Delta_{\mathbf{C}}$ and $\Delta_{\mathbf{C}}^+$ respectively.

Let $\alpha\in\Delta_I$. The root $\alpha$ is called compact if and only if $(\mathfrak{g}_\alpha+\mathfrak{g}_{-\alpha}+\mathbf{C}H_\alpha)\cap\mathfrak{h}$ is isomorphic to $\mathfrak{su}(2)$, respectively noncompact if and only if $(\mathfrak{g}_\alpha+\mathfrak{g}_{-\alpha}+\mathbf{C}H_\alpha)\cap\mathfrak{h}$ is isomorphic to $\mathfrak{sl}(2,\mathbf{R})$. The set of imaginary noncompact roots is denoted by $\Delta_{\mathrm{Inc}}$.

We will in the following assume that there is only one $H$-conjugacy class of Cartan subalgebras of $\mathfrak{h}$. This is obviously equivalent to $H$ having only one conjugacy class of Cartan subalgebras (or Cartan subgroups). So fix a $\theta$-invariant Cartan subalgebra $i\mathfrak{a}$ of $\mathfrak{h}$. Then $\Delta_{\mathbf{R}}=\Delta_{\mathrm{Inc}}=\emptyset$, see [13, Proposition 11.16]. Let $A$ denote the Cartan subset of $\mathbf{X}$ associated to the Cartan subspace $\mathfrak{a}$ for $\mathbf{X}$. Then $A$ is given by

$$A=\exp\mathfrak{a}=\varphi(p(\exp\mathfrak{a})),$$

see [9, Corollaire 1.7], i.e. $A$ is a connected real abelian Lie subgroup of $G$ with Lie algebra $\mathfrak{a}$. Let $S(\mathfrak{a})$ denote the symmetric algebra of the complexification of $\mathfrak{a}$. This algebra can be identified with the algebra of differential operators $\mathbf{D}(A)$ on $A$ with constant coefficients, by means of the action generated by

$$Xf(a)=\frac{d}{dt}f(\exp tX\cdot a)|_{t=0}$$

for $X\in\mathfrak{a}$, where $f\in C^\infty(A)$ and $a\in A$.

## Regular elements

Put $n=\operatorname{rank}\mathfrak{h}$ and let $x\in\mathbf{X}$. The characteristic polynomial of the $\mathbf{C}$-linear endomorphism $Ad(x)-I$ on $\mathfrak{g}=\mathfrak{q}_{\mathbf{C}}=\mathfrak{h}_{\mathbf{C}}$ can be written as

$$\det{}_{\mathbf{C}}((1+z)I-Ad(x))\equiv z^n D_{\mathbf{X}}(x) \mod z^{n+1}$$

for all $z\in\mathbf{C}$. The function $D_{\mathbf{X}}$ so defined is an $H$-invariant analytic function on $\mathbf{X}$. An element $x$ in $\mathbf{X}$ is called regular (cf. [14, §1]) if $D_{\mathbf{X}}(x)\neq0$, and the set of regular elements in any subset $U\subset\mathbf{X}$ will be denoted by $U'$.

Define for every root $\alpha$ a function $\xi_\alpha$ on $A$ by

$$\xi_\alpha(\exp X)=e^{\alpha(X)}$$

for $X\in\mathfrak{a}$. We see, using the root space decomposition of $\mathfrak{g}$, that

$$D_{\mathbf{X}}(a)=\prod_{\alpha\in\Delta}(1-\xi_{-\alpha}(a))$$

for $a\in A$. We furthermore, for all subsets $S\subset\Delta$, define the function

$$b_S=\prod_{\alpha\in S}\frac{(1-\xi_{-\alpha})}{|1-\xi_{-\alpha}|}$$

on the set $A'$. We note, since all of the functions $\xi_\alpha$, $\alpha\in\Delta_I^+$, are real, that the function $b_{\Delta_I^+}$ is $\pm1$ on the connected components of $A'$.

We easily see that $Z_H(\mathfrak{a})=Z_H(a)$ if $a\in A'$ (since $a=\exp iX$, where $X$ is a regular element of $\mathfrak{h}$) and that $Z_H(\mathfrak{a})=Z_H(A)$ (since $A$ is connected), i.e. the quotient $N_H(A)/Z_H(A)$ is finite and equal to $N_H(\mathfrak{a})/Z_H(\mathfrak{a})$. The subgroup $Z_H(\mathfrak{a})=Z_H(A)$ is a Cartan subgroup of $H$. The map from $H/Z_H(A)\times A'$ into $\mathbf{X}$ defined by $(hZ_H(A),a)\mapsto h\cdot a$, is an everywhere regular $|N_H(A)/Z_H(A)|$-to-one map onto $\mathbf{X}'$, see [14, Theorem 2(ii)], and we thus have the decomposition $\mathbf{X}'=\bigcup_{h\in H}h\cdot A'$. Let $U\subset A'$ be a compact subset. Since $D_{\mathbf{X}}$ is an $H$-invariant continuous function on $\mathbf{X}$, we conclude from regularity of the map $(hZ_H(A),a)\mapsto h\cdot a$, that the subset $H[U]=\bigcup_{h\in H}h\cdot U$ is closed in $\mathbf{X}$. We see in particular that the $H$-orbit $H[a]$ through any regular element $a\in A'$ is closed in $\mathbf{X}$.

## Orbital integrals

*Definition* 1. Let $f\in C_c^\infty(\mathbf{X})$. The orbital integral $K_f$ of $f$, relative to the Cartan subset $A$, is the function defined on the regular elements $a\in A'$ by

$$K_f(a)=|D_{\mathbf{X}}(a)|^{1/2}\int_{H/Z_H(A)}f(h\cdot a)\,d\dot{h},$$

where $d\dot{h}$ is an $H$-invariant measure on $H/Z_H(A)$.

*Remarks.* Let $a\in A'$ and let $f\in C_c^\infty(\mathbf{X})$, then $\operatorname{supp} f\cap H[a]\subset\mathbf{X}$ is compact, and the above integral converges. We also easily see that $K_f\in C^\infty(A')$.

Consider the space $I(A)$ of functions $F\in C^\infty(A')$ satisfying the properties:

$I_1(A)$: $\sup_{a\in V\cap A'}|XF(a)|<\infty$ for all compact subsets $V\subset A$ and for all $X\in S(\mathfrak{a})$.

$I_2(A)$: The function $b_{\Delta_I^+}F$ extends to a $C^\infty$-function on $A$.

$I_4(A)$: There exists a compact subset $V\subset A$ such that $F(a)\equiv0$ for $a\in A'\setminus V$.

We note that the space $I(A)$ is isomorphic to the space $I(\mathbf{X})$, see [10, pp. 9ff.], of $H$-invariant differentiable functions on $\mathbf{X}'$ satisfying the (similar) properties $I_i(\mathbf{X})$, $i\in\{1,2,3,4\}$, since condition $I_3(\mathbf{X})$ is empty when there is only one conjugacy class of Cartan subspaces.

Let $U\subset\mathbf{X}$ and $V\subset A$ be compact subsets, and consider the Fréchet spaces

$$C_U^\infty(\mathbf{X})=\{f\in C_c^\infty(\mathbf{X})\,|\,\operatorname{supp} f\subset U\},$$
$$C_V^\infty(A)=\{F\in C_c^\infty(A)\,|\,\operatorname{supp} F\subset V\},$$
$$C_V^\infty(A')=\{F\in I(A)\,|\,F(a)\equiv 0 \text{ for } a\in A'\backslash V\}.$$

**Theorem 2.** *Let $U\subset\mathbf{X}$ be compact. There exists a compact subset $V\subset A$, only depending on $U$, such that $K_f(a)\equiv 0$ for $a\in A'\backslash V$ for all $f\in C_U^\infty(\mathbf{X})$; and the map $f\mapsto K_f$ is a continuous map from $C_U^\infty(\mathbf{X})$ into $C_V^\infty(A')$.*

*Proof.* By mimicking [4, §2.2] for the space $\mathbf{X}$, see also [4, §8.1], we see that $\overline{H[U]}\cap A$ is a bounded and closed subset of $A$, hence compact. The orbital integral $K_f$ is obviously identically zero outside $\overline{H[U]}\cap A'$, so we can choose the subset $V\subset A$ as $\overline{H[U]}\cap A$. There exists around every $a\in A'$ a completely $G$-invariant neighbourhood $\mathcal{V}$ in $G$, see [4, §8] for the construction and definition of completely $G$-invariant neighbourhoods. We conclude from Harish-Chandra's method of descent, [4, Lemme 8.2.1], and properties $I_1(\mathfrak{m})$, $I_2(\mathfrak{m})$ and $I_4(\mathfrak{m})$ of the orbital integral $J_\mathfrak{m}$ defined on the Lie algebra $\mathfrak{m}$, see [3, §3] and [4, §4], that $K_f$ satisfies the properties $I_1(A)$, $I_2(A)$ and $I_4(A)$ listed above, since they are all of local nature. Let $a\in A'$, then the map $f\mapsto K_f(a)$ is a continuous functional on $C_U^\infty(\mathbf{X})$ (a Radon measure on $C_c^\infty(\mathbf{X})$), and continuity of the map $f\mapsto K_f$ thus follows from the closed graph theorem. □

**Corollary 3.** *Let $U\subset\mathbf{X}$ be compact and let $V\subset A$ be a compact subset as in Theorem 2. The map $f\mapsto b_{\Delta_I^+}K_f$ is a continuous map from $C_U^\infty(\mathbf{X})$ into $C_V^\infty(A)$.*

*Proof.* The map $f\mapsto b_{\Delta_I^+}K_f$ is a continuous map from $C_U^\infty(\mathbf{X})$ into $C_V^\infty(A')$ since $b_{\Delta_I^+}\equiv\pm 1$ on the connected components of $A'$. The map extends to a continuous map from $C_U^\infty(\mathbf{X})$ into $C_V^\infty(A)$ by Theorem 2, since $A'$ is dense in $A$. □

Let $\mathbf{D}(\mathbf{X})$ denote the algebra of $G$-invariant differential operators on $\mathbf{X}$. This algebra is isomorphic to the center $Z(\mathfrak{h})$ of the universal enveloping algebra of the complexification of $\mathfrak{h}$, see [2, Théorème 2.1] for details (valid in the general case as well), and we identify the two algebras. Let $\Gamma$ denote the Harish-Chandra isomorphism of $\mathbf{D}(\mathbf{X})=Z(\mathfrak{h})$ onto $S(\mathfrak{a})^W$, the Weyl group invariant elements of $S(\mathfrak{a})$, see e.g. [13, p. 220].

Let $D\in\mathbf{D}(\mathbf{X})$, then we have

$$(K_{Df})(a)=\Gamma(D)K_f(a)$$

for all $a\in A'$, see [16, Lemma 12.1]. Since $b_{\Delta_I^+}\equiv\pm1$ on the connected components of $A'$, we also get

$$(b_{\Delta_I^+}K_{Df})(a)=\Gamma(D)b_{\Delta_I^+}K_f(a)$$

for all $a\in A'$, and hence by density and continuity for all $a\in A$.

Let $\Omega=\prod_{\alpha\in\Delta^+}H_\alpha\in S(\mathfrak{a})$ and let $\delta\in\mathcal{D}'(\mathbf{X})^H$ denote the Dirac measure at the origin of $\mathbf{X}$.

**Theorem 4.** *Let $f\in C_c^\infty(\mathbf{X})$. There exists a constant $c\neq0$ such that*

$$\langle\delta,f\rangle=f(e)=c\Omega b_{\Delta_I^+}K_f(e),$$

*where $e$ denotes the identity element of $G$ $(A)$.*

*Proof.* It follows from [10, Lemme 7.1(ii)], since $i\mathfrak{a}$ is a fundamental Cartan subalgebra of $\mathfrak{h}$. □

## Fundamental solutions and solvability

Let $D\in\mathbf{D}(\mathbf{X})$. An invariant fundamental solution for $D$ is a solution $T\in\mathcal{D}'(\mathbf{X})^H$ to the differential equation $DT=\delta$. Consider $\Gamma(D)\in S(\mathfrak{a})$ as a differential operator on $A$, then a fundamental solution for $\Gamma(D)$ is a solution $T_A\in\mathcal{D}'(A)$, the space of distributions on $A$, to the differential equation $\Gamma(D)T_A=\delta_A$, where $\delta_A$ denotes the Dirac measure on $A$ at the origin.

Both the symmetric space $\mathbf{X}$ and the Lie group $A$ carry invariant measures, which in a natural way induce bilinear pairings of $C_c^\infty(\mathbf{X})$ and $C_c^\infty(A)$ with themselves. We denote these linear pairings by $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_A$ respectively. Let $D\in\mathbf{D}(\mathbf{X})$ $(D_A\in\mathbf{D}(A))$ and let $D^*$ $(D_A^*)$ denote the adjoint of $D$ $(D_A)$ with respect to the pairing $\langle\cdot,\cdot\rangle$ $(\langle\cdot,\cdot\rangle_A)$.

Define the isomorphism $\gamma_\mathfrak{a}$ from $\mathbf{D}(\mathbf{X})$ onto $S(\mathfrak{a})^W$ as on [2, p. 59]. This isomorphism is identical to the isomorphism $\gamma$ from $\mathbf{D}(\mathbf{X})$ onto $S(\mathfrak{a})^W$ defined on [1, p. 304] (with $\mathfrak{a}=\mathfrak{a}_1$ and $W\simeq W_1$), see [8, pp. 15ff.] for further details. We have the following identities: $2^{\mathrm{order}\,D}\Gamma(D)=\gamma_\mathfrak{a}(D)=\gamma(D)$ for $D\in\mathbf{D}(\mathbf{X})$ homogeneous, see [2, p. 59]. It follows from [1, Lemma 3] that $\Gamma(D)^*=\Gamma(D^*)$.

**Theorem 5.** *Let $D \in \mathbf{D}(\mathbf{X})$. Then $D$ has an invariant fundamental solution on $\mathbf{X}$ if $\Gamma(D)$ has a fundamental solution on $A$.*

*Proof.* Let $T_A \in \mathcal{D}'(A)$ be a fundamental solution for $\Gamma(D)$. Define a distribution $T \in \mathcal{D}'(\mathbf{X})^H$ by

$$\langle T, f\rangle = \langle T_A, c\Omega b_{\Delta_I^+} K_f\rangle_A$$

for $f \in C_c^\infty(\mathbf{X})$. Continuity follows from Corollary 3, and $H$-invariance from $H$-invariance of $K_f$. We easily see that:

$$\begin{aligned}\langle DT, f\rangle &= \langle T, D^* f\rangle = \langle T_A, c\Omega b_{\Delta_I^+} K_{D^* f}\rangle_A \\ &= \langle T_A, c\Omega\Gamma(D^*) b_{\Delta_I^+} K_f\rangle_A = \langle T_A, \Gamma(D^*) c\Omega b_{\Delta_I^+} K_f\rangle_A \\ &= \langle \Gamma(D^*)^* T_A, c\Omega b_{\Delta_I^+} K_f\rangle_A = c\Omega b_{\Delta_I^+} K_f(e) = f(e) = \langle \delta, f\rangle,\end{aligned}$$

since $\Gamma(D^*)^* = \Gamma(D)$. □

We decompose $\mathfrak{a}$ according to the Cartan decomposition as $\mathfrak{a} = \mathfrak{a}_{\mathfrak{k}} \oplus \mathfrak{a}_{\mathfrak{p}} = \mathfrak{a} \cap \mathfrak{k} \oplus \mathfrak{a} \cap \mathfrak{p}$. Let $A_K = A \cap K = \exp \mathfrak{a}_{\mathfrak{k}}$ and $A_{\mathfrak{p}} = \exp \mathfrak{a}_{\mathfrak{p}}$ be the compact, respectively the euclidean, part of $A = A_K A_{\mathfrak{p}}$. We similarly decompose the complex dual of $\mathfrak{a}$ as $\mathfrak{a}^*_{\mathbf{C}} = \mathfrak{a}^*_{\mathfrak{k},\mathbf{C}} \times \mathfrak{a}^*_{\mathfrak{p},\mathbf{C}}$, the product of the complex duals of $\mathfrak{a}_{\mathfrak{k}}$ and $\mathfrak{a}_{\mathfrak{p}}$. The lattice of characters of the compact abelian group $A_K$ is canonically identified with the lattice $\Lambda$ of analytically integral elements $\lambda \in \mathfrak{a}^*_{\mathfrak{k},\mathbf{C}}$. Consider now the elements of $S(\mathfrak{a})$ in the natural setup as polynomials on $\mathfrak{a}^*_{\mathbf{C}}$. Let $X \in S(\mathfrak{a})$ and let $\lambda \in \mathfrak{a}^*_{\mathfrak{k},\mathbf{C}}$, then we define the polynomial $X_\lambda$ on $\mathfrak{a}^*_{\mathfrak{p},\mathbf{C}}$ as $X_\lambda(\nu) = X(\lambda, \nu)$ for $\nu \in \mathfrak{a}^*_{\mathfrak{p},\mathbf{C}}$. Let $\{X_1, \ldots, X_m\}$ be a basis for $\mathfrak{a}_{\mathfrak{p}}$, and define a norm on $S(\mathfrak{a}_{\mathfrak{p}})$, the symmetric algebra of the complexification of $\mathfrak{a}_{\mathfrak{p}}$, as $\|X\|^2 = \sum_\alpha (\alpha!)^2 |a_\alpha|^2$ for $X = \sum_\alpha a_\alpha X_1^{\alpha_1} \ldots X_m^{\alpha_m}$ written in the multi-index notation. Let $|\cdot|_*$ denote any norm on $\mathfrak{a}^*_{\mathbf{C}}$.

**Proposition 6.** *Let $X \in S(\mathfrak{a})$. The differential operator $X$ on $A$ has a fundamental solution on $A$ if and only if there exists a constant $C > 0$ and an integer $N \in \mathbf{N} \cup \{0\}$ such that*

$$(**) \qquad \|X_\lambda\| \geq C(1 + |\lambda|_*)^{-N}$$

*for all $\lambda \in \Lambda$.*

*Proof.* See e.g. [5, §7] or [15, Proposition 3.2]. □

*Remark.* The inequality $\|X \cdot Y\| \geq C_N \|X\| \, \|Y\|$ holds for all $X, Y \in S(\mathfrak{a}_{\mathfrak{p}})$ of degree $\leq N$, with $N \in \mathbf{N}$, where $C_N > 0$ is a constant only depending on $N$. It follows, that if $X$ and $Y$ satisfy $(**)$ for some $\lambda \in \Lambda$, then so does the product $X \cdot Y$. Let $D_1, D_2 \in \mathbf{D}(\mathbf{X})$ and assume that $\Gamma(D_1)$ and $\Gamma(D_2)$ both have fundamental solutions on $A$, then it follows, that the product $D_1 \cdot D_2$ has an invariant fundamental solution on $\mathbf{X}$. In particular, if $D \in \mathbf{D}(\mathbf{X})$ and $\Gamma(D)$ has a fundamental solution on $A$, then all powers $D^m$, $m \in \mathbf{N}$, have invariant fundamental solutions on $\mathbf{X}$.

**Theorem 7.** *Let* $0\neq D\in\mathbf{D}(\mathbf{X})$. *Then* $D$ *is solvable, i.e.* $DC^\infty(\mathbf{X})=C^\infty(\mathbf{X})$, *if* $\deg\Gamma(D)=\deg\Gamma(D)_\lambda$ *for some* $\lambda\in\mathfrak{a}^*_{\mathfrak{k},\mathbf{C}}$.

*Proof.* We first notice that if $\deg\Gamma(D)=\deg\Gamma(D)_\lambda$ for some $\lambda\in\mathfrak{a}^*_{\mathfrak{k},\mathbf{C}}$, then $\deg\Gamma(D)=\deg\Gamma(D)_\lambda$ for all $\lambda\in\mathfrak{a}^*_{\mathfrak{k},\mathbf{C}}$. Let $\{Y_1,\dots,Y_l\}$ be a basis for $\mathfrak{a}_\mathfrak{k}$, and write $\Gamma(D)$ as $\sum_{\alpha,\beta}a_{\alpha,\beta}Y_1^{\beta_1}\dots Y_l^{\beta_l}X_1^{\alpha_1}\dots X_m^{\alpha_m}$. There exists a coefficient $a_{\alpha,0}\neq 0$ with $|\alpha|=\deg\Gamma(D)$, and we have the estimate $\|\Gamma(D)_\lambda\|\geq\|a_{\alpha,0}X_1^{\alpha_1}\dots X_m^{\alpha_m}\|=\alpha!|a_{\alpha,0}|>0$ for all $\lambda\in\mathfrak{a}^*_{\mathfrak{k},\mathbf{C}}$. We conclude from Theorem 5 and Proposition 6 that $D$ has an invariant fundamental solution on $\mathbf{X}$.

The algebra $\mathfrak{a}_\mathfrak{p}=\mathfrak{a}\cap\mathfrak{p}$ is a maximal abelian subalgebra of $\mathfrak{p}\cap\mathfrak{q}$, the so-called split part of $\mathfrak{a}$. Let $\Delta_\mathfrak{p}$ and $\Delta_\mathbf{C}$ denote the root systems of $\mathfrak{a}_\mathfrak{p}$ and $\mathfrak{a}$ in $\mathfrak{g}^\mathbf{C}$ (the complexification of $\mathfrak{g}$) respectively, and denote by $W_\mathfrak{p}$ and $W_\mathbf{C}$ the corresponding Weyl groups. We note that the pair $(\Delta_\mathbf{C},W_\mathbf{C})$ is isomorphic to the pair $(\Delta,W)$, the roots having multiplicity 2 respectively 1 in the two root systems. Define as on [1, pp. 305–306] homomorphisms $\gamma$ and $\eta$, of $\mathbf{D}(\mathbf{X})$ onto $S(\mathfrak{a})^{W_\mathbf{C}}$, respectively into $S(\mathfrak{a}_\mathfrak{p})^{W_\mathfrak{p}}$. We note again that $\Gamma(D)(2\lambda)=\gamma(D)(\lambda)$, $\lambda\in\mathfrak{a}^*_\mathbf{C}$. The correspondence between $\eta(D)$ and $\gamma(D)$ can be expressed as $\eta(D)(\nu)=\gamma(D)(\nu-\varrho_m)=\gamma(D)_{-\varrho_m}(\nu)$ for $\nu\in\mathfrak{a}^*_{\mathfrak{p},\mathbf{C}}\subset\mathfrak{a}^*_\mathbf{C}$, where $\varrho_m$ is a fixed element of $\mathfrak{a}^*_{\mathfrak{k},\mathbf{C}}\subset\mathfrak{a}^*_\mathbf{C}$, see [1, Lemma 1]. We conclude that $\deg\eta(D)=\deg\gamma(D)$, and hence, by [1, Theorem 2], that $\mathbf{X}$ is $D$-convex. It now follows, by [1, p. 301], that $D$ is solvable. □

## Examples and further results

(1) Let $\Delta\in\mathbf{D}(\mathbf{X})$ denote the Casimir operator on $\mathbf{X}$, then it is easily seen that $\Gamma(\Delta)(\lambda,\nu)=\lambda\cdot\lambda+\nu\cdot\nu-\varrho\cdot\varrho$ for $\lambda\in\mathfrak{a}^*_{\mathfrak{k},\mathbf{C}}$, $\nu\in\mathfrak{a}^*_{\mathfrak{p},\mathbf{C}}$, where $\varrho$ is half the sum of the positive roots of $\Delta_\mathbf{C}$. Assume that $\mathfrak{a}_\mathfrak{p}\neq\{0\}$, then we see that $\deg\Gamma(\Delta)=\deg\Gamma(\Delta)_\lambda$ for all $\lambda\in\mathfrak{a}^*_{\mathfrak{k},\mathbf{C}}$, and we conclude from the above that $\Delta$ has a fundamental solution and that it is solvable. Solvability of the Casimir operator was proved, for general semisimple symmetric spaces, by Chang in [7]. Let $D\in\mathbf{D}(\mathbf{X})$ be a differential operator of the form $\Delta^m+D_1$, with $m\in\mathbf{N}$, where $\deg D_1<\deg D=2m$. Again assuming that $\mathfrak{a}_\mathfrak{p}\neq\{0\}$, we see that $\Gamma(D)$ satisfies the conditions in Theorem 5 and Theorem 7, i.e. $D$ has a fundamental solution and it is solvable.

(2) Let $K$ be a compact Lie group and let $K_\mathbf{C}$ denote the complexification of $K$, then $K_\mathbf{C}/K$ is a Riemannian symmetric space (of type $G_\mathbf{C}/G_\mathbf{R}$) with only one conjugacy class of Cartan subspaces. Since $\mathfrak{a}=\mathfrak{a}_\mathfrak{p}$, we easily see from the above, that every non-zero invariant differential operator $D\in\mathbf{D}(K_\mathbf{C}/K)$ has a $K$-invariant fundamental solution and that it is solvable. Helgason obtained these results for general Riemannian symmetric spaces in [11] and [12], as mentioned in the introduction.

(3) Let $H$ be a complex connected semisimple Lie group. By choosing a suitable complexification $H_{\mathbf{C}}$ of $H$, we can view $H \simeq H_{\mathbf{C}}/H$ as a symmetric space of type $G_{\mathbf{C}}/G_{\mathbf{R}}$ with only one conjugacy class of Cartan subspaces. Then Theorem 5 is a well-known result by Cérézo and Rouvière, see [6, Proposition 1]. In this case however, it follows that $D$ is solvable if $\Gamma(D)$ has a fundamental solution on $A$, see [6, Proposition 2]. These results are also valid on general connected semisimple Lie groups with only one conjugacy class of Cartan subalgebras, see [15, Theorem 4.2].

(4) There are up to coverings two families and one exceptional example of non-complex, non-compact connected semisimple Lie groups with one conjugacy class of Cartan subalgebras, namely $SO_o(2n+1,1)$, $n \geq 0$, ($\dim \mathfrak{a} = n+1$, $\dim \mathfrak{a}_{\mathfrak{p}} = 1$); $SU^*(2n)$, $n \geq 3$, ($\dim \mathfrak{a} = 2n-1$, $\dim \mathfrak{a}_{\mathfrak{p}} = n-1$) and $\mathfrak{e}_{6(-26)}$ ($\dim \mathfrak{a} = 6$, $\dim \mathfrak{a}_{\mathfrak{p}} = 2$).

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