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Boundary behavior of the pluricomplex Green function

Dan Coman

Abstract. Let Ω be a bounded domain in $\mathbf{C}^n$. This paper deals with the study of the behavior of the pluricomplex Green function $g_\Omega(z,w)$ when the pole w tends to a boundary point w_0 of Ω. We find conditions on Ω which ensure that $\lim_{w\to w_0} g_\Omega(z,w)=0$, uniformly with respect to z on compact subsets of $\bar{\Omega}\setminus\{w_0\}$. Our main result is Theorem 5; it gives a sufficient condition for the above property to hold, formulated in terms of the existence of a plurisubharmonic peak function for Ω at w_0 which satisfies a certain growth condition.

1. Introduction and statement of results

Let Ω be a bounded open set in $\mathbf{C}^n$ and let w be a point in Ω. A plurisubharmonic function v on Ω is said to have a logarithmic pole at w if $v(z)\le\log\|z-w\|+c$, for some constant c and for z in a neighborhood of w. The pluricomplex Green function $g_\Omega(z,w)$ of Ω with pole at w is defined by $g_\Omega(z,w)=\sup v(z)$, where the supremum is taken over the set of negative plurisubharmonic functions v on Ω which have a logarithmic pole at w. This definition, given by Klimek [K1], is in analogy to the one dimensional case, where one obtains in this way the (negative) Green function for the Laplace operator. The function $g_\Omega(\cdot,w)$ is negative and plurisubharmonic in Ω and it has a logarithmic pole at w. It is also decreasing with respect to holomorphic mappings, i.e. $g_{\Omega'}(f(z),f(w))\le g_\Omega(z,w)$, where Ω' is a bounded open set in $\mathbf{C}^m$ and $f:\Omega\to\Omega'$ is a holomorphic mapping. It follows that g_Ω is biholomorphically invariant. If Ω is a hyperconvex domain (i.e. it is bounded and it has a negative continuous plurisubharmonic exhaustion function) and if for $z\in\partial\Omega$ and $w\in\Omega$ we define $g_\Omega(z,w)=0$, then $g_\Omega:\bar{\Omega}\times\Omega\to[-\infty,0]$ is continuous. This result was obtained by Demailly [D].

Let Δ denote the unit disk in $\mathbf{C}$ and let $\varrho(z,w)$ be the Poincaré distance between $z,w\in\Delta$, $\varrho(z,w)=\tanh^{-1}(|z-w|/|1-\bar{w}z|)$. Let $\delta_\Omega(z,w)=\inf\varrho(\xi,\eta)$, where the infimum is taken over all $\xi,\eta\in\Delta$ for which there is an analytic disk $f:\Delta\to\Omega$ with $f(\xi)=z$ and $f(\eta)=w$. In general [K2], for a bounded domain Ω in $\mathbf{C}^n$ one has $g_\Omega(z,w) \le \log\tanh\delta_\Omega(z,w)$; equality holds for all $z\in\Omega$ and for a fixed $w\in\Omega$ if and only if the function $z\mapsto\log\tanh\delta_\Omega(z,w)$ is plurisubharmonic. The results of Lempert [L] show that if Ω is a bounded convex domain in $\mathbf{C}^n$ then $g_\Omega(z,w)=\log\tanh\delta_\Omega(z,w)$ for all $z,w\in\Omega$.

In this paper we study the behavior of the pluricomplex Green function $g_\Omega(z,w)$ as the pole w approaches a boundary point w_0 of Ω. In the remainder of this section we state our results. The proofs of these results and an example are given in Section 2.

Let us start with some definitions. Let Ω be a bounded open set in $\mathbf{C}^n$ and let $w_0\in\partial\Omega$.

Definition. We say that Ω has the property (P) at w_0 if for every sequence of points $\{w_m\}_{m>0}\subset\Omega$ which converges to w_0 and for every compact set $K\subset\bar{\Omega}\setminus\{w_0\}$ one has $g_\Omega(z,w_m)\to 0$ as $m\to\infty$, uniformly for $z\in K\cap\Omega$.

Definition. We say that w_0 is a weak peak point for Ω if there exists a holomorphic map $h:\Omega\to\Delta$ such that $\lim_{z\to w_0}|h(z)|=1$ and $\limsup_{z\to q}|h(z)|<1$, for every $q\in\partial\Omega$, $q\neq w_0$. We call w_0 a local weak peak point for Ω if there exists a neighborhood U of w_0 such that w_0 is a weak peak point for $\Omega\cap U$.

Our first result is that the property (P) of Ω at $w_0\in\partial\Omega$ is local. We have the following theorem.

Theorem 1. *Let Ω be a bounded open set in $\mathbf{C}^n$ and let $w_0\in\partial\Omega$. The following are equivalent:*

(i) *Ω has the property* (P) *at w_0;*

(ii) *for every neighborhood U of w_0, $\Omega\cap U$ has the property* (P) *at w_0;*

(iii) *there is a neighborhood U of w_0 such that $\Omega\cap U$ has the property* (P) *at w_0.*

This theorem yields the following corollary.

Corollary 2. *If Ω is a bounded open set in $\mathbf{C}^n$ and if $w_0\in\partial\Omega$ is a local weak peak point for Ω then Ω has the property* (P) *at w_0.*

The next two results come from the relation between the functions g_Ω and δ_Ω. The first one gives examples of domains for which property (P) fails. The second result characterizes completely the convex domains which satisfy property (P), and it can also be viewed as a partial converse to the first one.

Proposition 3. *Let Ω be a bounded domain in $\mathbf{C}^n$, let $w_0\in\partial\Omega$, and assume that $\partial\Omega$ is of class C^1 near w_0 and that there exists a nonconstant analytic disk $f:\Delta\to\bar{\Omega}$ such that $w_0\in f(\Delta)$. Then Ω does not have the property* (P) *at w_0.*

Proposition 4. *Let Ω be a bounded domain in $\mathbf{C}^n$ and assume that $g_\Omega(z,w) = \log\tanh\delta_\Omega(z,w)$ for all $z,w\in\Omega$. (By Lempert's theorem this is the case if Ω is convex.) For $w_0\in\partial\Omega$ let A_{w_0} be the union of all analytic disks contained in $\bar\Omega$ and passing through w_0. If $\{w_m\}$ is a sequence of points in Ω which converges to w_0 and if K is a compact subset of $\bar\Omega$ such that $K\cap A_{w_0}=\emptyset$ then $g_\Omega(z,w_m)\to 0$ uniformly on $K\cap\Omega$ as $m\to\infty$. In particular, if $A_{w_0}=\{w_0\}$ then Ω has the property* (P) *at w_0.*

Finally, our last result gives a sufficient condition for Ω to have the property (P) at $w_0\in\partial\Omega$ in terms of the existence of a plurisubharmonic peak function ϱ at w_0. Let us recall first the following definition.

Definition. Let Ω be a bounded domain in $\mathbf{C}^n$ and let $w_0\in\partial\Omega$. A function ϱ is called a plurisubharmonic peak function for Ω at w_0 if ϱ is plurisubharmonic in Ω and continuous on $\bar\Omega$, $\varrho(w_0)=0$ and $\varrho(z)<0$ for $z\in\bar\Omega\setminus\{w_0\}$.

If ϱ is a plurisubharmonic peak function for Ω at w_0 and $r\in\left(0,\frac{1}{2}\right)$, we define $N_\varrho(r)$ by

$$N_\varrho(r) = \max\left\{\frac{\log|\varrho(z)|}{\log\|z-w_0\|} : z\in\bar\Omega,\ r\le\|z-w_0\|\le\frac{1}{2}\right\}.$$

This number is the smallest exponent N for which the inequality

$$\varrho(z)\le -\|z-w_0\|^N$$

holds for all $z\in\bar\Omega$ with $r\le\|z-w_0\|\le\frac{1}{2}$. We have the following theorem.

Theorem 5. *Let Ω be a bounded domain in $\mathbf{C}^n$ and let w_0 be a boundary point of Ω. Assume that there exists a plurisubharmonic peak function ϱ for Ω at w_0 such that:*

(i) *ϱ is Hölder continuous at w_0, i.e. there are constants $c>0$ and $\gamma\in(0,1]$ such that $\varrho(z)\ge -c\|z-w_0\|^\gamma$ for all $z\in\bar\Omega$;*

(ii) *$N_\varrho(r)=O(\log\log(1/r))$ as $r\searrow 0$.*

Then Ω has the property (P) *at w_0.*

The hypotheses of this theorem are satisfied for bounded pseudoconvex domains with real analytic boundary (the existence of the required plurisubharmonic peak function follows from Theorems 2 and 3 in [DF]). They are also satisfied in the more general case when Ω is pseudoconvex with smooth boundary and w_0 is a point of finite type (here plurisubharmonic peak functions exist by a theorem in [C]). In both these cases the quantity $N_\varrho(r)$ is bounded as r tends to 0. It is not hard to construct a bounded pseudoconvex domain Ω in $\mathbf{C}^2$ such that $w_0=0\in\partial\Omega$, $\partial\Omega$ is C^∞ smooth near 0, 0 is not a point of finite type, Ω is not convex near 0 (recall that the case of bounded convex domains is settled in Proposition 4), but Theorem 5 applies and Ω has the property (P) at 0. This is outlined in the example at the end of Section 2.

Remark. Let Ω be a bounded domain in $\mathbf{C}^n$ and let $z_0 \in \partial\Omega$. As in the one dimensional case, one can show that if there exists a plurisubharmonic peak function for Ω at z_0, then $\lim_{z\to z_0} g_\Omega(z,w)=0$, uniformly for $w\in K\cap\Omega$, for any compact $K\subset\bar{\Omega}\setminus\{z_0\}$. So when $\Omega\subset\mathbf{C}$ this is equivalent to property (P) at z_0, by the symmetry of the Green function ($g_\Omega(z,w)=g_\Omega(w,z)$). In dimensions $n>1$ it is known that the pluricomplex Green function g_Ω is in general not symmetric, not even when Ω is a smoothly bounded strongly pseudoconvex domain [BD]. Theorem 5 shows that a sufficient condition for property (P) can still be given in terms of plurisubharmonic peak functions which have some special properties.

2. Proofs

Proof of Theorem 1. The implication (ii) $\Rightarrow$ (iii) is obvious and the implication (i) $\Rightarrow$ (ii) is clearly true, since $g_\Omega(z,w)\le g_{\Omega\cap U}(z,w)$ for all $z,w\in\Omega\cap U$. So we only need to prove that (iii) implies (i).

Let K be a compact in $\bar{\Omega}$ such that w_0 is not in K and let U be a neighborhood of w_0 such that $\Omega\cap U$ has the property (P) at w_0. Let r and r' be positive numbers such that $B(w_0,r)\subseteq U$ and $K\cap\bar{B}(w_0,r')=\emptyset$; here $B(w_0,r)$ is the open ball centered at w_0 and of radius r in $\mathbf{C}^n$. Let $\Omega_r=\Omega\cap B(w_0,r)$. Then $\Omega_r\subseteq\Omega\cap U$, so Ω_r has the property (P) at w_0. We choose $R>0$ big enough so that $\Omega\subseteq B(w_0,R)$. Finally we let $\{w_m\}_{m>0}$ be a sequence of points in Ω such that $w_m\to w_0$ as $m\to\infty$.

We fix two positive numbers r_1 and r_2 such that

$$0<r_1<r_2<\min\{r,r'\},$$

and we let $S_j=\{z\in\Omega:\|z-w_0\|=r_j\}$, $j=1,2$. Let

$$c_m=\inf_{z\in S_1} g_{\Omega_r}(z,w_m),$$

where m is large enough so that $\|w_m-w_0\|<r_1$. Let

$$v_m(z)=c_m\frac{\log(\|z-w_0\|/r_2)}{\log(r_1/r_2)}.$$

The functions v_m are plurisubharmonic and

$$\begin{aligned} v_m(z)&=c_m\le g_{\Omega_r}(z,w_m), && \text{if } z\in S_1,\\ v_m(z)&=0>g_{\Omega_r}(z,w_m), && \text{if } z\in S_2. \end{aligned}$$

Hence the function

$$u_m(z)=\begin{cases} g_{\Omega_r}(z,w_m), & \text{if } z\in\Omega_{r_1},\\ \max\{v_m(z),g_{\Omega_r}(z,w_m)\}, & \text{if } z\in\Omega_{r_2}\backslash\Omega_{r_1},\\ v_m(z), & \text{if } z\in\Omega\backslash\Omega_{r_2},\end{cases}$$

is plurisubharmonic in Ω with a logarithmic pole at w_m. Also

$$u_m(z)<d_m=c_m\frac{\log(R/r_2)}{\log(r_1/r_2)}\quad \text{for all } z\in\Omega,$$

so $g_\Omega(z,w_m)\ge u_m(z)-d_m$. If $\|z-w_0\|\ge r_2$, hence in particular if $z\in K$, then $u_m(z)>0$. We conclude that

$$g_\Omega(z,w_m)\ge -d_m\quad \text{for all } z\in K\cap\Omega.$$

Since Ω_r has the property (P) at w_0 it follows that $c_m\to 0$ as $m\to\infty$, so $d_m\to 0$ and thus $g_\Omega(z,w_m)\to 0$ uniformly on $K\cap\Omega$ when $m\to\infty$. □

Proof of Corollary 2. Let U be a neighborhood of w_0 and let $f:\Omega\cap U\to\Delta$ be a holomorphic function satisfying

$$\lim_{z\to w_0}|f(z)|=1,\quad \limsup_{z\to q}|f(z)|<1 \tag{2.1}$$

for all the points $q\neq w_0$ in the boundary of $\Omega\cap U$. It is enough to show that $\Omega\cap U$ has the property (P) at w_0, so let K be a compact subset of $\overline{\Omega\cap U}$ which does not contain w_0 and let $\{w_m\}$ be a sequence of points in $\Omega\cap U$ which converges to w_0.

We have that

$$g_\Delta(f(z),f(w_m))=\tfrac{1}{2}\log(1+E(z,w_m))\le g_{\Omega\cap U}(z,w_m),$$

where

$$E(z,w_m)=\left|\frac{f(z)-f(w_m)}{1-\overline{f(w_m)}f(z)}\right|^2-1=\frac{(|f(z)|^2-1)(1-|f(w_m)|^2)}{|1-\overline{f(w_m)}f(z)|^2}.$$

Since the compact K does not contain w_0 it follows from (2.1) that there exists a positive number $\alpha<1$ such that $|f(z)|<\alpha$ for all $z\in K\cap\Omega\cap U$. For such z we see that

$$0\ge E(z,w_m)\ge\frac{(|f(z)|^2-1)(1-|f(w_m)|^2)}{(1-\alpha)^2}\ge\frac{|f(w_m)|^2-1}{(1-\alpha)^2},$$

so $E(z,w_m)\to 0$ uniformly on $z\in K\cap\Omega\cap U$ as $m\to\infty$, hence the same is true for $g_{\Omega\cap U}(z,w_m)$. □

Proof of Proposition 3. Let $\vec{\nu}$ be the inward pointing normal at w_0 to $\partial\Omega$. By restricting f around some $t_0 \in f^{-1}(w_0)$ we may assume that there is a $\delta>0$ such that the functions $f_t = f + t\vec{\nu}$, $0<t\leq\delta$, carry Δ into Ω. We reparametrize Δ such that $f(0)=q\neq w_0$ for some $q\in\bar{\Omega}$. Then there is an $\alpha\in\Delta$, $\alpha\neq 0$, such that $f(\alpha)=w_0$. Hence

$$g_\Omega(q+t\vec{\nu}, w_0+t\vec{\nu}) \leq \log\tanh\delta_\Omega(q+t\vec{\nu}, w_0+t\vec{\nu}) \leq \log|\alpha| < 0$$

for all t, $0<t\leq\delta$. We finally let $K=\{q+t\vec{\nu}: 0\leq t\leq\delta\}$ and $w_j = w_0 + (1/j)\vec{\nu}$. Then $g_\Omega(z, w_j)$ does not converge uniformly to 0 on $K\cap\Omega$ as $j\to\infty$. □

Proof of Proposition 4. We assume that $g_\Omega(z, w_m)$ does not converge uniformly to 0 on $K\cap\Omega$. It follows that there exists an $\varepsilon>0$ such that, after passing to a subsequence, there are points $z_m\in K\cap\Omega$ satisfying $g_\Omega(z_m, w_m) < -2\varepsilon$. Since $g_\Omega = \log\tanh\delta_\Omega$ we see from the definition of δ_Ω that for each m there is an analytic disk $f_m: \Delta\to\Omega$ such that $f_m(0)=z_m$, $f_m(t_m)=w_m$ for some $t_m\in\Delta$, and $\log|t_m| < -\varepsilon$. Since Ω is bounded, K is compact and $\{t_m\}\subset\subset\Delta$, it follows that, after passing to a subsequence, $\{f_m\}$ converges locally uniformly to a function $f: \Delta\to\bar{\Omega}$, $\{z_m\}$ converges to some $z\in K$ and $\{t_m\}$ converges to some $t\in\Delta$. Hence $f(0)=z\in K$, $f(t)=w_0$, so $z\in K\cap A_{w_0}$, a contradiction. □

Proof of Theorem 5. We fix $R>0$ such that the diameter of Ω is less than R and we let $\{w_m\}_{m\geq 1}\subset\Omega$ be a sequence of points converging to w_0. For $a>0$ we set $\Omega_a = \Omega\cap B(w_0, a)$. For $r\in(0,\frac{1}{2})$ we define

$$\alpha(r) = 1 - \frac{\gamma}{N_\varrho(r)} \in (0,1). \tag{2.2}$$

Let K be a compact subset of $\bar{\Omega}$ which does not contain w_0 and fix $R'=R'(K)>0$ such that $R'<\frac{1}{2}$ and $K\cap\bar{B}(w_0, R')=\emptyset$. The proof is done by constructing for each $\varepsilon>0$ and $m\geq m(\varepsilon)$, where $m(\varepsilon)$ is large enough, a plurisubharmonic function ψ_m on Ω such that $\psi_m(z)\leq g_\Omega(z, w_m)$ for $z\in\Omega$, and $\psi_m(z) > -\varepsilon$ for all $z\in K\cap\Omega$.

We proceed in three steps. In the first step, given two radii r, r', $0<r<r'<\frac{1}{2}$, we use the function ϱ to construct a plurisubharmonic function $v_m(z; r, r')$ which satisfies $v_m(z; r, r')\leq g_\Omega(z, w_m)$ on Ω, and $v_m(z; r, r')\geq -h_m(r, r')\log(R/r')$ for $z\in\Omega$, $\|z-w_0\|\geq r'$; the number $h_m(r, r')$ is given by

$$h_m(r, r') = \frac{\alpha(r)\log\dfrac{1}{r} + c + \dfrac{4\|w_m - w_0\|}{r}}{\log\dfrac{r'}{r}} \tag{2.3}$$

and has the property that $0<h_m(r, r')<1$ if $r\ll r'$ and m is large enough or $m=0$.

In the second step, given a sequence of radii $0<r_j<r_{j-1}<...<r_1<R'$, we use the functions v_m to construct by induction on k, $1\le k\le j$, a plurisubharmonic function $\omega_m^j(z)$ which satisfies $\omega_m^j(z)\le g_\Omega(z,w_m)$ on Ω, and $\omega_m^j(z)\ge -H_m^j\log(R/R')$ for $z\in\Omega$ with $\|z-w_0\|\ge R'$ and for m such that $\|w_m-w_0\|\le\frac{1}{4}r_j$. Here H_m^j is given by

$$(2.4)\qquad H_m^j=h_m(r_j,r_{j-1})h_0(r_{j-1},r_{j-2})\dots h_0(r_1,R').$$

Finally, in the third step we use hypothesis (ii) of the theorem to show that for any $\varepsilon>0$ we can choose an integer j large enough and radii $r_j\ll r_{j-1}\ll\dots\ll r_1\ll R'$ such that if $\|w_m-w_0\|\le\frac{1}{4}r_j$ then $H_m^j<\varepsilon/\log(R/R')$. To complete the proof we just set $\psi_m=\omega_m^j$.

Step 1. We fix two radii r and r' such that $0<r<r'<\frac{1}{2}$, and we define for $m\ge 0$

$$u_m(z;r,r')=\log\frac{\|z-w_m\|}{r}+\frac{1}{r^\gamma}\varrho(z)-\alpha(r)\log\frac{1}{r},$$

where $z\in\Omega$ and $\alpha(r)$ is defined in (2.2). Since $|\log(1+x)|\le 2|x|$ for all real numbers x with $|x|\le\frac{1}{2}$, we note that for z with $\|z-w_0\|\ge r$ we have

$$\begin{aligned}\left|\log\frac{\|z-w_m\|}{r}-\log\frac{\|z-w_0\|}{r}\right| &= \left|\log\left(1+\frac{\|z-w_m\|-\|z-w_0\|}{\|z-w_0\|}\right)\right|\\ &\le\frac{2}{r}\big|\|z-w_m\|-\|z-w_0\|\big|\le\frac{2\|w_m-w_0\|}{r},\end{aligned}$$

provided that m is sufficiently large. It follows that

$$(2.5)\qquad u_0(z;r,r')-\frac{4\|w_m-w_0\|}{r}\le u_m(z;r,r')-\frac{2\|w_m-w_0\|}{r}\le u_0(z;r,r')$$

for z as specified above. The functions u_m are plurisubharmonic in Ω and for $m>0$ they have a logarithmic pole at w_m. We claim that u_0 is negative in $\Omega_{1/2}$. This is obvious for z with $\|z-w_0\|\le r$ and if we set $x=\|z-w_0\|$, for $\|z-w_0\|\ge r$, and use the definition of $N_\varrho(r)$ we see that

$$u_0(z;r,r')\le f(x)=\log\frac{x}{r}-\frac{1}{r^\gamma}x^{N_\varrho(r)}-\alpha(r)\log\frac{1}{r}.$$

But f has an absolute maximum on the positive real axis at the point x_0 given by $x_0^{N_\varrho(r)}=r^\gamma/N_\varrho(r)$ and $f(x_0)=-(1+\log N_\varrho(r))/N_\varrho(r)<0$, so the claim is proved. We also note that for $z\in\bar{\Omega}$ with $\|z-w_0\|=r$ hypothesis (i) yields

$$u_0(z;r,r')\ge-\alpha(r)\log\frac{1}{r}-c$$

and hence

$$u_0(z;r,r')-\frac{4\|w_m-w_0\|}{r}\geq -\alpha(r)\log\frac{1}{r}-c-\frac{4\|w_m-w_0\|}{r}.$$

By (2.5) and by the above relation it now follows that, for $m=0$ or for m such that $\|w_m-w_0\|\leq\frac{1}{4}r$, the function $\tilde{v}_m$ defined below is plurisubharmonic in Ω:

$$\tilde{v}_m(z;r,r')=\begin{cases}\tilde{u}(z), & \text{if } z\in\Omega_r,\\ \max\left\{\tilde{u}(z), h_m(r,r')\log\frac{\|z-w_0\|}{r'}\right\}, & \text{if } z\in\Omega_{r'}\setminus\Omega_r,\\ h_m(r,r')\log\frac{\|z-w_0\|}{r'}, & \text{if } z\in\Omega\setminus\Omega_{r'}.\end{cases}\tag{2.6}$$

Here $\tilde{u}(z)=u_m(z;r,r')-2\|w_m-w_0\|/r$ and $h_m(r,r')$ is defined by (2.3) for m as specified above. We note that $\tilde{v}_m(\,\cdot\,;r,r')$ is negative if $\|z-w_0\|<r'$ and positive if $\|z-w_0\|>r'$ but the function

$$v_m(z;r,r')=\tilde{v}_m(z;r,r')-h_m(r,r')\log\frac{R}{r'}\tag{2.7}$$

is negative and plurisubharmonic on Ω. Also, for $m>0$, v_m has a logarithmic pole at w_m and by (2.6) we have that

$$v_m(z;r,r')\leq -h_m(r,r')\log\frac{R}{r'}\tag{2.8}$$

for $z\in\Omega$ with $\|z-w_0\|\leq r'$. As $u_0(z;r,r')<0$ for all $z\in\Omega_{1/2}$ it follows by (2.5) and (2.6) that there exists a positive number $\nu(r,r')$ such that

$$\tilde{v}_m(z;r,r')=h_m(r,r')\log\frac{\|z-w_0\|}{r'}\tag{2.9}$$

provided that $z\in\Omega$, $\|z-w_0\|\geq r'-\nu(r,r')$, and m is as specified above.

Step 2. Let us fix a positive integer j and a sequence of radii $r_1,r_2,\dots,r_j$ satisfying $0<r_j<r_{j-1}<\dots<r_1<R'$. We also fix an integer m such that $\|w_m-w_0\|\leq\frac{1}{4}r_j$. For $k\in\{1,\dots,j\}$ we set

$$\widetilde{H}_k=\begin{cases}h_m(r_j,r_{j-1}), & \text{if } k=j,\\ h_m(r_j,r_{j-1})h_0(r_{j-1},r_{j-2})\dots h_0(r_k,r_{k-1}), & \text{if } 1\leq k\leq j-1,\end{cases}$$

where r_0 is taken to be R'. Note that $\widetilde{H}_1 = H^j_m$, where H^j_m was defined in (2.4). By induction, we construct for each $k \in \{1, \dots, j\}$ a negative plurisubharmonic function ϕ_k on Ω with a logarithmic pole at w_m and such that if $z \in \Omega$ then

$$\phi_k(z) = \widetilde{H}_k \log \frac{\|z-w_0\|}{R} \quad \text{for } \|z-w_0\| \geq r_{k-1} - \nu(r_k, r_{k-1}), \tag{2.10}$$

$$\phi_k(z) \leq -\widetilde{H}_k \log \frac{R}{r_{k-1}} \quad \text{for } \|z-w_0\| \leq r_{k-1}, \tag{2.11}$$

$$\phi_k(z) \geq -\widetilde{H}_k \log \frac{R}{r_{k-1}} \quad \text{for } \|z-w_0\| \geq r_{k-1}. \tag{2.12}$$

Note that (2.12) is an immediate consequence of (2.10).

We start with $k=j$ and we set $\phi_j(z) = v_m(z; r_j, r_{j-1})$. By (2.8), relation (2.11) holds for ϕ_j and (2.10) is also satisfied, as it is easily seen from the definition of v_m (relation (2.7)) and from (2.9).

We assume now that for $k \in \{1, \dots, j-1\}$ we have constructed a negative plurisubharmonic function ϕ_{k+1} on Ω, with a logarithmic pole at w_m and such that ϕ_{k+1} satisfies (2.10) and (2.11). Then for $z \in \Omega$ with $\|z-w_0\| \geq r_k - \nu(r_{k+1}, r_k)$ we note by (2.10) and by the definition of u_0 that

$$\phi_{k+1}(z) + \widetilde{H}_{k+1} \log \frac{R}{r_k} + \widetilde{H}_{k+1} \frac{\varrho(z)}{r_k^\gamma} - \alpha(r_k) \widetilde{H}_{k+1} \log \frac{1}{r_k} = \widetilde{H}_{k+1} u_0(z; r_k, r_{k-1}).$$

Let $E(z)$ denote the left-hand side of the above equality. It follows from this and from the definition (2.6) of $\tilde{v}_0$ that the function

$$\tilde{\phi}_k(z) = \begin{cases} E(z), & \text{if } \|z-w_0\| < r_k, \\ \widetilde{H}_{k+1} \tilde{v}_0(z; r_k, r_{k-1}), & \text{if } \|z-w_0\| \geq r_k, \end{cases}$$

is well defined and plurisubharmonic in Ω, with a logarithmic pole at w_m. Inequality (2.11) for ϕ_{k+1} shows that $\tilde{\phi}_k$ is negative for $z \in \Omega$ with $\|z-w_0\| \leq r_k$, and by the definition (2.6) of $\tilde{v}_0$ it follows that $\tilde{\phi}_k$ is negative for $z \in \Omega$ with $r_k < \|z-w_0\| < r_{k-1}$, and is increasing in $\|z-w_0\|$ for $z \in \Omega$ with $\|z-w_0\| \geq r_{k-1}$. We set

$$\phi_k(z) = \tilde{\phi}_k(z) - \widetilde{H}_k \log \frac{R}{r_{k-1}}, \quad z \in \Omega.$$

The function ϕ_k is then negative and plurisubharmonic on Ω, with a logarithmic pole at w_m. Equality (2.9) for $\tilde{v}_0(z; r_k, r_{k-1})$ and the definition of $\widetilde{H}_k$ show that (2.10) holds for ϕ_k. Since $\tilde{\phi}_k$ is negative for $z \in \Omega$ with $\|z-w_0\| < r_{k-1}$, it follows that ϕ_k satisfies (2.11) as well.

We conclude by induction that $\omega_m^j(z)=\phi_1(z)$ is a negative plurisubharmonic function on Ω with a logarithmic pole at w_m and such that, by (2.12) (recall that $\widetilde{H}_1=H_m^j$),

$$\omega_m^j(z)\geq -H_m^j\log\frac{R}{R'}$$

for all $z\in\Omega$ with $\|z-w_0\|\geq R'$, and thus for all $z\in K\cap\Omega$. The definition of the Green function $g_\Omega(z,w_m)$ shows that $g_\Omega(z,w_m)\geq\omega_m^j(z)$ for all $z\in\Omega$.

Step 3. Let us fix again two radii r, r', $0<r<r'<\frac{1}{2}$. We first note that if $m=0$, or if m is such that $\|w_m-w_0\|\leq\frac{1}{4}r$, then

$$h_m(r,r')\leq l(r,r')=\frac{\alpha(r)\log\dfrac{1}{r}+c+1}{\log\dfrac{r'}{r}}. \tag{2.13}$$

We claim that for any $r'>0$ the inequality

$$l(r,r')\leq 1-\frac{\gamma}{2N_\varrho(r)} \tag{2.14}$$

holds provided that r is sufficiently small. Indeed, the above inequality is equivalent to

$$\log\frac{1}{r'}+c+1\leq\frac{\gamma}{2N_\varrho(r)}\left(\log\frac{1}{r}+\log\frac{1}{r'}\right),$$

which holds if and only if $N_\varrho(r)=o(\log(1/r))$ as $r\searrow 0$. Thus hypothesis (ii) shows that our claim is true.

Next, we will construct a decreasing sequence of radii $\{r_j\}_{j\geq 0}$, $0<...<r_j<r_{j-1}<...<r_1<r_0=R'$, by choosing $r_1\ll R'$ and then inductively defining $r_j\ll r_{j-1}$ such that for each j inequality (2.14) holds with $r=r_j$ and $r'=r_{j-1}$. In view of the above, it suffices to choose $r_j\ll r_{j-1}$ such that

$$\log\frac{1}{r_{j-1}}+c+1\leq\frac{\gamma}{2N_\varrho(r_j)}\left(\log\frac{1}{r_j}+\log\frac{1}{r_{j-1}}\right). \tag{2.15}$$

Since $N_\varrho(r)=O(\log\log(1/r))$ as $r\searrow 0$ we can find a positive constant p such that $N_\varrho(r)\leq p\log\log(1/r)$ for all r sufficiently small. We define r_j by

$$\log\log\frac{1}{r_j}=2(j+j_0)\log(j+j_0)$$

for all $j\geq 1$, where j_0 is a fixed large integer such that $r_1<R'$. Then $\log(1/r_j)=e^{2(j+j_0)\log(j+j_0)}$, so (2.15) holds if the following inequality is true:

$$e^{2(j+j_0-1)\log(j+j_0-1)}+c+1\leq\frac{\gamma(e^{2(j+j_0)\log(j+j_0)}+e^{2(j+j_0-1)\log(j+j_0-1)})}{4p(j+j_0)\log(j+j_0)}.$$

This last inequality is equivalent to

$$1+(c+1)e^{-2(j+j_0-1)\log(j+j_0-1)} \leq \frac{\gamma[1+e^{2(j+j_0)\log(j+j_0/j+j_0-1)}(j+j_0-1)^2]}{4p(j+j_0)\log(j+j_0)},$$

which is clearly satisfied for all $j\geq 1$ provided that j_0 is large enough. For $\{r_j\}$ defined in this way we have by (2.13) that

$$h_m(r_j, r_{j-1}) \leq 1-\frac{\gamma}{2N_\varrho(r_j)} \leq 1-\frac{\gamma}{2p\log\log\dfrac{1}{r_j}} = 1-\frac{\gamma}{4p(j+j_0)\log(j+j_0)}$$

for all $j\geq 1$. Hence for such j we have

$$H_m^j \leq \prod_{k=1}^{j}\left(1-\frac{\gamma}{4p(k+j_0)\log(k+j_0)}\right).$$

We finally note that $\prod_{k=2}^{\infty}(1-\gamma/4pk\log k)=0$, since $\sum_{k=2}^{\infty}1/k\log k=\infty$, and the proof is complete. □

Example. For $z=(z_1,z_2)\in\mathbf{C}^2$ we write $z_j=x_j+iy_j$, $j=1,2$. Let $f,g\colon\mathbf{R}\to[0,\infty)$ be C^∞ even functions vanishing to infinite order at 0 and satisfying

(i) $f''(x)>0$, $g''(x)>0$ for all $x\neq 0$;

(ii) $g'(x^2)>f''(x)$ for $x>0$;

(iii) $f(x)\geq x^{\log\log(1/x)}$ for $0<x<\delta$, where δ is some positive number.

We set

$$\phi(z_1,z_2)=f(x_1)+f(y_1)+g(x_1y_1)+f(x_2)+f(y_2)+g(x_2y_2),$$
$$\Phi(z_1,z_2)=x_2+\phi(z_1,z_2).$$

A simple computation of the Levi form of Φ shows that for $z,t\in\mathbf{C}^2$

$$\langle L\Phi(z)t,t\rangle \geq \tfrac{1}{4}f''(\alpha(z))\|t\|^2,$$

where $\alpha(z)=\min\{\max(|x_1|,|y_1|),\max(|x_2|,|y_2|)\}$. Thus Φ is plurisubharmonic on $\mathbf{C}^2$, and actually it is strictly plurisubharmonic at all the points z with $z_1\neq 0$ and $z_2\neq 0$.

The open set $\{z\in\mathbf{C}^2:\Phi(z)<0\}$ is pseudoconvex, it contains the origin on its boundary, and its boundary is C^∞ smooth near the origin. So we can choose a constant $a>0$ such that the set Ω defined by $\Omega=\{z\in B(0,a):\Phi(z)<0\}$ is a bounded pseudoconvex domain and $\partial\Omega$ is a C^∞ smooth hypersurface near $0\in\partial\Omega$, described by the equation $\Phi(z)=0$.

Let $z=(z_1,z_2)$, with $z_1=x_1(1-i)$, be a point on the boundary of Ω (note that one can choose points of this form arbitrarily close to 0). Since f', g' are odd functions we have $(\partial\Phi/\partial x_1+\partial\Phi/\partial y_1)(z)=0$, so the vector $v=(1,1,0,0)\in\mathbf{R}^4$ is in the real tangent plane of $\partial\Omega$ at z. Using again the fact that f', g' are odd functions and f'', g'' are even functions, we see that the real Hessian of Φ evaluated at z and v is

$$\langle H\Phi(z)v,v\rangle=2[f''(x_1)-g'(x_1^2)]<0,$$

so Ω is not convex near 0.

We note that $\phi(z)\geq 0$, since f, g are nonnegative, and $\phi(z)=0$ if and only if $z=0$. Hence the function $\varrho(z_1,z_2)=\operatorname{Re} z_2+\frac{1}{2}\phi(z_1,z_2)$ is a plurisubharmonic peak function for Ω at 0. Indeed, for $z\in\bar{\Omega}$ we have

$$\varrho(z)=\operatorname{Re} z_2+\tfrac{1}{2}\phi(z)\leq -\tfrac{1}{2}\phi(z). \tag{2.16}$$

As f is even, convex and increasing for $x>0$ and $g\geq 0$ we have

$$f(x_1)+f(y_1)+g(x_1y_1)\geq 2f\big(\tfrac{1}{2}(|x_1|+|y_1|)\big)\geq 2f\big(\tfrac{1}{2}|z_1|\big),$$

and hence

$$\phi(z_1,z_2)\geq 2\big[f\big(\tfrac{1}{2}|z_1|\big)+f\big(\tfrac{1}{2}|z_2|\big)\big]\geq 4f\big(\tfrac{1}{4}(|z_1|+|z_2|)\big)\geq 4f\big(\tfrac{1}{4}\|z\|\big).$$

By the choice of f and by (2.16) we get

$$\varrho(z)\leq -2f\big(\tfrac{1}{4}\|z\|\big)\leq -2\big(\tfrac{1}{4}\|z\|\big)^{\log\log(4/\|z\|)},$$

so $N_\varrho(r)=O(\log\log(1/r))$ as $r\searrow 0$ and Theorem 5 applies. Finally, by considering the analytic disks $\gamma_m(\zeta)=(\zeta,\zeta^m)$, for $m=1,2,\dots$, we see that the vanishing order of $\Phi\circ\gamma_m$ at $\zeta=0$ is m, so 0 is not a point of finite type of $\partial\Omega$.

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Dan Coman
Department of Mathematics
"Babes–Bolyai" University
RO-3400 Cluj-Napoca
Romania
Current address:
Department of Mathematics
University of Notre Dame
Notre Dame, IN 46556-5683
U.S.A.
email: Dan.F.Coman.2@nd.edu