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Holomorphic vector fields and proper holomorphic self-maps of Reinhardt domains

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1. Introduction

In 1977, H. Alexander discovered a typical fact of several complex variables.

Theorem. ([1]) *The proper holomorphic self-mappings of the euclidean unit ball of* $\mathbf{C}^{n+1}$ $(n\geq 1)$ *are automorphisms.*

This result has been generalized to certain pseudoconvex domains with regular boundaries and well behaved sets of weakly pseudoconvex points. For instance, S. Pinchuk [20] extended it to $\mathcal{C}^2$-strictly pseudoconvex domains and E. Bedford and S. Bell [4] to pseudoconvex domains with real-analytic boundaries. K. Diederich and J. E. Fornaess proved that proper holomorphic maps, from domains with sufficiently small sets of weakly pseudoconvex points (in the Hausdorff measure sense) to smoothly bounded ones, do not branch [12].

It then became a question to know if the phenomenon discovered by H. Alexander occurs for any smoothly bounded domain in $\mathbf{C}^{n+1}$. This question is still largely open, even for pseudoconvex domains of finite type. However, the problem is easier to handle for domains presenting some symmetries. For instance, Y. Pan [18] gave a positive answer for pseudoconvex Reinhardt domains of finite type (see also [10]). The presence of symmetries may even allow to relax some boundary conditions and to consider mappings between different domains. S. Bell proved that mappings between circular domains are algebraic as soon as they preserve the origin [6]. Mappings between particular classes of Reinhardt domains were studied by G. Dini and A. Selvaggi [13], M. Landucci and G. Patrizio [15] and M. Landucci and S. Pinchuk [16]. In [9], F. Berteloot and S. Pinchuk classified the proper maps between bounded complete Reinhardt domains in $\mathbf{C}^2$ and characterized the bidisc as being the only domain in this class which admits non-injective proper holomorphic self-maps (see also [17]). In this paper, we solve the above problem for complete Reinhardt domains in $\mathbf{C}^{n+1}$. Our approach also works for Reinhardt domains in

$(\mathbf{C}^*)^{n+1}$ and gives a new proof of a result of E. Bedford (see [3, p. 160]) for the specific case of $\mathcal{C}^2$-smooth boundaries. Our main result is the following theorem.

Theorem 1.1. *Let Ω be a bounded complete Reinhardt domain in $\mathbf{C}^{n+1}$, or a Reinhardt domain in $(\mathbf{C}^*)^{n+1}$, with boundary of class $\mathcal{C}^2$. Then every proper holomorphic self-map of Ω is an automorphism.*

To this purpose, we study the Lie algebra of holomorphic tangent vector fields of strictly pseudoconvex Reinhardt hypersurfaces. We essentially need to show that it is finite dimensional and has a simple structure. As we consider $\mathcal{C}^2$-smooth hypersurfaces, we cannot use the works of S. Chern and J. Moser [11]. It could appear more promising to adapt the results obtained by N. Stanton for $\mathcal{C}^\infty$-smooth rigid hypersurfaces [22]. However, as we do not deal with a purely local situation, we shall get a rather precise description of this Lie algebra by using elementary tools.

Theorem 1.2. *The Lie algebra of holomorphic tangent vector fields of a $\mathcal{C}^2$-strictly pseudoconvex Reinhardt hypersurface in $\mathbf{C}^{n+1}$ is finite dimensional and consists of rational vector fields of $(\mathbf{C}^*)^{n+1}$.*

We then use the results on holomorphic extension due to S. Bell [5] and D. Barrett [2] to investigate the effect of proper mappings on the rotation vector fields. We may then show that the structure of the branch locus is naturally related to the structure of the Lie algebra of holomorphic tangent vector fields associated to some strictly pseudoconvex part of the boundary. By using the above theorem, we obtain precise information on the branch locus which eventually implies that the self-maps are unbranched and thus yield Theorem 1.1. By combining this method with some techniques introduced in [9], we obtain the following local statement.

Theorem 1.3. *Let Ω be a bounded complete Reinhardt domain in $\mathbf{C}^{n+1}$ such that $b\Omega$ is somewhere $\mathcal{C}^\infty$-strictly pseudoconvex. Then every proper holomorphic self-map of Ω is an automorphism.*

Notation.

- We denote by $(z_0, z_1, \dots, z_n) = (z_0, z')$ the coordinates in $\mathbf{C}^{n+1}$.
- For $z = (z_0, z_1, \dots, z_n)$ and $w = (w_0, w_1, \dots, w_n)$ we shall define zw by $zw = (z_0 w_0, z_1 w_1, \dots, z_n w_n)$.
- For any multi-index $(k_1, \dots, k_n, l) := (K, l) \in \mathbf{Z}^n \times \mathbf{Z}$; $(z')^K z_0^l := z_0^l z_1^{k_1} \dots z_n^{k_n}$.
- For any integer j, j^* denotes the multi-index $(\delta_{1,j}, \dots, \delta_{n,j})$ where $\delta_{i,j}$ is the Kronecker symbol.
- The set Δ denotes the unit disc in $\mathbf{C}$ and Δ^k the corresponding polydisc in $\mathbf{C}^k$.

- We set $\mathbf{C}^*=\mathbf{C}\setminus\{0\}$ and, for any subset A of $\mathbf{C}^{n+1}$, A^* denotes the intersection $A\cap(\mathbf{C}^*)^{n+1}$.
- For any $z\in\mathbf{C}^{n+1}$, we denote by e^{iz} the point $(e^{iz_0},\dots,e^{iz_n})$. For any $\eta\in\mathbf{C}^{n+1}$, T_η denotes the torus $\{e^{i\theta}\eta;\theta\in\mathbf{R}^{n+1}\}$.
- For any $0\le j\le n$, $\mathcal{H}_j$ denotes the hyperplane $\{z_j=0\}$ and $\mathcal{H}:=\bigcup_{j=0}^n\mathcal{H}_j$, for a domain Ω we set $\mathcal{H}_{\Omega j}:=\Omega\cap\mathcal{H}_j$ and $\mathcal{H}_\Omega:=\Omega\cap\mathcal{H}$.
- The field $\mathcal{R}_j$ denotes the rotation vector field $iz_j(\partial/\partial z_j)$.
- We denote by V_f the branch locus of any holomorphic map f.
- For any self map f, f^k denotes the k^{th} iterate of f.

2. The Lie algebra of holomorphic tangent vector fields

The aim of this section is to establish a precise version of Theorem 1.2.

A vector field

$$X:=B(z_0,z')\frac{\partial}{\partial z_0}+\sum_{j=1}^n A_j(z_0,z')\frac{\partial}{\partial z_j}$$

is said to be holomorphic on a domain $\mathcal{U}\subset\mathbf{C}^{n+1}$ if the functions B and A_j are holomorphic on $\mathcal{U}$. We say that X is tangent to some real hypersurface $\mathcal{S}\subset\mathcal{U}$ if the vector field $X+\overline{X}$ is tangent to $\mathcal{S}$. Thus, if $\mathcal{S}$ is defined by $\{\sigma=0\}$, the tangency condition may be written as

$$0\equiv\operatorname{Re}\langle X,\sigma\rangle:\equiv\operatorname{Re}\left(B\frac{\partial\sigma}{\partial z_0}+\sum_{j=1}^n A_j\frac{\partial\sigma}{\partial z_j}\right)\quad\text{on }\mathcal{S}.$$

We shall need the following definition.

Definition. For every multi-index $(K,l)\in\mathbf{Z}^n\times\mathbf{Z}$, $\mathcal{E}_{K,l}$ is the space of holomorphic vector fields on $(\mathbf{C}^*)^{n+1}$ which are of the following form:

$$(b(z')^K z_0^{l+1}+\tilde{b}(z')^{-K}z_0^{-l+1})\frac{\partial}{\partial z_0}+\sum_{j=1}^n(a_j(z')^{K+j^*}z_0^l+\tilde{a}_j(z')^{-K+j^*}z_0^{-l})\frac{\partial}{\partial z_j},$$

where b, $\tilde{b}$, a_j, $\tilde{a}_j$ are complex constants.

We are now in order to state the main result of this section.

Proposition 2.1. *Let $\mathcal{S}$ be a Reinhardt hypersurface in $\mathbf{C}^{n+1}$ which is defined by $\mathcal{S}=\{|z_0|^2=\phi(|z_1|^2,\dots,|z_n|^2)\}$, where the function $\phi(t)$ is of class $\mathcal{C}^2$ on some neighbourhood of $t_0:=(t_{0_1},\dots,t_{0_n})\in(\mathbf{R}^{+*})^n$.*

Assume that $\mathcal{S}$ is strictly pseudoconvex at $\eta_0:=(\phi(t_0)^{1/2},t_{0_1}^{1/2},\dots,t_{0_n}^{1/2})$.

Then there exists a finite family of multi-indices $\mathcal{M}\subset\mathbf{Z}^n\times\mathbf{Z}$ such that every holomorphic vector field X, defined on some neighbourhood of the torus T_{η_0} and tangent to $\mathcal{S}$, admits a finite decomposition

$$X=\sum_{(K,l)\in\mathcal{M}} X_{K,l},$$

where the $X_{K,l}$ are holomorphic vector fields on $(\mathbf{C}^*)^{n+1}$ which belong to $\mathcal{E}_{K,l}$ and are tangent to $\mathcal{S}$ near T_{η_0}.

Proof. Let X be a holomorphic vector field which is defined on some neighbourhood of T_{η_0},

(1)
$$X:=B(z_0,z')\frac{\partial}{\partial z_0}+\sum_{j=1}^{n}A_j(z_0,z')\frac{\partial}{\partial z_j}.$$

By expanding the holomorphic functions B and A_j $(j=1,\dots,n)$ in Laurent series on a suitable neighbourhood of T_{η_0}, one may rewrite X as

(2)
$$X:=\Big(\sum_{K,l} b_{K,l}(z')^K z_0^l\Big)\frac{\partial}{\partial z_0}+\sum_{j=1}^{n}\Big(\sum_{K,l} a_{j,K,l}(z')^K z_0^l\Big)\frac{\partial}{\partial z_j},$$

where $(K,l)\in\mathbf{Z}^n\times\mathbf{Z}$ and $b_{K,l}$, $a_{j,K,l}$ are complex numbers. After setting $\sigma(z_0,z'):=|z_0|^2-\phi(|z_1|^2,\dots,|z_n|^2)$, the identity (2) gives

(3)
$$\langle X,\sigma\rangle=\sum_{K,l}\Big[b_{K,l}(z')^K z_0^l\bar z_0-\sum_{j=1}^{n}\Big(a_{j,K,l}\bar z_j(z')^K z_0^l\frac{\partial\phi}{\partial t_j}(|z_1|^2,\dots,|z_n|^2)\Big)\Big].$$

Let V_0 be a sufficiently small neighbourhood of t_0. One may parametrize $\mathcal{S}$ by $z_0=\varrho e^{iv}$, $z_j=r_je^{iu_j}$ $(j=1,\dots,n)$, where $(u,v)\in[0,2\pi]^n\times[0,2\pi]$, $r=(r_1,\dots,r_n)\in V_0$ and $\varrho^2=\phi({r_1}^2,\dots,{r_n}^2)$. Using this parametrization and the identity (3), the tangency condition becomes

(4)
$$\operatorname{Re}\Big[\sum_{K,l}e^{i(K\cdot u+lv)}\Big(b_{K,l+1}r^K\varrho^{l+2}-\sum_{j=1}^{n}a_{j,K+j^*,l}r^{K+2j^*}\varrho^l\frac{\partial\phi}{\partial t_j}(r_1^2,\dots,r_n^2)\Big)\Big]$$
$$\equiv\operatorname{Re}\langle X,\sigma\rangle|_{\mathcal{S}}\equiv 0.$$

We now introduce the notation

(5)
$$E_{K,l}(X,r)=b_{K,l+1}r^K\varrho^{l+2}-\sum_{j=1}^{n}\Big(a_{j,K+j^*,l}r^{K+2j^*}\varrho^l\frac{\partial\phi}{\partial t_j}(r_1^2,\dots,r_n^2)\Big)$$
$$+\bar b_{-K,-l+1}r^{-K}\varrho^{-l+2}-\sum_{j=1}^{n}\Big(\bar a_{j,-K+j^*,-l}r^{-K+2j^*}\varrho^{-l}\frac{\partial\phi}{\partial t_j}(r_1^2,\dots,r_n^2)\Big).$$

Then (4) is equivalent to

$$0 \equiv 2\operatorname{Re}\langle X, \sigma\rangle|_{\mathcal{S}} \equiv \sum_{K,l} e^{i(K\cdot u+lv)} E_{K,l}(X,r). \tag{6}$$

For any $(K,l)\in\mathbf{Z}^n\times\mathbf{Z}$, let $X_{K,l}$ be the element of $\mathcal{E}_{K,l}$ which is defined by

$$\begin{aligned} X_{K,l} &= (b_{K,l+1}(z')^K z_0^{l+1} + b_{-K,-l+1}(z')^{-K} z_0^{-l+1})\frac{\partial}{\partial z_0} \\ &\quad + \sum_{j=1}^n (a_{j,K+j^*,l}(z')^{K+j^*} z_0^l + a_{j,-K+j^*,-l}(z')^{-K+j^*} z_0^{-l})\frac{\partial}{\partial z_j}. \end{aligned} \tag{7}$$

Then, using the same parametrization of $\mathcal{S}$ as above we get the identity

$$2\operatorname{Re}\langle X_{K,l}, \sigma\rangle|_{\mathcal{S}} \equiv e^{i(K\cdot u+lv)} E_{K,l}(X,r) + e^{-i(K\cdot u+lv)} E_{-K,-l}(X,r). \tag{8}$$

Thus, it follows from (6), (7) and (8) that X is tangent to $\mathcal{S}$ if and only if each $X_{K,l}$ itself is tangent to $\mathcal{S}$. To end the proof it remains to show that there exist at most a finite number of multi-indices $(K,l)\in\mathbf{Z}^n\times\mathbf{Z}$ for which $\mathcal{E}_{K,l}$ contains non-trivial vector fields which are tangent to $\mathcal{S}$. To this purpose, we shall derive a simple condition of tangency for the fields in $\mathcal{E}_{K,l}$.

Assume that X is a non-trivial element of $\mathcal{E}_{K,l}$ which is tangent to $\mathcal{S}$. Then, we may assume that $X=X_{K,l}$, where $X_{K,l}$ is given by (7). According to (8), the function $E_{K,l}(X,r)$ must identically vanish on V_0. Thus, after multiplying both sides of (5) by $r^K\varrho^{l-2}$, we obtain the following necessary condition for X to be tangent to $\mathcal{S}$,

$$\begin{aligned} 0 &\equiv r^{2K}\varrho^{2l}\left(b_{K,l+1} - \sum_{j=1}^n a_{j,K+j^*,l} r_j^2 \varrho^{-2}\frac{\partial\phi}{\partial t_j}(r_1^2,\dots,r_n^2)\right) \\ &\quad + \left(\bar{b}_{-K,-l+1} - \sum_{j=1}^n \bar{a}_{j,-K+j^*,-l} r_j^2\varrho^{-2}\frac{\partial\phi}{\partial t_j}(r_1^2,\dots,r_n^2)\right). \end{aligned} \tag{9}$$

After replacing r_j^2 by t_j, ϱ^2 by $\phi(t)$ and simplifying the notation, we may rewrite (9) as follows

$$0 \equiv t^K\phi^l(t)\left(b - \sum_{j=1}^n a_j \frac{t_j}{\phi(t)}\frac{\partial\phi}{\partial t_j}(t)\right) + \left(\tilde{b} - \sum_{j=1}^n \tilde{a}_j\frac{t_j}{\phi(t)}\frac{\partial\phi}{\partial t_j}(t)\right). \tag{10}$$

We shall now use logarithmic coordinates. Let us set $x_j := \log t_j$ for $j=1,\dots,n$ and $\psi(x) := \log\phi(e^x)$. Then (10) becomes

$$e^{K\cdot x}e^{l\psi}(b + L(\operatorname{grad}\psi)) + (\tilde{b} + \tilde{L}(\operatorname{grad}\psi)) \equiv 0, \tag{11}$$

where L and $\tilde{L}$ are elements of $(\mathbf{C}^n)'$ which are respectively defined by $L(u) := -\sum_{j=1}^n a_j u_j$ and $\tilde{L}(u) := -\sum_{j=1}^n \tilde{a}_j u_j$.

As is well known, the strict pseudoconvexity of the Reinhardt hypersurface $\mathcal{S}$ is equivalent to the strict convexity of the function ψ. Thus the conclusion will be directly obtained by using (11) and the following lemma. □

Lemma 2.2. *Let ψ be a function of class $\mathcal{C}^2$ which is defined on a neighbourhood V of $x_0 \in (\mathbf{R}^{+*})^n$. Assume that ψ is strictly convex at x_0. Then there exists at most a finite number of multi-indices $(K,l) \in \mathbf{Z}^n \times \mathbf{Z}$ such that ψ satisfies a partial differential equation of the following type on V:*

$$e^{K\cdot x} e^{l\psi}(L(\operatorname{grad}\psi)+b) = \tilde{L}(\operatorname{grad}\psi)+\tilde{b}, \tag{E}$$

where $b, \tilde{b} \in \mathbf{C}$ and $L, \tilde{L} \in (\mathbf{C}^n)'$ do not vanish simultaneously.

Proof of the lemma. Let σ be the map defined by $\sigma(x) := \operatorname{grad}(\psi)$ on V. Then, since ψ is strictly convex at x_0, the hessian $\det[(\partial^2\psi/\partial x_i \partial x_j)(x_0)]$ is strictly positive and the map σ is open at x_0. Let U be an open ball which is contained in the range of σ. If $\tilde{L}+\tilde{b} \equiv 0$ (resp. $L+b\equiv 0$) then the equation (E) shows that $L+b$ (resp. $\tilde{L}+\tilde{b}$) is vanishing on U and therefore on $\mathbf{C}^n$. Thus $L+b$ and $\tilde{L}+\tilde{b}$ do not both vanish identically since otherwise we would have $L\equiv\tilde{L}\equiv 0$ and $b=\tilde{b}=0$.

We now proceed by contradiction and suppose that there exists infinite families $(K_j, l_j)_{j>0} \subset \mathbf{Z}^n \times \mathbf{Z}$, $(L_j, \tilde{L}_j)_{j>0} \subset (\mathbf{C}^n)' \times (\mathbf{C}^n)'$ and $(b_j, \tilde{b}_j)_{j>0} \subset \mathbf{C}\times\mathbf{C}$ such that the following equations hold on V:

$$e^{K_j\cdot x} e^{l_j\psi} = \frac{\tilde{L}_j(\operatorname{grad}\psi)+\tilde{b}_j}{L_j(\operatorname{grad}\psi)+b_j}. \tag{E$_j$}$$

Without loss of generality, we may assume that $(K_j, l_j) \neq (0,0)$ and $(K_j, l_j) + (K_{j'}, l_{j'}) \neq (0,0)$ for every $j>0$. Let us set $\mathcal{A}_j := \{u \in \mathbf{C}^n ; L_j(u)+b_j = 0\}$ and $\tilde{\mathcal{A}}_j := \{u \in \mathbf{C}^n ; \tilde{L}_j(u)+\tilde{b}_j = 0\}$. In general, $\mathcal{A}_j$ is an affine hyperplane in $\mathbf{C}^n$ but, when $L_j \equiv 0$, $\mathcal{A}_j$ is the empty set.

Observe that if $\mathcal{A}_j = \mathcal{A}_{j'}$, then $L_j + b_j \equiv \alpha(L_{j'}+b_{j'})$ for some $\alpha \in \mathbf{C}$. In particular, if $\mathcal{A}_j = \tilde{\mathcal{A}}_j$ then the equation (E$_j$) shows that the function $e^{K_j\cdot x}e^{l_j\psi}$ is constant on V, which is impossible since $(K_j, l_j) \neq (0,0)$ and σ is open at x_0. We also have $\mathcal{A}_j \cup \tilde{\mathcal{A}}_j \neq \mathcal{A}_{j'} \cup \tilde{\mathcal{A}}_{j'}$ for $j \neq j'$ since otherwise the equations (E$_j$) and (E$_{j'}$) would imply that the function $e^{(K_j+\varepsilon K_{j'})\cdot x} e^{(l_j+\varepsilon l_{j'})\psi}$ is constant on V for $\varepsilon \in \{1,-1\}$, which is impossible for the same reasons as above. Thus, after replacing $(K_j, l_j)_{j>0}$ by some subsequence, we may assume that one of the following conditions is satisfied:

(i) $\mathcal{A}_j$, $\tilde{\mathcal{A}}_j$, $\mathcal{A}_{j'}$ and $\tilde{\mathcal{A}}_{j'}$ are four distinct sets for $j \neq j'$;
(ii) $\mathcal{A}_j = \mathcal{A}_1$ for every $j>0$ and $\tilde{\mathcal{A}}_j \neq \tilde{\mathcal{A}}_{j'}$ for $j \neq j'$;
(iii) $\tilde{\mathcal{A}}_j = \tilde{\mathcal{A}}_1$ for every $j>0$ and $\mathcal{A}_j \neq \mathcal{A}_{j'}$ for $j \neq j'$.

Now, by elementary linear algebra, we may find an integer N and a non-zero vector $\alpha:=(\alpha_1,\dots,\alpha_N)\in\mathbf{Q}^N$ such that $(K_{N+1},l_{N+1})=\sum_{j=1}^N\alpha_j(K_j,l_j)$. Let A be an integer such that $A\alpha_j=:n_j\in\mathbf{Z}$ for every $j\in\{1,\dots,N\}$, then

$$e^{A(K_{N+1}\cdot x+l_{N+1}\psi)}=\prod_{j=1}^N\left(e^{n_j(K_j\cdot x+l_j\psi)}\right)$$

and, taking into account the equations (E_j) for $j\in\{1,\dots,N+1\}$,

$$\left(\frac{\tilde L_{N+1}(\operatorname{grad}\psi)+\tilde b_{N+1}}{L_{N+1}(\operatorname{grad}\psi)+b_{N+1}}\right)^A=\prod_{j=1}^N\left(\frac{\tilde L_j(\operatorname{grad}\psi)+\tilde b_j}{L_j(\operatorname{grad}\psi)+b_j}\right)^{n_j}. \tag{12}$$

Let us define G_j $(0\le j\le N+1)$ on $\mathbf{C}^n$ by $G_j(u):=(\tilde L_j(u)+\tilde b_j)/(L_j(u)+b_j)$. The functions G_j and $1/G_j$ are meromorphic on $\mathbf{C}^n$ and holomorphic on the connected open set $\Omega:=\mathbf{C}^n\setminus\bigcup_{j=1}^{N+1}(\tilde{\mathcal{A}}_j\cup\mathcal{A}_j)$. Let us again consider a non-empty open set U of $\mathbf{R}^n$ which is contained in the range of $\sigma:=\operatorname{grad}\psi$ at x_0. The set U cannot be contained in $\mathcal{H}_j$ (or $\tilde{\mathcal{A}}_j$) for $j\in\{1,\dots,N+1\}$ since otherwise the holomorphic function L_j+b_j, or $\tilde L_j+\tilde b_j$, would identically vanish on $\mathbf{C}^n$. Thus we may assume that $U\subset\Omega$. Let V be a non-empty open set in $\mathbf{C}^n$ such that $V\cap\mathbf{R}^n\subset U$. According to (12), the functions G_{N+1}^A and $\prod_{j=1}^N G_j^{n_j}$ coincide on U and therefore on V. Thus, these functions do actually coincide on $\mathbf{C}^n$,

$$\left(\frac{\tilde L_{N+1}(u)+\tilde b_{N+1}}{L_{N+1}(u)+b_{N+1}}\right)^A\equiv\prod_{j=1}^N\left(\frac{\tilde L_j(u)+\tilde b_j}{L_j(u)+b_j}\right)^{n_j}.$$

But this is impossible in view of the conditions (i) to (iii). □

3. The structure of the branch locus

The aim of this section is to apply the results on holomorphic tangent vector fields to the study of rigidity properties of proper holomorphic mappings. The following proposition describes how these objects are related.

Proposition 3.1. *Let Ω_1 and Ω_2 be two Reinhardt domains in $\mathbf{C}^{n+1}$. Assume that $b\Omega_1$ is $\mathcal{C}^2$-strictly pseudoconvex at some point η_0 which belongs to $b\Omega_1^*$. Then there exists a finite family of multi-indices $\mathcal{I}\subset\mathbf{Z}^n\times\mathbf{Z}$ and an associated space of rational functions*

$$\mathcal{G}:=\bigoplus_{(K,l)\in\mathcal{I}}\mathbf{C}(z')^K z_0^l$$

such that, for any proper holomorphic mapping $f:\Omega_1\to\Omega_2$*, which holomorphically extends to some neighbourhood of* T_{η_0} *and does not branch on* T_{η_0}*, there exists an* $(n+1,n+1)$ *matrix* $Q_f:=[(Q_f)_{k,p}]$ *with entries in* $\mathcal{G}$ *which satisfies the identity*

$$\left[\frac{\partial f_k}{\partial z_p}\right][(Q_f)_{k,p}]=i[\delta_{k,p}f_k]. \tag{13}$$

Proof. By assumption the map f holomorphically extends to some neighbourhood of T_{η_0}, we shall still denote this extension by f. As f induces a local biholomorphism at any point of the torus T_{η_0}, we may define $n+1$ holomorphic vector fields $(Q_f)_p$ on some neighbourhood of T_{η_0} by pulling back the rotation vector fields $\mathcal{R}_p$: $(Q_f)_p:=f^*(\mathcal{R}_p)$ for $p\in\{0,\dots,n\}$. By construction the vector fields $(Q_f)_p$ are tangent to $b\Omega_1$ on some neighbourhood of T_{η_0}. Thus, as Proposition 2.1 shows, there exists a finite family of multi-indices $\mathcal{M}\subset\mathbf{Z}^n\times\mathbf{Z}$ which only depends on $b\Omega_1$ and such that $(Q_f)_p\in\bigoplus_{(K,l)\in\mathcal{M}}\mathcal{E}_{K,l}$ for $p\in\{0,\dots,n\}$. Then, according to the definition of the spaces $\mathcal{E}_{K,l}$, it suffices to set

$$\mathcal{I}=\bigcup_{1\le j\le n}\ \bigcup_{(K,l)\in\mathcal{M}}\{(K,l+1),(-K,-l+1),(K+j^*,l),(-K+j^*,-l)\}$$

and

$$\mathcal{G}:=\bigoplus_{(K,l)\in\mathcal{I}}\mathbf{C}(z')^K z_0^l$$

to get

$$(Q_f)_p=\sum_{k=0}^{n}(Q_f)_{k,p}(z)\frac{\partial}{\partial z_k},\quad \text{where } (Q_f)_{k,p}\in\mathcal{G}.$$

Consider now the holomorphic $(n+1,n+1)$ matrix $Q_f:=[(Q_f)_{k,p}]$ on $(\mathbf{C}^{n+1})^*$. Since, by construction, the linear tangent map of f maps $(Q_f)_p(\eta)$ to $\mathcal{R}_p(f(\eta))$ for every η near T_{η_0}, the identity (13) is clearly satisfied on some neighbourhood of T_{η_0}. By analytic continuation, the same identity occurs on Ω_1^*. □

We shall now use the above proposition in order to describe the structure of the branch locus of proper holomorphic mappings between certain Reinhardt domains. The following lemma will be the key point in our argumentation.

Lemma 3.2. *Let* Ω_1 *and* Ω_2 *be two Reinhardt domains in* $\mathbf{C}^{n+1}$*. Let* $f:\Omega_1\to\Omega_2$ *be a proper holomorphic mapping. Assume that the hypothesis of Proposition* 3.1 *are satisfied. Let* Q_f *be the matrix with entries in* $\mathcal{G}$ *which is defined in Proposition* 3.1*. Then*

(i) $$z\in V_f^* \implies f(z)\in\mathcal{H},$$

(ii) $$z\in V_f^* \implies \det Q_f(z)=0.$$

Proof. By taking the determinant of identity (13) we get

$$\det\left[\frac{\partial f_k}{\partial z_p}\right]\det Q_f(z) = i^{n+1}\prod_{k=0}^{n} f_k(z)$$

for every $z\in\Omega_1^*$. The assertion (i) follows immediately.

Let us now prove the assertion (ii). For this we assume that $\zeta\in V_f^*$ and $\det Q_f(\zeta)\neq 0$ for some $\zeta\in\Omega_1^*$ and seek a contradiction. Let us first notice that there exists a neighbourhood U of ζ and a holomorphic matrix $\widetilde{Q}(z)$ defined on U such that $\widetilde{Q}(z)=[Q_f(z)]^{-1}$ for every $z\in U$. On the other hand, the identity (13) shows that

$$\left[\frac{1}{if_k}\frac{\partial f_k}{\partial z_p}(z)\right][(Q_f)_{k,p}(z)] = \mathrm{Id}$$

on $U\setminus(V_f\cup f^{-1}(\mathcal{H})\cap U)$. Thus we must have $\widetilde{Q}(z)=[(1/if_k)(\partial f_k/\partial z_p)]$ on U which means that the functions $(1/f_k)(\partial f_k/\partial z_p)(z)$ are holomorphically extendable to U.

According to the assertion (i) we may assume that $f_0(\zeta)=0$ and, since f is proper, we may pick a complex linear disc $\sigma:\Delta\to\Omega_1$ ($\sigma(u)=\zeta+u\xi$) through ζ such that $\tilde\sigma:=f_0\circ\sigma$ does not identically vanish on Δ. Thus for u close to the origin and $u\neq 0$ we have

$$\frac{1}{\tilde\sigma(u)}\frac{d\tilde\sigma(u)}{du} = \sum_{p=0}^{n}\frac{\xi_p}{f_0(\sigma(u))}\frac{\partial f_0}{\partial z_p}(\sigma(u)).$$

As we have previously seen, the right-hand side of the above identity is a well-defined holomorphic function on some neighbourhood of the origin in Δ. Since $(1/\tilde\sigma(u))(d\tilde\sigma(u)/du)$ is singular at the origin we have reached a contradiction. □

We end this section by giving a precise description of the branch locus of proper holomorphic self-mappings of certain Reinhardt domains.

Proposition 3.3. *Let Ω be a complete Reinhardt domain in $\mathbf{C}^{n+1}$ and let $f:\Omega\to\Omega$ be a proper holomorphic self-map such that $V_f\neq\emptyset$. Assume that $b\Omega$ is $\mathcal{C}^2$-strictly pseudoconvex at some point $\eta_0\in b\Omega^*$ and that f^k holomorphically extends without branching to some neighbourhood of T_{η_0} for every $k\geq 1$. Then, after replacing f by some iterate and permuting the variables, we have*

$$V_f=\bigcup_{j=1}^{m}\mathcal{H}_{\Omega j}\quad\text{and}\quad f^{-1}(\mathcal{H}_{\Omega j})=\mathcal{H}_{\Omega j}=f(\mathcal{H}_{\Omega j})$$

for some $m\in\{0,\dots,n\}$ and every $j\in\{1,\dots,m\}$.

Proof. Let $\mathcal{F}$ be the set of sections by the coordinate hyperplanes $\mathcal{H}_{\Omega j}$ such that $f^k(\mathcal{H}_{\Omega j})\subset\mathcal{H}_{\Omega j}$ for some $k\geq 1$. Since $\mathcal{F}$ is finite we may replace f by some iterate and assume that $f(\mathcal{H}_{\Omega j})\subset\mathcal{H}_{\Omega j}$ for every $\mathcal{H}_{\Omega j}\in\mathcal{F}$. We now proceed in three steps.

First step. The inclusion $f^k(V_{f^k})\subset\mathcal{H}_\Omega$ holds for every $k\geq 1$.

According to the first assertion of Lemma 3.2, one has $f^k(V_{f^k}\cap\Omega^*)\subset\mathcal{H}$ for every $k\geq 1$. It therefore remains to show that if $\mathcal{H}_{\Omega j}\subset V_{f^k}$ for some $k\geq 1$ then $f^k(\mathcal{H}_{\Omega j})\subset\mathcal{H}$. We argue by contradiction.

Assume that $\mathcal{H}_{\Omega j_1}\subset V_g$ where $g:=f^{k_0}$ and $g(\mathcal{H}_{\Omega j_1})\not\subset\mathcal{H}$. Let us then construct a sequence $(\mathcal{H}_{\Omega j_q})_{q\geq 1}$ such that $g(\mathcal{H}_{\Omega j_{q+1}})=\mathcal{H}_{\Omega j_q}$ for $q\geq 1$. Assume that $\mathcal{H}_{\Omega j_1},\dots,\mathcal{H}_{\Omega j_q}$ have already been constructed. Then $g^{q-1}(\mathcal{H}_{\Omega j_q})\subset\mathcal{H}_{\Omega j_1}\subset V_g$ and therefore

$$g^{-1}(\mathcal{H}_{\Omega j_q})\subset(g^q)^{-1}(V_g)\subset V_{g^{q+1}}. \tag{14}$$

If $g^{-1}(\mathcal{H}_{\Omega j_q})$ does not contain any coordinate hyperplane, then the holomorphic function $(\prod_{j=0}^n z_j)$ restricted to the analytic set $g^{-1}(\mathcal{H}_{\Omega j_q})$ has a negligeable zero set and thus $g^{-1}(\mathcal{H}_{\Omega j_q})\cap\Omega^*$ is dense in $g^{-1}(\mathcal{H}_{\Omega j_q})$. It follows that $g^q(\mathcal{H}_{j_q})\subset g^{q+1}[g^{-1}(\mathcal{H}_{\Omega j_q})]\subset\overline{g^{q+1}[g^{-1}(\mathcal{H}_{\Omega j_q})\cap\Omega^*]}$, where the first inclusion occurs because g is onto. Then, by using (14) and the first assertion of Lemma 3.2, we get

$$g^q(\mathcal{H}_{\Omega j_q})\subset\overline{g^{q+1}(V_{g^{q+1}}\cap\Omega^*)}\cap\Omega\subset\mathcal{H}_\Omega. \tag{15}$$

Since $g^q(\mathcal{H}_{\Omega j_q})=g(\mathcal{H}_{\Omega j_1})$ and $g(\mathcal{H}_{\Omega j_1})\not\subset\mathcal{H}$, the last inclusion of (15) is not possible and therefore $g^{-1}(\mathcal{H}_{\Omega j_q})$ must contain some coordinate hyperplane $\mathcal{H}_{\Omega j_{q+1}}$. As $g|_{\mathcal{H}_{\Omega j_{q+1}}}$ is a proper map from $\mathcal{H}_{\Omega j_{q+1}}$ to $\mathcal{H}_{\Omega j_q}$, it is onto and thus we actually have $g(\mathcal{H}_{\Omega j_{q+1}})=\mathcal{H}_{\Omega j_q}$.

Let $r>s>0$ be some integers such that $\mathcal{H}_{\Omega j_r}=\mathcal{H}_{\Omega j_s}$. Then $g^{r-s}(\mathcal{H}_{\Omega j_r})=\mathcal{H}_{\Omega j_s}$ and therefore $\mathcal{H}_{\Omega j_r}=\mathcal{H}_{\Omega j_s}\in\mathcal{F}$. It follows that $g^k(\mathcal{H}_{\Omega j_r})=\mathcal{H}_{\Omega j_r}$ for every k and in particular $g(\mathcal{H}_{\Omega j_1})=\mathcal{H}_{\Omega j_r}$, which is a contradiction.

Second step. The inclusions $(f^k)^{-1}(V_f^k)\subset V_{f^k}\subset\mathcal{H}_\Omega$ hold for every $k\geq k_0$ and some $k_0\in\mathbf{N}$.

Since $V_{f^{k+1}}=(f^k)^{-1}(V_f)\cup V_{f^k}$ for any $k\geq 1$, the sequence $(V_{f^k})_{k\geq 1}$ is increasing. By using both Proposition 3.1 and the assertion (ii) of Lemma 3.2, we may find an integer N and a sequence of polynomials Q_k,

$$Q_k(z):=z_0^N\dots z_n^N\det Q_{f^k}(z)$$

whose degrees are uniformly bounded and such that

$$V_{f^k}\subset\{Q_k=0\}. \tag{16}$$

Without loss of generality we may assume that $\|Q_k\|=1$ for every k and, after taking some subsequence, that Q_k is converging to Q_∞. Since $Q_\infty\neq 0$, it follows from (16) that the number of irreducible components of V_{f^k} is uniformly bounded. Thus, there exists some k_0 such that $V_{f^k}=V_{f^{k+1}}$ for $k\geq k_0$. In particular, for $k\geq k_0$, one has $V_{f^k}=V_{f^{2k}}=(f^k)^{-1}(V_{f^k})\cup V_{f^k}$ and therefore $(f^k)^{-1}(V_{f^k})\subset V_{f^k}$. Then, as f^k is onto, we get $V_{f^k}\subset f^k[(f^k)^{-1}(V_{f^k})]\subset f^k(V_{f^k})$ and the conclusion follows from the first step.

Third step. Let $g:=f^{k_0}$, then after a possible permutation of the variables, one has $V_g=\mathcal{H}_{\Omega 1}\cup...\cup\mathcal{H}_{\Omega m}$ for some $0\leq m\leq n$ and $g^{-1}(\mathcal{H}_{\Omega j})=\mathcal{H}_{\Omega j}=g(\mathcal{H}_{\Omega j})$ for $0\leq j\leq m$.

By the second step one has $V_g=\mathcal{H}_{\Omega j_1}\cup...\cup\mathcal{H}_{\Omega j_m}$ and $g^{-1}(V_g)\subset V_g$. Thus there exists $l_2\in\{j_1,...,j_m\}$ such that $g(\mathcal{H}_{\Omega l_2})\subset\mathcal{H}_{\Omega j_1}$. Then, $g^{-1}(\mathcal{H}_{\Omega l_2})\subset(g^2)^{-1}(V_g)\subset V_{g^3}\subset\mathcal{H}_\Omega$, and one finds l_3 such that $g(\mathcal{H}_{\Omega l_3})\subset\mathcal{H}_{l_2}$. By iterating one obtains a sequence $(l_k)_{k\geq 2}$ such that $g(\mathcal{H}_{\Omega l_{k+1}})\subset\mathcal{H}_{l_k}$. The sequence $(\mathcal{H}_{\Omega l_k})_{k\geq 2}$ must contain an element $\mathcal{H}_{\Omega l_r}$ of $\mathcal{F}$. Thus $g(\mathcal{H}_{l_r})\subset\mathcal{H}_{\Omega l_r}$ and, since the map $g|_{\Omega\cap\mathcal{H}_{\Omega l_r}}$ is proper one actually has $g(\mathcal{H}_{\Omega l_r})=\mathcal{H}_{\Omega l_r}$. Thus $\mathcal{H}_{l_r}=\mathcal{H}_{\Omega j_1}$ and the conclusion follows. □

Remark. The following example, which is due to G. Dini and A. Selvaggi [13], shows that one cannot expect such a simple control of the branch locus for proper mappings between different Reinhardt domains. Let $\Omega_1:=\{|z|^4+|w|^4<1\}$ and $\Omega_2:=\{|z|+|w|^{1/p}<1\}$ be two Reinhardt domains in $\mathbf{C}^2$ and $f:\Omega_1\to\Omega_2$ a proper holomorphic map which is defined by $f(z,w)=\left(\frac{1}{2}(z^2+w^2)^2, \left(\sqrt{2}\right)^{-2p}(w^2-z^2)^{2p}\right)$ for $p>1$. Then the branch locus of f contains the intersection of Ω_1 with the line $\{z=w\}$ and thus is not contained in $\mathcal{H}$.

4. Proper holomorphic self-maps

Proof of Theorem 1.1. Let $f:\Omega\to\Omega$ be proper and holomorphic. As Ω is bounded and complete or contained in $(\mathbf{C}^*)^{n+1}$, f holomorphically extends to some neighbourhood of $\bar{\Omega}$ (see [6] and [2]), we shall still denote this extension by f.

We first establish the existence of some point $\eta_0\in b\Omega^*$ such that $b\Omega$ is $\mathcal{C}^2$-strictly pseudoconvex at η_0 and f does not branch on the torus T_{η_0}. As is well known, the existence of strictly pseudoconvex regions of $b\Omega$ follows from its global smoothness (see [21, Proposition 15.5.2]). Then, if T_{η_0} is a strictly pseudoconvex torus in $b\Omega^*$, f cannot branch at any point of T_{η_0} since it maps these points on smooth ones (see [12, Lemma 4]).

We shall now prove that V_f is empty. If $\Omega\subset(\mathbf{C}^*)^{n+1}$, it directly follows from Lemma 3.2 that V_f is empty and, since $b\Omega$ is smooth, this implies that f is an automorphism by a result of S. Pinchuk ([19]). We now assume that Ω is complete and proceed by contradiction. By conjugating f with some dilation, we may also assume that $\Omega \subset \Delta^{n+1}$. As the above arguments also apply to the iterates of f, we may use Proposition 3.3 and assume that there exists $0 \le m \le n$ such that $V_f = \bigcup_{j=1}^m \mathcal{H}_{\Omega j}$ and $f^{-1}(\mathcal{H}_{\Omega j}) = \mathcal{H}_{\Omega j} = f(\mathcal{H}_{\Omega j})$ for $0 \le j \le m$. Let us set $V := \bigcup_{j=1}^m \mathcal{H}_{\Omega j}$. Then, $f^k : \Omega \setminus V \to \Omega \setminus V$ is a covering map for every k and moreover, one may pick a map $p(z) := (z_0^{\alpha_0}, \dots, z_m^{\alpha_m}, z_{m+1}, \dots, z_n)$ ($\alpha_j \in \mathbf{N}$, $\alpha_j > 1$) such that p^k maps $\widetilde{\Omega}_k := (p^k)^{-1}(\Omega)$ onto Ω and $(p^k)_*$ coincides with $(f^k)_*$ on the homotopy groups $\Pi_1(\widetilde{\Omega}_k \setminus V) = \Pi_1(\Omega \setminus V)$. This implies the existence of a sequence of homeomorphisms $(\phi_k) : (\widetilde{\Omega}_k \setminus V) \to (\Omega \setminus V)$ such that $f^k \circ \phi_k = p^k$. One easily sees that the ϕ_k are actually biholomorphic and thus, by the Riemann removable singularities theorem, extend as biholomorphisms between $\widetilde{\Omega}_k$ and Ω. Then, by a theorem of W. Kaup and J. P. Vigué [14], the complete Reinhardt domains $\widetilde{\Omega}_k$ and Ω are actually linearly equivalent. Thus there exists a sequence of linear biholomorphisms $L_k : \widetilde{\Omega}_k \to \Omega$. By Montel's theorem, some subsequence is uniformly converging on compact subsets of $\mathbf{C}^{n+1}$ to some linear map L which induces a biholomorphism $L : \Delta^m \times \Omega' \to \Omega$. This is impossible since $b\Omega$ is somewhere strictly pseudoconvex. Thus $V_f = \emptyset$ and, as Ω is simply connected, f is an automorphism. □

Proof of Theorem 1.3. Let $T_{\eta_0} \subset b\Omega^*$ be a strictly pseudoconvex torus. The pull-back $(f^k)^*(\mathcal{R}_p)$ are well defined at some point on T_{η_0}, using Proposition 3.3 of [9] one sees that these fields are defined along T_{η_0}. The conclusion is then obtained as for Theorem 1.1 by using Proposition 3.3. □

Remark. Theorem 1.3 is not true for circular domains: some complete circular basins of attraction in $\mathbf{C}^2$ are spherical outside a finite number of circles and do admit non-injective proper holomorphic self-maps (see [8], other examples related with dynamics are in [7]).

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