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Sobolev embeddings into BMO, VMO, and L_∞

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Abstract. Let X be a rearrangement-invariant Banach function space on $\mathbf{R}^n$ and let V^1X be the Sobolev space of functions whose gradient belongs to X. We give necessary and sufficient conditions on X under which V^1X is continuously embedded into BMO or into L_∞. In particular, we show that $L_{n,\infty}$ is the largest rearrangement-invariant space X such that V^1X is continuously embedded into BMO and, similarly, $L_{n,1}$ is the largest rearrangement-invariant space X such that V^1X is continuously embedded into L_∞. We further show that V^1X is a subset of VMO if and only if every function from X has an absolutely continuous norm in $L_{n,\infty}$. A compact inclusion of V^1X into C^0 is characterized as well.

1. Introduction

The space BMO of functions having *bounded mean oscillation*, introduced by John and Nirenberg [JN], has proved to be particularly useful in various areas of analysis, especially harmonic analysis (see [To] or [S, Chapter 4] and the references given there) and interpolation theory (see [BS, Chapter 5]), as an appropriate substitute for L_∞ when L_∞ does not work.

The main objective of the present paper is to establish criteria for the membership of a function to BMO or to L_∞ in terms of the summability properties of its gradient. More precisely, we characterize all rearrangement-invariant (r.i.) Banach function spaces X on $\mathbf{R}^n$ such that the corresponding Sobolev space V^1X of functions whose gradient belongs to X is continuously embedded into BMO or into L_∞. Furthermore, we show that the Marcinkiewicz space $L_{n,\infty}$ is the largest rearrangement-invariant space X such that V^1X is continuously embedded into BMO, whereas the Lorentz space $L_{n,1}$ is the largest rearrangement-invariant space X such that V^1X is continuously embedded into L_∞. Our conclusions bring some new information to the study of the gap between L_∞ and BMO (cf. [GJ], [LaP] or [LP]). We also give a necessary and sufficient condition for V^1X to be uniformly included into the space of functions with *vanishing mean oscillation* (VMO), which has recently found applications in the theory of partial differential equations—

see e.g. [CFL], [Ch], [BC]. Moreover, we prove that $V^1X \subset \mathrm{VMO}$ if and only if $X \subset (L_{n,\infty})_a$, the subspace of $L_{n,\infty}$ containing functions with absolutely continuous norms. Finally, we present a characterization of the compact embedding of W^1X into C^0.

Roughly speaking, in an r.i. space the norm depends only on the measure of level sets of a function. This class of function spaces includes, for example, Lebesgue, Lorentz, Lorentz–Zygmund, and Orlicz spaces. Thus, in particular, our characterization reproduces the well-known results about embeddings of V^1X into BMO, VMO, and L_∞ when X is a Lebesgue space or a Lorentz space, and enables us to deal with a more general class of Lorentz–Zygmund spaces. We also show that L_n is the largest Orlicz space L_A such that $V^1L_A \hookrightarrow \mathrm{BMO}$, recovering thus a recent result of [F], and that there does not exist any largest Orlicz space L_A such that $V^1L_A \hookrightarrow L_\infty$.

The paper is organized as follows. Basic definitions and elementary properties of the relevant function spaces are collected in Section 2. In Section 3 we state the main results of the paper. Two inequalities, possibly of independent interest, extending the Pólya–Szegő principle for rearrangements (see e.g. [BZ], [T3]) and the classical Poincaré inequality, are proved in Section 4. Section 5 contains proofs of the main results. Some applications are presented in Section 6. We wish to thank Nicola Fusco for pointing out to us the question that is dealt with in Theorem 3.3(i).

2. Preliminaries

Throughout the paper, λ_n stands for the n-dimensional Lebesgue measure. The letter C will denote various constants independent of appropriate quantities. By the symbol "$\hookrightarrow$" we mean a continuous embedding between (quasi-)normed linear spaces. Let $\mathbf{R}^n$ denote the Euclidean space of dimension n which will be assumed ≥ 2 throughout the paper. Let Q be a cube in $\mathbf{R}^n$. The space $\mathrm{BMO}(Q)$ is the class of real-valued integrable functions on Q such that

$$\|f\|_{*,Q} = \sup_{Q' \subset Q} \frac{1}{\lambda_n(Q')} \int_{Q'} |f(x) - f_{Q'}|\,dx < \infty,$$

where $f_{Q'} = \lambda_n(Q')^{-1} \int_{Q'} f$, and the supremum is extended over all subcubes Q' of Q. Let us recall that BMO is not a Banach space, although it can be turned into one by introducing the norm

$$\|f\|_{\mathrm{BMO}(Q)} = \|f\|_{*,Q} + \|f\|_{L_1(Q)}.$$

We say that a function $f\colon Q\to\mathbf{R}$ belongs to VMO(Q), the space of functions with *vanishing mean oscillation*, if $\lim_{s\to 0_+}\varrho_f(s)=0$, where

$$\varrho_f(s)=\sup_{\lambda_n(Q')\le s}\frac{1}{\lambda_n(Q')}\int_{Q'}|f(x)-f_{Q'}|\,dx. \tag{2.1}$$

Occasionally we shall work with the function ϱ_f^B, defined in the same way as ϱ_f but with cubes replaced by balls.

The following relations hold: $L_\infty\subsetneq$ BMO, VMO$\subsetneq$BMO, $L_\infty\not\subset$VMO, and VMO$\not\subset L_\infty$ (the non-equalities and non-inclusions can be demonstrated for example with the functions $\log|x|$, $\log|x|$, $\sin(\log|x|)$, and $(\log|x|)^{1/2}$, respectively).

Let G be a measurable subset of $\mathbf{R}^n$ and let f be a real-valued measurable function on $\mathbf{R}$. The *nonincreasing rearrangement* of f is given by

$$f^*(t)=\sup\{s\ge 0\mid \lambda_n(\{x\in G\mid |f(x)|>s\})>t\},\quad 0<t<\lambda_n(G),$$

and the *signed nonincreasing rearrangement* of f is given by

$$f^\circ(t)=\sup\{s\in\mathbf{R}\mid \lambda_n(\{x\in G\mid f(x)>s\})>t\},\quad 0<t<\lambda_n(G).$$

We also denote by $G^\star$ the ball, centered at the origin, and having the same measure as G, and by $f^\star$ the *spherically symmetric rearrangement* of f, namely, the radially decreasing function on $G^\star$ equidistributed with f. Observe that $f^\star(x)=f^*(C_n|x|^n)$, where $C_n=\pi^{n/2}/\Gamma\big(1+\tfrac12 n\big)$, the measure of the n-dimensional unit ball.

Let X be a Banach space of functions defined on $\mathbf{R}^n$, equipped with the norm $\|\cdot\|$. We say that X is a *rearrangement-invariant Banach function space*, or briefly an r.i. space, if the following five axioms hold:

(P1) $0\le g\le f$ a.e. implies $\|g\|_X\le\|f\|_X$;

(P2) $0\le f_n\nearrow f$ a.e. implies $\|f_n\|_X\nearrow\|f\|_X$;

(P3) $\|\chi_E\|_X<\infty$ for any $E\subset\mathbf{R}^n$ such that $\lambda_n(E)<\infty$ (here χ_E denotes the characteristic function of E);

(P4) for every $E\subset\mathbf{R}^n$ with $\lambda_n(E)<\infty$, there exists a constant C_E such that $\int_E f\le C_E\|f\|_X$ for all $f\in X$;

(P5) $\|f\|_X=\|g\|_X$ whenever $f^*=g^*$.

A function $f\in X$ is said to have an *absolutely continuous* (*a.c.*) *norm* if $\|f\chi_{E_k}\|_X\to 0$ whenever $\chi_{E_k}\to 0$ a.e. If every function in X has an a.c. norm, then we say that X has an a.c. norm. We denote by X_a the subspace of X containing functions with a.c. norms in X.

If X is an r.i. space, then the set

$$X'=\left\{f\colon\mathbf{R}^n\to\mathbf{R}\;\middle|\;\int_{\mathbf{R}^n}|fg|<\infty\text{ for all }g\in X\right\},$$

endowed with the norm

$$\|f\|_{X'} = \sup_{g\neq 0} \frac{\int |fg|}{\|g\|_X},$$

is called the *associate space* of X. Recall that X' is again an r.i. space and $(X')'=X$. The Hölder inequality

$$\int |fg| \le \|f\|_X \|g\|_{X'} \tag{2.2}$$

holds and, moreover,

$$\|f\|_X = \sup_{g\neq 0} \frac{\int |fg|}{\|g\|_{X'}}. \tag{2.3}$$

For every r.i. space X there exists an r.i. space $\overline{X}$ on $(0,\infty)$, satisfying $\|f\|_X=\|f^*\|_{\overline{X}}$ for every $f\in X$. Such a space is called a *representation space of* X. The norm in $\overline{X}$ is given by $\|f\|_{\overline{X}}=\sup\{\int_0^\infty f^*g^* \mid \|g\|_{X'}\le 1\}$. Since $\lambda_n(\mathbf{R}^n)=\infty$ and λ_n is non-atomic, $\overline{X}$ is in fact unique. For proofs of properties of r.i. spaces and further results we refer the reader to [BS, Chapter 2].

The simplest example of an r.i. space is the *Lebesgue space* L_p, $1\le p\le\infty$. The *generalized Lorentz–Zygmund* (GLZ) *space* $L_{p,q;\alpha,\beta}$, equipped with the norm

$$\|f\|_{L_{p,q;\alpha,\beta}} = \|t^{1/p-1/q} l^\alpha(t) ll^\beta(t) f^*(t)\|_{L_q(0,\infty)},$$

is an r.i. space, when $1\le p,q\le\infty$, $\alpha,\beta\in\mathbf{R}$, $l(t)=1+|\log t|$, $ll(t)=1+\log(l(t))$, and one of the following conditions holds:

$$\begin{cases} 1<p<\infty; \\ p=1,\ q=1,\ \alpha>0; \\ p=1,\ q=1,\ \alpha=0,\ \beta\ge 0; \\ p=\infty,\ q=\infty,\ \alpha<0; \\ p=\infty,\ q=\infty,\ \alpha=0,\ \beta\le 0; \end{cases} \tag{2.4}$$

(cf. [EOP]). If $\alpha=\beta=0$, then $L_{p,q;\alpha,\beta}$ coincides with the usual Lorentz space $L_{p,q}$, and, in particular, with the Lebesgue space L_p if $p=q$. Let us further recall that $(L_{p,q;\alpha,\beta})'=L_{p',q';-\alpha,-\beta}$, where $1/p+1/p'=1$.

Another example of an r.i. space is the *Orlicz space* L_A, generated by a *Young function* A, i.e., a convex increasing function on $(0,\infty)$ satisfying $\lim_{t\to 0_+} A(t)/t= \lim_{t\to\infty} t/A(t)=0$. Recall that L_A contains all measurable functions $f\colon\mathbf{R}^n\to\mathbf{R}$ such that $\int_{\mathbf{R}^n} A(|f(x)|/K)\,dx<\infty$ for some $K>0$. The *Luxemburg norm* in L_A is given by

$$\|f\|_{L_A}=\inf\left\{K>0\;\middle|\;\int A\left(\frac{|f(x)|}{K}\right)dx\le 1\right\}. \tag{2.5}$$

Moreover, $(L_A)'=L_{\tilde{A}}$ (with equivalent norms), where $\tilde{A}$ is the *complementary function* of A, defined as $\tilde{A}(t)=\sup\{st-A(s)\,|\,s>0\}$.

Let $G\subset\mathbf{R}^n$. We define the extension operator acting on real-valued functions on G by

$$E_G f(x)=\begin{cases} f(x), & x\in G,\\ 0, & x\in\mathbf{R}^n\setminus G.\end{cases}$$

For a measurable subset G of $\mathbf{R}^n$ and an r.i. space X we set

$$X(G)=\{f \text{ real-valued on } G\,|\,E_G f\in X\},\qquad \|f\|_{X(G)}=\|E_G f\|_X.$$

If in addition G is open, we define the following spaces and norms of Sobolev type:

$$\begin{aligned}
&V^1X(G)=\{u\colon G\to\mathbf{R}\,|\,u \text{ is weakly differentiable on } G,\ |Du|\in X(G)\};\\
&W^1X(G)=X(G)\cap V^1X(G),\quad \|u\|_{W^1X(G)}=\|u\|_{X(G)}+\|Du\|_{X(G)};\\
&V_0^1X(G)=\{u\colon G\to\mathbf{R}\,|\,E_G u \text{ is weakly differentiable on } \mathbf{R}^n,\ |Du|\in X(G)\};\\
&W_0^1X(G)=X(G)\cap V_0^1X(G),\quad \|u\|_{W_0^1X(G)}=\|u\|_{X(G)}+\|Du\|_{X(G)},
\end{aligned}$$

where D stands for the gradient and $|\cdot|$ for the n-dimensional Euclidean norm.

3. Main results

Our main result is the following characterization of the embedding of V^1X into BMO.

3.1. Theorem. *Let X be an r.i. space, and let $Q\subset\mathbf{R}^n$ be a cube. Let $m\in(0,\infty)$ and let*

$$B_m=\sup_{0<s<m}\frac{1}{s}\|r^{1/n}\chi_{(0,s)}(r)\|_{\overline{X}'}.$$

(i) *There is a constant $C>0$ such that*

$$\|u\|_{*,Q}\le C\|Du\|_{X(Q)},\quad u\in V^1X(Q), \tag{3.1}$$

if and only if

$$B_1<\infty. \tag{3.2}$$

Moreover, if (3.2) *holds, then the best constant* C *in* (3.1) *satisfies* $C \le K B_{\lambda_n(Q)/2}$, *where* K *depends only on* n.

(ii) *The space* $L_{n,\infty}(Q)$ *is the largest r.i. space* $X(Q)$ *that renders* (3.1) *true. In other words,* (3.1) *holds if and only if* $X(Q) \hookrightarrow L_{n,\infty}(Q)$.

Note that B_1 in (3.2) can be equivalently replaced by B_m with any positive m.

3.2. *Remark.* We can also show that (3.2) is equivalent to

$$\|u\|_{\mathrm{BMO}(Q)} \le C\|u\|_{W^1X(Q)}, \quad u \in W^1X(Q).$$

As for the embedding into VMO, we have the following result.

3.3. Theorem. *Let* X *be an r.i. space and let* $Q \subset \mathbf{R}^n$ *be a cube. Then*

(i) $V^1X(Q) \subset \mathrm{VMO}(Q)$ *uniformly, i.e.,*

$$\lim_{s\to 0+} \sup_{\|Du\|_X \le 1} \varrho_u(s) = 0, \tag{3.3}$$

if and only if

$$\lim_{s\to 0+} \frac{1}{s}\|r^{1/n}\chi_{(0,s)}(r)\|_{\overline{X}'} = 0; \tag{3.4}$$

(ii) $V^1X(Q) \subset \mathrm{VMO}(Q)$ *if and only if* $X(Q) \subset (L_{n,\infty})_a(Q)$.

3.4. *Remark.* The proof of Theorem 3.1(i) (see Section 5 below) immediately gives a sufficient condition for the inclusion of $V^1X(Q)$ into $\mathrm{VMO}(Q)$. Assume that $B_1 < \infty$ and X has an a.c. norm. Then $V^1X(Q) \subset \mathrm{VMO}(Q)$, and, moreover, $\varrho_u(s) \le K B_{s/2}\|Du\|_{X(Q)}$ for $0 < s < \lambda_n(Q)$ and $u \in V^1X(Q)$, where K only depends on n. However, the inclusion is not necessarily uniform in the sense of (3.3), as shown by the example $X = L^n$.

Our next aim is to characterize the embeddings of V_0^1X and V^1X into L_∞ and C^0. For this type of embeddings we can consider more general domains than cubes. Actually, any open set G is admissible for embeddings of $V_0^1X(G)$, whereas a suitable class of sets for embeddings of $V^1X(G)$ can be defined as follows in terms of isoperimetric inequalities.

We call $\mathcal{G}$ the collection of all open sets $G \subset \mathbf{R}^n$ of finite measure such that, for some $K > 0$,

$$K h_G(s) \ge \min\{s, \lambda_n(G) - s\}^{1/n'}, \quad 0 < s < \lambda_n(G), \tag{3.5}$$

where $h_G(s)$ is the isoperimetric function of G, defined as

$$h_G(s) = \inf_{\lambda_n(E)=s} P(E,G).$$

Here, the infimum is extended over all measurable subsets of G having finite perimeter $P(E,G)$ relative to G. Recall that $P(E,G)$ is defined as the total variation on G of the vector-valued measure $D\chi_E$; in particular, if $\partial E \cap G$ is smooth, then $P(E,G)$ agrees with the $(n-1)$-dimensional Hausdorff measure of $\partial E \cap G$. The smallest constant K such that (3.5) holds will be denoted by $K(G)$ and called the *relative isoperimetric constant* of G. Note that $\mathcal{G}$ includes, e.g., connected open sets having the cone property, in particular cubes and balls.

3.5. Theorem. *Let X be an r.i. space, and let $m \in (0,\infty)$. Set*

$$\mathcal{A}_m = \|r^{-1/n'} \chi_{(0,m)}(r)\|_{\overline{X}'}.$$

(i) *Let $G \subset \mathbf{R}^n$ be open and let $\lambda_n(G) < \infty$. Then there is a constant $C > 0$ such that*

$$\|u\|_{L_\infty(G)} \le C \|Du\|_{X(G)}, \quad u \in V_0^1 X(G), \tag{3.6}$$

if and only if

$$\mathcal{A}_1 < \infty. \tag{3.7}$$

Moreover, if (3.7) *holds, the best constant C in* (3.6) *satisfies $C \le (nC_n^{1/n})^{-1} \mathcal{A}_{\lambda_n(G)}$.*

(ii) *Let $G \in \mathcal{G}$. Then there is a constant $C > 0$ such that*

$$\operatorname{ess\,sup} u - \operatorname{ess\,inf} u \le C \|Du\|_{X(G)}, \quad u \in V^1 X(G), \tag{3.8}$$

if and only if (3.7) *holds. Moreover, if* (3.7) *holds, then the best constant C in* (3.8) *satisfies $C \le 2^{1/n'} K(G) \mathcal{A}_{\lambda_n(G)/2}$.*

(iii) *Let $G \subset \mathbf{R}^n$ be open and let $\lambda_n(G) < \infty$. Then $L_{n,1}(G)$ is the largest r.i. space $X(G)$ that makes* (3.6) *true. In other words,* (3.6) *holds if and only if $X(G) \hookrightarrow L_{n,1}(G)$. An analogous statement holds for the inequality* (3.8).

As in (3.2), the choice of the index 1 in (3.7) is immaterial.

3.6. *Remark.* The same argument as in the proof of Theorem 3.5(i) (see Section 5 below) shows that if $G = \mathbf{R}^n$ and if X is such that $u \in X$ implies $\lambda_n(\{|u| > t\}) < \infty$ for every $t > 0$, then (3.6) is equivalent to (3.7) with $m = \infty$. Thus, in particular, if (3.7) holds with $m = \infty$, then (3.6) is true for sets of infinite measure as well.

The fact that $L_{n,1}$ is the optimal r.i. domain for the Sobolev embedding (the statement (iii) of Theorem 3.5) can be obtained also by a method from [EKP].

3.7. *Remark.* In fact, we can also show that the following three statements are equivalent:

(i) $\mathcal{A}_1<\infty$;

(ii) for every $G\in\mathcal{G}$ we have

$$\|u-u_G\|_{L_\infty(G)}\le C\|Du\|_{X(G)}, \quad u\in V^1X(G);$$

(iii) for every G which is a finite union of sets from $\mathcal{G}$ we have

$$\|u\|_{L_\infty(G)}\le C\|u\|_{W^1X(G)}, \quad u\in W^1X(G).$$

3.8. *Remark.* Obviously, (3.7) implies (3.2) for any r.i. space X. The converse is not true. For instance, $X=L_n$ satisfies (3.2) but not (3.7). For more examples and more delicate results see Corollary 6.2(ii), Theorem 3.1(ii), and Theorem 3.5(iii).

Recall that a bounded open subset G of $\mathbf{R}^n$ is called *strongly Lipschitz* if for every $x\in\partial G$ there exists a neighbourhood U of x such that $G\cap U$ is the epigraph of a Lipschitz continuous function.

3.9. Theorem. *Let X be an r.i. space and let G be a bounded strongly Lipschitz domain. Then the following three statements are equivalent*:

(i) $\lim_{s\to 0_+}\|r^{-1/n'}\chi_{(0,s)}(r)\|_{\overline{X}'}=0$;

(ii) $\lim_{s\to 0_+}\sup_{\|u\|_{W^1X(G)}\le 1}\sup_{|x-y|<s}|u(x)-u(y)|=0$;

(iii) $W^1X(G)\hookrightarrow C^0(G)$ *compactly.*

Moreover, if any of (i)–(iii) *holds, then there exists a constant C such that for $s>0$*

$$\sup_{|x-y|<s}|u(x)-u(y)|\le C\|u\|_{W^1X(G)}\|r^{-1/n'}\chi_{(0,s^n)}(r)\|_{\overline{X}'}. \tag{3.9}$$

3.10. *Remark.* It is worth noting that the embedding $W^1X(G)\hookrightarrow C^0(G)$ holds also if $\mathcal{A}_1<\infty$ and X has an a.c. norm. For example, this is the case when $X=L_{n,1}(G)$.

4. Generalized Pólya–Szegő principle and Poincaré inequality

In this section we state and prove two lemmas which will play a key role in the proof of our embedding theorems. The first of them provides a generalized Pólya–Szegő inequality.

4.1. Lemma. *Let X be an r.i. space and let $G \subset \mathbf{R}^n$ be open.*

(i) *Let $u \in V_0^1 X(G)$ be such that $\lambda_n(\{|u|>t\})<\infty$ for $t>0$. Then u^* is locally absolutely continuous and*

$$nC_n^{1/n}\left\| -\frac{du^*}{ds} s^{1/n'}\right\|_{\overline{X}(0,\lambda_n(G))} = \|Du^\star\|_{X(G^\star)} \le \|Du\|_{X(G)}. \tag{4.1}$$

(ii) *Assume that G is connected and $\lambda_n(G)<\infty$. Let $u \in V^1X(G)$. Then u° is locally absolutely continuous and*

$$\left\| h_G(s)\left(-\frac{du^\circ}{ds}\right)\right\|_{\overline{X}(0,\lambda_n(G))} \le \|Du\|_{X(G)}. \tag{4.2}$$

Proof. (i) The identity in (4.1) is a consequence of the very definition of $u^\star$. As for the inequality in (4.1), let us set

$$\phi(s) = nC_n^{1/n}\left(-\frac{du^*}{ds}\right)s^{1/n'}, \quad 0<s<\lambda_n(G).$$

If we show that

$$\int_0^s \phi^*(r)\,dr \le \int_0^s |Du|^*(r)\,dr, \quad 0<s<\lambda_n(G), \tag{4.3}$$

then the inequality in (4.1) will follow on applying [BS, Chapter 2, Theorem 4.6]. In order to prove (4.3), we make use of an argument from [T2], [T3]. Let $0 \le a < b$ and let v be the function defined by

$$v(x) = \begin{cases} 0 & \text{if } |u(x)| \le u^*(b), \\ u(x) - u^*(b) & \text{if } u^*(b) < |u(x)| < u^*(a), \\ u^*(a) - u^*(b) & \text{if } u^*(a) \le |u(x)|. \end{cases}$$

Since $|Du| \in X(G)$, and $X(G) \subset L_1(G) + L_\infty(G)$ (cf. [BS, Chapter 2, Theorem 6.6]), we have

$$\int_0^s |Du|^*(r)\,dr < \infty \quad \text{for } s>0,$$

and therefore $Du \in L_1(G')$ for every $G' \subset G$ having finite measure. Hence, as

$$\lambda_n(\{x \in G \mid u^*(a) > |u(x)| > u^*(b)\}) \le b-a, \tag{4.4}$$

we have that $v \in W^1L^1(\mathbf{R}^n)$. The coarea formula (in its form for functions of bounded variation) applied to v yields

$$(4.5)\qquad \int_{\{x\in G \mid u^*(a)>|u(x)|>u^*(b)\}} |Du|\,dx = \int_{u^*(b)}^{u^*(a)} P(\{|u|>t\},\mathbf{R}^n)\,dt.$$

The standard isoperimetric theorem tells us that

$$(4.6)\qquad P(\{|u|>t\},\mathbf{R}^n) \ge nC_n^{1/n}\lambda_n^{1/n'}(\{|u|>t\}).$$

Now, the last two inequalities easily imply that

$$(4.7)\qquad \int_{\{x\in G \mid u^*(a)>|u(x)|>u^*(b)\}} |Du|\,dx \ge nC_n^{1/n}a^{1/n'}[u^*(a)-u^*(b)].$$

The estimates (4.4) and (4.7) ensure that u^* is locally absolutely continuous. Moreover, the inequalities (4.5) and (4.6) yield, via a change of variables,

$$\int_a^b \phi(r)\,dr \le \int_{\{x\in G \mid u^*(a)>|u(x)|>u^*(b)\}} |Du|\,dx.$$

Thus, by (4.4) and by the Hardy–Littlewood inequality for rearrangements, we obtain for every countable family $\{(a_i,b_i)\}_{i=1}^\infty$ of disjoint intervals in $(0,\lambda_n(G))$,

$$\int_{\bigcup(a_i,b_i)} \phi(r)\,dr \le \int_0^{\Sigma(b_i-a_i)} |Du|^*(r)\,dr.$$

The last estimate yields

$$(4.8)\qquad \sup_{\lambda_1(E)=s} \int_E \phi(r)\,dr \le \int_0^s |Du|^*(r)\,dr,$$

since every measurable set $E\subset(0,\lambda_n(G))$ can be approximated from outside by sets of the form $\bigcup(a_i,b_i)$. Hence (4.3) follows, as its left-hand side coincides with that of (4.8).

(ii) The absolute continuity of u° is proved in [CEG, Lemma 6.6]. The proof of (4.2) is analogous to that of (4.1). In particular, for the analogue of (4.4), we use [CEG, (6.22)]. □

The next result extends the Poincaré inequality to the context of r.i. spaces.

4.2. Lemma. *Let X be an r.i. space and let $G\subset\mathbf{R}^n$ be open. Let $\lambda_n(G)<\infty$. Then*

$$\|u\|_{X(G)} \le \left(\frac{\lambda_n(G)}{C_n}\right)^{1/n} \|Du\|_{X(G)}, \quad u\in V_0^1X(G). \tag{4.9}$$

Proof. We define the linear operator

$$Tg(s)=\int_s^{\lambda_n(G)} \frac{g(r)}{r^{1/n'}}\,dr$$

for functions $g\colon[0,\lambda_n(G))\to\mathbf{R}$. It is not hard to verify that

$$\|Tg\|_{L_1(0,\lambda_n(G))} \le \lambda_n(G)^{1/n}\|g\|_{L_1(0,\lambda_n(G))},$$

and

$$\|Tg\|_{L_\infty(0,\lambda_n(G))} \le n\lambda_n(G)^{1/n}\|g\|_{L_\infty(0,\lambda_n(G))}.$$

An interpolation theorem of Calderón ([C], cf. also [BS, Chapter 3, Theorem 2.12]) now yields

$$\|Tg\|_{\overline{X}(0,\lambda_n(G))} \le n\lambda_n(G)^{1/n}\|g\|_{\overline{X}(0,\lambda_n(G))}, \quad g\in\overline{X}(0,\lambda_n(G)). \tag{4.10}$$

Now, if $u\in V_0^1X(G)$, then

$$u^*(s)=\int_s^{\lambda_n(G)} -\frac{du^*}{dr}\,dr.$$

We thus get (4.9) from (4.10) and Lemma 4.1(i). □

5. Proofs of the main results

Proof of Theorem 3.1. (i) Assume that (3.2) holds and let $u\in V^1X(Q)$. Then (cf. the proof of Lemma 4.1) $|Du|\in L_1(Q)$. Since Q is a Lipschitz domain, Sobolev's embedding theorem ensures that $u\in L_{n'}(Q)$, and, the more so, $u\in L_1(Q)$. Let Q' be a subcube of Q. Denote the restriction of u to Q' by v and set $a=\lambda_n(Q')$. Then we have by [CEG, (6.30)]

$$v^\circ(s)-v_{Q'}=\int_0^a \left(\chi_{(s,a)}(r)-\frac{r}{a}\right)\left(-\frac{dv^\circ}{dr}\right)dr, \quad 0<s<a.$$

By Fubini's theorem, this yields

$$\int_0^a |v^\circ(s)-v_{Q'}|\,ds \le \int_0^a -\frac{dv^\circ}{dr}\left(\int_0^a \left|\chi_{(s,a)}(r)-\frac{r}{a}\right| ds\right) dr$$
$$=2\int_0^{a/2} \frac{r(a-r)}{a}\left(-\frac{dv^\circ}{dr}\right) dr + 2\int_{a/2}^a \frac{r(a-r)}{a}\left(-\frac{dv^\circ}{dr}\right) dr.$$

Using (2.2), we easily obtain

$$\int_0^a |v^\circ(s)-v_{Q'}|\,ds \le 2\left\| r^{1/n'}\left(-\frac{dv^\circ}{dr}\right)\chi_{(0,a/2)}(r)\right\|_{\overline{X}} \|r^{1/n}\chi_{(0,a/2)}(r)\|_{\overline{X}'}$$
$$+2\left\|(a-r)^{1/n'}\left(-\frac{dv^\circ}{dr}\right)\chi_{(a/2,a)}(r)\right\|_{\overline{X}} \|(a-r)^{1/n}\chi_{(a/2,a)}(r)\|_{\overline{X}'}.$$

Now, on taking into account the identity

$$((a-r)^{1/n}\chi_{(a/2,a)}(r))^* = (r^{1/n}\chi_{(0,a/2)}(r))^*,$$

the inequality (3.5), and the fact that $K(Q')=K(Q)$, we get from Lemma 4.1(ii),

$$\int_0^a |v^\circ(s)-v_{Q'}|\,ds \le 4K(Q)\|r^{1/n}\chi_{(0,a/2)}(r)\|_{\overline{X}'}\|Du\|_{X(Q')}. \tag{5.1}$$

Since $(v-v_{Q'})^*=(v^\circ-v_{Q'})^*$, we have

$$\int_{Q'} |u(x)-u_{Q'}|\,dx = \int_0^a |v^\circ(s)-v_{Q'}|\,ds. \tag{5.2}$$

By (5.1) and (5.2),

$$\|u\|_{*,Q} \le 2K(Q)\|Du\|_{X(Q)} B_{a/2},$$

and (3.1) follows with $K=2K(Q)$.

Conversely, let (3.1) hold. We claim that, for some $C_1>0$,

$$\sup_{0<s<\lambda_n(Q)} \frac{1}{s}\int_0^s (u^*(r)-u^*(s))\,dr \le C_1\|u\|_{W^1X(Q)}, \quad u\in W^1X(Q). \tag{5.3}$$

Indeed, an argument analogous to that of [BS, Chapter 5, Theorem 7.10] shows that there is a constant $C_2>0$ such that

$$\sup_{0<s<\lambda_n(Q)} \frac{1}{s}\int_0^s (u^*(r)-u^*(s))\,dr \le C_2(\|u\|_{*,Q}+\|u\|_{L_1(Q)}). \tag{5.4}$$

Using (3.1) and the estimate $\|u\|_{L_1(Q)}\le C\|u\|_{X(Q)}$, which follows from (P4) (or (2.2)), we obtain (5.3) from (5.4). Now, Lemma 4.2 and (5.3) yield

$$\sup_{0<s<\lambda_n(Q)} \frac{1}{s}\int_0^s (u^*(r)-u^*(s))\,dr \le C_3\|Du\|_{X(Q)}, \quad u\in V_0^1X(Q),$$

with $C_3=C_1(1+(\lambda_n(Q)/C_n)^{1/n})$. Since $\int_0^s(u^*(r)-u^*(s))\,dr=\int_0^s -r(du^*/dr)\,dr$, we have

$$\sup_{u\in V_0^1X(Q)}\ \sup_{0<s<\lambda_n(Q)} \frac{\displaystyle\frac{1}{s}\int_0^s -r\frac{du^*}{dr}\,dr}{\|Du\|_{X(Q)}} \le C_3.$$

For s small enough there is a ball $B\subset Q$ such that $\lambda_n(B)=s$. Considering radially decreasing (r.d.) functions $u\in V_0^1X(B)\subset V_0^1X(Q)$, we obtain from the last inequality and Lemma 4.1(i) that

$$C_4\ge \sup_{\substack{u\in V_0^1X(B)\\ u\text{ r.d.}}} \frac{\displaystyle\frac{1}{s}\int_0^s -r\frac{du^*}{dr}\,dr}{\displaystyle\left\|r^{1/n'}\left(-\frac{du^*}{dr}\right)\right\|_{\overline{X}(0,s)}}. \tag{5.5}$$

By (2.3), the right-hand side of (5.5) equals $s^{-1}\|r^{1/n}\chi_{(0,s)}(r)\|_{\overline{X}'}$, and (3.2) follows for small (hence for all) $m>0$.

(ii) Let $\lambda_n(Q)=m$. Assume that $X(Q)\hookrightarrow L_{n,\infty}(Q)$. Then $L_{n',1}(Q)\hookrightarrow X'(Q)$, and therefore

$$\begin{aligned}\sup_{0<s<m}\frac{1}{s}\|r^{1/n}\chi_{(0,s)}(r)\|_{\overline{X}'} &\le C\sup_{0<s<m}\frac{1}{s}\|r^{1/n}\chi_{(0,s)}(r)\|_{n',1}\\ &= C\sup_{0<s<m}\frac{1}{s}\int_0^s\frac{(s-r)^{1/n}}{r^{1/n}}\,dr<\infty,\end{aligned}$$

and (3.1) follows from (i).

Conversely, assume that $X(Q)\not\hookrightarrow L_{n,\infty}(Q)$. Since both X and $L_{n,\infty}$ are r.i. spaces, this implies $X(Q)\not\subset L_{n,\infty}(Q)$ ([BS, Chapter 1, Theorem 1.8]). That is, there exists a function g from $X(Q)$ such that $\|g\|_{X(Q)}=1$, and $g\notin L_{n,\infty}(Q)$. Thus there exists a sequence $t_k\in(0,m)$ such that $t_k^{1/n}g^*(t_k)\to\infty$ as $k\to\infty$. But then,

$$\begin{aligned}\sup_{0<s<m}\frac{1}{s}\|r^{1/n}\chi_{(0,s)}(r)\|_{\overline{X}'} &\ge \sup_{k\in\mathbf{N}}\frac{1}{t_k}\sup_{\|f\|_X\le1}\int_0^{t_k} f^*(r)(t_k-r)^{1/n}\,dr\\ &\ge \sup_{k\in\mathbf{N}}\frac{1}{t_k}\int_0^{t_k} g^*(r)(t_k-r)^{1/n}\,dr\\ &\ge \sup_{k\in\mathbf{N}}\frac{g^*(t_k)}{t_k}\int_0^{t_k}(t_k-r)^{1/n}\,dr=\infty,\end{aligned}$$

hence (3.2) is not satisfied, and, by (i), $V^1X(Q)\not\hookrightarrow \mathrm{BMO}(Q)$. □

Proof of Theorem 3.3. (i) That (3.4) implies (3.3) follows easily from a close inspection of the proof of Theorem 3.1(i).

Conversely, assume that (3.3) is true. Let $x_0\in Q$ and denote by B_t the ball centered at x_0 and having measure t. Let t be small enough so that $B_t\subset Q$ and consider radially decreasing functions u with respect to x_0, supported in $B_{t/2}$. Then, similarly as in the proof of Theorem 3.1(i),

$$\frac{1}{\lambda_n(B_t)}\int_{B_t}|u(x)-\bar u_{B_t}|\,dx=\frac{1}{t}\int_0^t\left|\int_0^t\left(\chi_{(s,t)}(r)-\frac{r}{t}\right)\left(-\frac{d\bar u^*}{dr}\right)dr\right|ds,$$

whence

$$\begin{aligned}\frac{1}{\lambda_n(B_t)}\int_{B_t}|\bar u(x)-\bar u_{B_t}|\,dx&\ge\frac{1}{t}\int_{t/2}^t\left|\int_0^t\left(\chi_{(s,t)}(r)-\frac{r}{t}\right)\left(-\frac{d\bar u^*}{dr}\right)dr\right|ds\\&=\frac{1}{t}\int_{t/2}^t\left(\int_0^{t/2}\frac{r}{t}\left(-\frac{d\bar u^*}{dr}\right)dr\right)ds\\&=\frac{1}{2t}\int_0^{t/2}r\left(-\frac{d\bar u^*}{dr}\right)dr.\end{aligned}$$

Now, by Lemma 4.1(i), we have

$$\|D\bar u\|_X=nC_n^{1/n}\left\|-\frac{d\bar u^*}{ds}s^{-1/n'}\right\|_{\bar X}.$$

Combining the last two estimates and setting $g(r)=nC_n^{1/n}r^{1/n'}(-d\bar u^*/dr)$, we obtain

$$\sup_{\|Du\|_X\le1}\varrho_u^B(t)\ge\sup_{\|g\|_{\bar X}\le1}\frac{1}{2nC_n^{1/n}t}\int_0^{t/2}r^{1/n}g(r)\,dr=\frac{1}{2nC_n^{1/n}t}\|\chi_{(0,t/2)}(r)r^{1/n}\|_{\bar X'},$$

and the assertion follows, since, as is readily verified, $\varrho_u^B(t)\le C_1\varrho(C_2t)$, where C_1 and C_2 are positive numbers depending only on n.

(ii) Let $X\subset(L_{n,\infty})_a$. Then, following the lines of the proof of Theorem 3.1(i), we obtain for every $u\in V^1X(Q)$ and every $Q'\subset Q$ such that $\lambda_n(Q')=a$

$$\begin{aligned}\frac{1}{\lambda_n(Q')}\int_{Q'}|u(x)-u_{Q'}|\,dx&\le\frac{4K(Q)}{a}\|r^{1/n}\chi_{(0,a/2)}(r)\|_{n',1}\|Du\chi_{Q'}\|_{n,\infty}\\&=C\|Du\chi_{Q'}\|_{n,\infty},\end{aligned}$$

which tends to zero as $a \to 0_+$, since $Du \in X(Q) \subset (L_{n,\infty})_a(Q)$. Hence $V^1X(Q) \subset$ VMO(Q).

Conversely, assume that $X \not\subset (L_{n,\infty})_a$. Then there exists a function $g \in X(Q)$, a sequence $t_k \to 0_+$, $k \in \mathbf{N}$, and a positive number δ such that

$$g^*(t_k)t_k^{1/n} \geq \delta, \quad k \in \mathbf{N}. \tag{5.6}$$

Let u be a radially decreasing function with respect to the centre of Q. Moreover, assume that the support of u has measure a and is contained in Q, and that

$$u^*(t) = \int_t^a \frac{g^*(r)}{r^{1/n'}}\, dr, \quad t \in (0,a).$$

Then, by Lemma 4.1(i), $u \in V^1X(Q)$, since $g \in X(Q)$. We claim that $u \notin$ VMO(Q).

For $k \in \mathbf{N}$ let B_k be balls concentric with Q and such that $\lambda_n(B_k) = 2t_k$. Similarly as in the proof of Theorem 3.1(i) we can show that, denoting by v_k the restriction of u to B_k,

$$\frac{1}{\lambda_n(B_k)} \int_{B_k} |u(x) - u_{B_k}|\, dx = \frac{1}{2t_k} \int_0^{2t_k} \left| \int_0^{2t_k} \left(\chi_{(s,2t_k)}(r) - \frac{r}{2t_k} \right) \left(-\frac{dv_k^\circ}{dr} \right) dr \right| ds$$

for large k. Now, $-dv_k^\circ/dr = -(du^*/dr)\chi_{(0,2t_k)}(r)$, because u is nonnegative and radially decreasing and B_k are concentric with Q. Therefore we in fact have

$$\frac{1}{\lambda_n(B_k)} \int_{B_k} |u(x) - u_{B_k}|\, dx = \frac{1}{2t_k} \int_0^{2t_k} |F(s)|\, ds, \tag{5.7}$$

where the function $F(s) = \int_0^{2t_k} -(du^*/dr)(r)(\chi_{(s,2t_k)}(r) - (r/2t_k))\, dr$. Observe that $dF/ds = du^*/ds$, which is negative and increasing on $(0, 2t_k)$. Hence F is strictly decreasing and convex on $(0,2t_k)$, and there is a unique $s_0 \in (0,2t_k)$ such that $F(s_0) = 0$.

Assume that $s_0 \leq t_k$. Then, since $-F(s)$ is positive and concave on $(t_k, 2t_k)$, we get from (5.6)

$$\begin{aligned} \frac{1}{2t_k} \int_0^{2t_k} |F(s)|\, ds &\geq \frac{1}{2t_k} \int_{t_k}^{2t_k} -F(s)\, ds \geq \frac{1}{4}(-F(2t_k)) = \frac{1}{8t_k} \int_0^{2t_k} -\frac{du^*}{dr} r\, dr \\ &\geq \frac{1}{8t_k} \int_0^{t_k} g^*(r) r^{1/n}\, dr \geq C \frac{g^*(t_k)}{t_k} t_k^{1+1/n} \geq C\delta. \end{aligned}$$

Now assume that $s_0 \geq t_k$. By the positivity and convexity of F on $(0, t_k)$ we have $F(s) \geq F'(t_k)(s - t_k)$ for every $s \in (0, t_k)$. Thus, using $F'(t_k) = (du^*/dt)(t_k)$ and (5.6), we get

$$\frac{1}{2t_k} \int_0^{2t_k} |F(s)|\, ds \geq \frac{F'(t_k)}{2t_k} \int_0^{t_k} (s - t_k)\, ds = \frac{1}{4} g^*(t_k) t_k^{1/n} \geq \frac{\delta}{4}.$$

The last two estimates combined with (5.7) show that $u \notin \mathrm{VMO}(Q)$, since $\lim_{s\to 0_+} \varrho_u^B(s)=0$ whenever $\lim_{s\to 0_+} \varrho_u(s)=0$. The proof is complete. □

Proof of Theorem 3.5. (i) Assume that (3.7) holds. Let $G\subset\mathbf{R}^n$ be open, $\lambda_n(G)=m$, and let $u\in V_0^1X(G)$. Then

$$\|u\|_{L_\infty(G)}=u^*(0)=\int_0^m -\frac{du^*}{dr}\,dr. \tag{5.8}$$

By (2.2),

$$\int_0^m -\frac{du^*}{dr}\,dr \le \frac{1}{nC_n^{1/n}}\|r^{-1/n'}\chi_{(0,m)}(r)\|_{\overline{X}'}\left\|nC_n^{1/n}r^{1/n'}\left(-\frac{du^*}{dr}\right)\chi_{(0,m)}(r)\right\|_{\overline{X}}. \tag{5.9}$$

Now, using (5.8), (5.9), and Lemma 4.1(i), we get (3.6) with $C=\mathcal{A}_m(nC_n^{1/n})^{-1}$.

Conversely, assume that (3.6) holds for some $G\subset\mathbf{R}^n$. Let B be a ball contained in G. We shall consider radially decreasing functions $u\in V_0^1X(B)\subset V_0^1X(G)$. By (3.6) and Lemma 4.1(i),

$$C\ge \sup_{\substack{u\in V_0^1X(B)\\ u \text{ r.d.}}}\frac{\|u\|_{L_\infty(B)}}{\|Du\|_{X(B)}}=\sup_{\substack{u\in V_0^1X(B)\\ u \text{ r.d.}}}\frac{\displaystyle\int_0^{\lambda_n(B)} -\frac{du^*}{dr}\,dr}{\left\|nC_n^{1/n}r^{1/n'}\left(-\dfrac{du^*}{dr}\right)\chi_{(0,m)}(r)\right\|_{\overline{X}}}.$$

By (2.3), the last supremum equals $\mathcal{A}_{\lambda_n(B)}$, and (3.7) follows for $m=\lambda_n(B)$, and therefore also for $m=\lambda_n(G)$.

(ii) Let (3.7) hold, let $G\in\mathcal{G}$ have measure m, and let $u\in V^1X(G)$. By (2.2), (3.5), and Lemma 4.1(ii), we have

$$\begin{aligned}
\operatorname{ess\,sup} u-\operatorname{ess\,inf} u&=\int_0^m -\frac{du^\circ}{dr}\,dr\\
&\le K(G)\|\min\{r,m-r\}^{-1/n'}\|_{\overline{X}'(0,m)}\\
&\quad\times\left\|\frac{1}{K(G)}\min\{r,m-r\}^{1/n'}\left(-\frac{du^\circ}{dr}\right)\right\|_{\overline{X}(0,m)}\\
&\le K(G)\|\min\{r,m-r\}^{-1/n'}\|_{\overline{X}'(0,m)}\|Du\|_{X(G)}.
\end{aligned}$$

Now, (3.8) follows with the desired C, as $\|\min\{r,m-r\}^{-1/n'}\|_{\overline{X}'(0,m)}=2^{1/n'}\mathcal{A}_{m/2}$.

Conversely, the necessity of (3.7) can be obtained as in (i) on noting that for a nonnegative radially decreasing function u vanishing outside the ball B we have $u^\circ=u^*$, $u^*(0)=\operatorname{ess\,sup} u$, and $u^*(m)=\operatorname{ess\,inf} u=0$.

(iii) Note that (3.7) holds if and only if $\sup_{\|f\|_{X(G)}\le 1}\int_0^m f^*(r)r^{-1/n'}\,dr<\infty$, in other words, $X(G)\hookrightarrow L_{n,1}(G)$. □

Proof of Theorem 3.9. (i) ⇒ (ii) Since G is a bounded strongly Lipschitz domain, (cf. e.g. [A, Lemma 5.17]), we may assume without loss of generality that G is a cube. On applying (3.8) to the restriction of u over subcubes of G containing x, y and having sides of length s, we obtain (3.9), and the assertion follows.

(ii) ⇒ (i) Let $x_0\in G$ and for t small let B_t be a ball centered at x_0 and such that $\lambda_n(B_t)=t$. Consider radially decreasing functions u with respect to x_0 and supported in B_t. Let t be small enough so that $B_t\subset G$, and set $\tau=(t/C_n)^{1/n}$. Then

$$\sup_{|x-y|<\tau}|\bar u(x)-\bar u(y)|=\bar u(x_0)=\int_0^t -\frac{d\bar u^*}{dr}\,dr. \tag{5.10}$$

Set $g(r)=nC_n^{1/n}(-d\bar u^*/dr)r^{1/n'}$. Then (5.10) yields

$$\begin{aligned}\sup_{\|g\|_{\bar X}\le 1}\sup_{|x-y|<\tau}|\bar u(x)-\bar u(y)| &= \frac{1}{nC_n^{1/n}}\sup_{\|g\|_{\bar X}\le 1}\int_0^t \frac{g(r)}{r^{1/n'}}\,dr\\ &= \frac{1}{nC_n^{1/n}}\|r^{-1/n'}\chi_{(0,t)}(r)\|_{\bar X'}.\end{aligned} \tag{5.11}$$

On the other hand, we have by Lemma 4.2 and the definition of the norm in W^1X,

$$\|\bar u\|_{W^1X(G)}\le K\|g\|_{\bar X}, \tag{5.12}$$

where $K=1+(\lambda_n(G)/C_n)^{1/n}$. We thus obtain from (5.11) and (5.12)

$$\|r^{-1/n'}\chi_{(0,t)}(r)\|_{\bar X'}\le nC_n^{1/n}\sup_{\|\bar u\|_{W^1X(G)}\le K}\ \sup_{|x-y|<\tau}|\bar u(x)-\bar u(y)|,$$

and the desired implication follows.

(i) ⇒ (iii) By (i) and Theorem 3.5, the set $\{\|u\|_{W^1X(G)}\le 1\}$ is equibounded, and by (ii) it is equicontinuous at every $x_0\in G$. We already know that (i) implies (ii), hence the desired implication follows via the Arzelà–Ascoli theorem.

(iii) ⇒ (i) If (iii) holds, then, by the Arzelà–Ascoli theorem, the set $\{\|u\|_{W^1X(G)}\le 1\}$ is equicontinuous at every $x_0\in G$. Now, (i) follows as in the proof of (ii) ⇒ (i). Note that a full proof of the implications (i) ⇒ (ii) and (i) ⇒ (iii) would require an approximation of u by a sequence of continuous functions (e.g. averages of u over subcubes whose measure goes to 0), converging a.e. to u. For the sake of brevity we omit the details. □

6. Examples

We conclude the paper with some applications of the results from Section 3. For the sake of simplicity, we consider spaces of functions defined on a cube $Q\subset\mathbf{R}^n$. We start with GLZ spaces.

6.1. Theorem. *Let* $1\le p,q\le\infty$, $\alpha,\beta\in\mathbf{R}$, *and assume that one of the conditions in* (2.4) *holds. Let* $X(Q)=L_{p,q;\alpha,\beta}(Q)$.

(i) *The embedding* $V^1X(Q)\hookrightarrow\mathrm{BMO}(Q)$ *holds if and only if one of the following conditions is satisfied* (*recall that* $n\ge2$):

$$\begin{cases} p>n; \\ p=n, \ \alpha>0; \\ p=n, \ \alpha=0, \ \beta\ge0. \end{cases} \tag{6.1}$$

(ii) *The embedding* $V^1X(Q)\hookrightarrow L_\infty(Q)$ *holds if and only if one of the following conditions holds*:

$$\begin{cases} p>n; \\ p=n, \ \alpha>\dfrac{1}{q'}; \\ p=n, \ \alpha=\dfrac{1}{q'}, \ \beta>\dfrac{1}{q'}; \\ p=n, \ q=1, \ \alpha=0, \ \beta=0. \end{cases} \tag{6.2}$$

(iii) *The inclusion* $V^1X(Q)\subset\mathrm{VMO}(Q)$ *holds if and only if one of the following conditions holds*:

$$\begin{cases} p>n; \\ p=n, \ \alpha>0; \\ p=n, \ \alpha=0, \ \beta>0; \\ p=n, \ 1\le q<\infty, \ \alpha=0, \ \beta=0. \end{cases} \tag{6.3}$$

The same inclusion holds uniformly in the sense of (3.3) *if and only if one of the first three conditions in* (6.3) *holds.*

6.2. Corollary. (i) *The only GLZ space of the form* $X=L_{p,q;\alpha,\beta}$ *such that* $V^1X(Q)\hookrightarrow\mathrm{BMO}(Q)$, *but* $V^1X(Q)\not\subset\mathrm{VMO}(Q)$, *is the Lorentz space* $L_{n,\infty}$.

(ii) *A GLZ space* $X=L_{p,q;\alpha,\beta}$ *satisfies* $V^1X(Q)\hookrightarrow\mathrm{BMO}(Q)$, *but* $V^1X(Q)\not\hookrightarrow L_\infty(Q)$, *if and only if one of the following conditions holds*:

$$\begin{cases} p=n, \ 1\le q\le\infty, \ 0<\alpha<\dfrac{1}{q'}; \\ p=n, \ 1<q\le\infty, \ \alpha=\dfrac{1}{q'}, \ \beta\le\dfrac{1}{q'}; \\ p=n, \ 1<q\le\infty, \ \alpha=0, \ \beta\ge0. \end{cases}$$

Proof of Theorem 6.1. The statements (i) and (ii) can be proved by elementary calculus, cf. also [EOP, Theorem 6.3].

(iii) By a direct calculation we can verify that if either $p>n$, or $p=n$ and $\alpha>0$, or $p=n$, $\alpha=0$, and $\beta>0$, then (3.3) holds. On the other hand, if $p=n$, $1\le q<\infty$, and $\alpha=\beta=0$, then $B_{\lambda_n(Q)}<\infty$ and X has an a.c. norm. Hence the assertion follows either by Theorem 3.3 or by Remark 3.4.

Conversely, if $V^1X(Q)\subset\mathrm{VMO}(Q)$, then also $V^1X(Q)\subset\mathrm{BMO}(Q)$, and, by (i), one of the conditions in (6.1) must be satisfied. However, the only possible choice of the parameters p, q, α, β such that (6.1) is true but (6.3) is false, is $p=n$, $q=\infty$, and $\alpha=\beta=0$. But in this case $V^1X(Q)\not\subset\mathrm{VMO}(Q)$, since, e.g., $\chi_Q(x)\log|x-x_Q|\in V^1X(Q)\setminus\mathrm{VMO}(Q)$, where x_Q is the centre of Q. The inclusion is indeed not uniform if $p=n$, $1\le q<\infty$, and $\alpha=\beta=0$; to see this, observe that (3.4) is not satisfied, and use Theorem 3.3(i). □

Our next example concerns Orlicz spaces. Part (ii) of the following theorem was proved in [Ci], cf. also [T1].

6.3. Theorem. *Let A be a Young function.*

(i) *The embedding $V^1L_A(Q)\hookrightarrow\mathrm{BMO}(Q)$ holds if and only if*

$$\int_0^t \tilde{A}(s)\,ds\le Ct^{n'+1} \quad \text{for some } C \text{ and all } t>1. \tag{6.4}$$

(ii) *The embedding $V^1L_A(Q)\hookrightarrow L_\infty(Q)$ holds if and only if*

$$\int^\infty \frac{\tilde{A}(s)}{s^{n'+1}}\,ds<\infty. \tag{6.5}$$

Proof. (i) Using (2.5) and elementary calculus, we can easily show that

$$\|r^{1/n}\chi_{(0,s)}(r)\|_{L_{\tilde{A}}}=\frac{s^{1/n}}{E^{-1}(1/s)},$$

where

$$E(t)=\frac{n}{t^n}\int_0^t \tilde{A}(y)y^{n-1}\,dy, \quad t>0.$$

Note that E is increasing, since $E(t)=n\int_0^1\tilde{A}(ty)y^{n-1}\,dy$. Thus, when $X=L_A$, (3.2) can be rewritten as

$$\int_0^t \tilde{A}(s)\Big(\frac{s}{t}\Big)^{n-1}ds\le ct^{n'+1}, \quad t>1. \tag{6.6}$$

Furthermore,

$$\int_0^t \tilde{A}(s)\Big(\frac{s}{t}\Big)^{n-1}\,ds \ge \int_{t/2}^t \tilde{A}(s)\Big(\frac{s}{t}\Big)^{n-1}\,ds \ge \frac{1}{2^{n-1}}\int_{t/2}^t \tilde{A}(s)\,ds \ge \frac{1}{2^n}\int_0^t \tilde{A}(s)\,ds,$$

whence (6.6) is equivalent to (6.4). This proves (i).

The proof of (ii) is a consequence of the fact that, in case $X{=}L_A$, (3.7) and (6.5) are equivalent, as a straightforward calculation shows. □

Theorem 6.3 enables us to prove two optimality results in the context of Orlicz spaces.

6.4. Theorem. (i) *The space $L_n(Q)$ is the largest Orlicz space $L_A(Q)$ such that $V^1L_A(Q)\hookrightarrow \mathrm{BMO}(Q)$ (and also such that $V^1L_A(Q)\subset \mathrm{VMO}(Q)$).*

(ii) *There does not exist any largest Orlicz space $L_A(Q)$ such that $V^1L_A(Q)\hookrightarrow L_\infty(Q)$.*

Proof. (i) Obviously (6.4) is satisfied when $A(t){=}t^n$.

Let now A be a Young function such that $L_n(Q)\subsetneq L_A(Q)$. That is (cf. [KR]), $\tilde{A}(t)\ge Ct^{n'}$ for some C and all $t{>}1$, and there exist sequences $t_k\nearrow\infty$, $\lambda_k\nearrow\infty$ such that

$$\tilde{A}(t_k)=\lambda_k t_k^{n'}. \tag{6.7}$$

Now, since $\tilde{A}$ is a Young function, we have

$$\tilde{A}(s)\ge \frac{\tilde{A}(t_k)}{t_k}s, \quad s\ge t_k. \tag{6.8}$$

For $k\in\mathbf{N}$ define

$$z_k=\lambda_k^{\alpha}t_k, \tag{6.9}$$

where α is any fixed number in $(0,n-1)$. Then, by (6.8), (6.7), and (6.9),

$$\int_0^{z_k}\tilde{A}(s)\,ds \ge \int_{t_k}^{z_k}\tilde{A}(s)\,ds \ge \frac{\tilde{A}(t_k)}{t_k}\int_{t_k}^{z_k} s\,ds = \frac{\tilde{A}(t_k)}{t_k}\,\frac{z_k^2-t_k^2}{2} \ge C\lambda_k^{2\alpha+1}t_k^{n'+1}$$

for some $C{>}0$. On the other hand, by (6.9), $z_k^{n'+1}{=}\lambda_k^{\alpha(n'+1)}t_k^{n'+1}$, and therefore (6.4) does not hold, since $\alpha{<}n-1$ is equivalent to $\alpha(n'+1){<}2\alpha+1$.

Let us recall that an alternative proof follows from Theorem 3.1(ii) combined with the well-known fact that there is no Orlicz space $L_A(Q)$ such that $L_n(Q)\subsetneq L_A(Q)\subset L_{n,\infty}(Q)$.

(ii) The proof of this part is patterned on a construction from [KP]. We may assume, with no loss of generality, that $\lambda_n(Q)=1$. Let A be a Young function such that (6.5) holds. We claim that then there is another Young function, B, such that $\widetilde{B}(t)\geq\tilde{A}(t)$ for all $t>0$, $\limsup_{t\to\infty}\widetilde{B}(t)/\tilde{A}(\beta t)=\infty$ for every $\beta>1$, and

$$\int_1^\infty \frac{\widetilde{B}(s)}{s^{n'+1}}\,ds<\infty. \tag{6.10}$$

For such B we would have $L_A(Q)\subsetneq L_B(Q)$ and $V^1L_B(Q)\hookrightarrow L_\infty(Q)$, as required.

To prove our claim, let us set $a_k=(k\log^2 k)^{-1}$, $k\in\mathbf{N}$. For $t\in[1,\infty)$ we define τ by the identity

$$\frac{\tilde{A}(\tau)}{\tau}=a_k t^{n'-1}, \quad t\in[k!,(k+1)!). \tag{6.11}$$

We note that τ is uniquely defined, since the function $\tilde{A}(t)/t$ strictly increases from 0 to ∞ as t goes from 0 to ∞. We claim that for every $\beta>1$

$$\limsup_{t\to\infty}\frac{\tilde{A}(\tau)}{\tau}\frac{t}{\tilde{A}(\beta t)}=\infty. \tag{6.12}$$

Indeed, assume the contrary. Then, for some $\beta>1$ and $K>0$,

$$K^{-1}a_k t^{n'}\leq\tilde{A}(\beta t), \quad t\in[k!,(k+1)!).$$

But then

$$\begin{aligned}\int_1^\infty \frac{\tilde{A}(s)}{s^{n'+1}}\,ds&=\sum_{k=1}^\infty\int_{\beta k!}^{\beta(k+1)!}\frac{\tilde{A}(s)}{s^{n'+1}}\,ds=\beta^{-n'}\sum_{k=1}^\infty\int_{k!}^{(k+1)!}\frac{\tilde{A}(\beta y)}{y^{n'+1}}\,dy\\&\geq\frac{1}{K\beta^{n'}}\sum_{k=1}^\infty a_k\log(k+1)=\infty,\end{aligned}$$

which contradicts (6.5). This proves (6.12).

Now, let $\beta_j\nearrow\infty$ be a fixed sequence. Then, by (6.12), there exists a sequence $t_j\nearrow\infty$ such that $t_j\geq j!$, $t_{j+1}>\tau_j$ (where τ_j corresponds to t_j in the sense of (6.11)), and

$$\lim_{j\to\infty}\frac{\tilde{A}(\tau_j)}{\tau_j}\frac{t_j}{\tilde{A}(\beta_j t_j)}=\infty. \tag{6.13}$$

We define

$$\widetilde{B}(t)=\begin{cases}\tilde{A}(t_j)+\dfrac{\tilde{A}(\tau_j)-\tilde{A}(t_j)}{\tau_j-t_j}(t-t_j), & t\in(t_j,\tau_j),\\ \tilde{A}(t), & \text{otherwise.}\end{cases}$$

Then $\widetilde{B}$ is a Young function and, evidently, $\widetilde{B}(t)\ge\tilde{A}(t)$ for $t\in(0,\infty)$. It follows easily from (6.13) that, for every $j\in\mathbf{N}$, $\tau_j>2t_j$, and therefore also $\tilde{A}(\tau_j)>2\tilde{A}(t_j)$. Hence, using (6.13), we get

$$\frac{\widetilde{B}(2t_j)}{\tilde{A}(\beta_j t_j)}=\frac{\tilde{A}(t_j)+\dfrac{\tilde{A}(\tau_j)-\tilde{A}(t_j)}{\tau_j-t_j}t_j}{\tilde{A}(\beta_j t_j)}\ge\frac{1}{2}\frac{\tilde{A}(\tau_j)t_j}{\tau_j\tilde{A}(\beta_j t_j)}\nearrow\infty.$$

It remains to show (6.10). We have

$$\int_1^\infty\frac{\widetilde{B}(s)}{s^{n'+1}}\,ds\le\int_1^\infty\frac{\tilde{A}(s)}{s^{n'+1}}\,ds+\sum_{j=1}^\infty\frac{\tilde{A}(\tau_j)-\tilde{A}(t_j)}{\tau_j-t_j}\int_{t_j}^{\tau_j}\frac{s-t_j}{s^{n'+1}}\,ds.$$

Further, using (6.11), $t_j\ge j!$, and the monotonicity of $\{a_j\}$, we obtain

$$\sum_{j=1}^\infty\frac{\tilde{A}(\tau_j)-\tilde{A}(t_j)}{\tau_j-t_j}\int_{t_j}^{\tau_j}\frac{s-t_j}{s^{n'+1}}\,ds\le C\sum_{j=1}^\infty\frac{\tilde{A}(\tau_j)}{\tau_j}t_j^{1-n'}\le C\sum_{j=1}^\infty a_j<\infty.$$

Therefore, we get (6.10) on recalling (6.5). The proof is complete. □

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