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The harmonic Bergman kernel and the Friedrichs operator

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Abstract. The harmonic Bergman kernel Q_Ω for a simply connected planar domain Ω can be expanded in terms of powers of the Friedrichs operator F_Ω if $\|F_\Omega\|<1$ in operator norm. Suppose that Ω is the image of a univalent analytic function ϕ in the unit disk with $\phi'(z)=1+\psi(z)$ where $\psi(0)=0$. We show that if the function ψ belongs to a space $\mathcal{D}_s(\mathbf{D})$, $s>0$, of Dirichlet type, then provided that $\|\psi\|_\infty<1$ the series for Q_Ω also converges pointwise in $\bar{\Omega}\times\bar{\Omega}\setminus\triangle(\partial\Omega)$, and the rate of convergence can be estimated. The proof uses the eigenfunctions of the Friedrichs operator as well as a formula due to Lenard on projections in Hilbert spaces. As an application, we show that for every $s>0$ there exists a constant $C_s>0$ such that if $\|\psi\|_{\mathcal{D}_s(\mathbf{D})}\le C_s$, then the biharmonic Green function for $\Omega=\phi(\mathbf{D})$ is positive.

1. Introduction

Let Ω be a bounded simply connected domain in the complex plane $\mathbf{C}$, and let $L^2(\Omega)$ denote the space of functions f on Ω for which

$$\|f\|_\Omega^2=\int_\Omega |f(z)|^2\,d\Sigma(z)<\infty,$$

where $d\Sigma(z)=\pi^{-1}\,dx\,dy$, $z=x+iy$, is the Lebesgue measure normalized so that the unit disk $\mathbf{D}$ has mass 1. We denote the Bergman spaces of analytic, anti-analytic, and harmonic functions on Ω by $A^2(\Omega)$, $\overline{A^2(\Omega)}$, and $HL^2(\Omega)$; they are defined as the intersection of $L^2(\Omega)$ with the corresponding class of functions on Ω. It is evident that the sum $A^2(\Omega)+\overline{A^2(\Omega)}$ is a subspace of the harmonic Bergman space $HL^2(\Omega)$ on Ω, and one can show that equality $A^2(\Omega)+\overline{A^2(\Omega)}=HL^2(\Omega)$ holds if and only if there exists a constant $\theta<1$ such that

$$\left|\int_\Omega f^2(z)\,d\Sigma(z)\right|\le\theta\int_\Omega |f(z)|^2\,d\Sigma(z) \tag{1.1}$$

for all $f \in A^2(\Omega)$ with zero mean value in Ω, that is, $\int_\Omega f\,d\Sigma = 0$ (see Proposition 4.6 in [7]). This is usually referred to as the *Friedrichs inequality*. Under the assumption that the boundary of Ω has a continuous tangent except at a finite number of corners, Friedrichs proved in [1] that the inequality (1.1) holds for $A^2(\Omega)$. Later it has been shown that it is enough that Ω satisfies the interior cone condition [8], [9].

Related to the inequality is the *Friedrichs operator* defined on $A^2(\Omega)$ as the conjugate-linear operator

$$F_\Omega f(z) = \int_\Omega K_\Omega(z,\zeta)\bar{f}(\zeta)\,d\Sigma(\zeta), \quad z \in \Omega, \tag{1.2}$$

where K_Ω is the analytic Bergman kernel. If the operator F_Ω is compact (as it is if the domain has no corners except for internal cusps [1]), it follows from the general theory of Hilbert spaces that there exists a complete orthonormal basis of analytic functions $\{e_n\}_{n=0}^\infty$ in $A^2(\Omega)$ and a corresponding sequence of positive constants $\{\lambda_n\}_{n=0}^\infty$, which tend monotonically to zero, as $n \to \infty$, with the property that for every function $f \in A^2(\Omega)$ we have

$$\int_\Omega f^2(z)\,d\Sigma(z) = \sum_{n=0}^\infty \lambda_n \langle f, e_n\rangle_\Omega^2, \tag{1.3}$$

where $\langle\cdot,\cdot\rangle_\Omega$ is the inner product in $L^2(\Omega)$. An equivalent way to write this is

$$F_\Omega f(z) = \sum_{n=0}^\infty \lambda_n \langle e_n, f\rangle_\Omega e_n(z), \quad f \in A^2(\Omega).$$

We will refer to $\{e_n\}_{n=0}^\infty$ as the eigenfunctions for the Friedrichs operator and $\{\lambda_n\}_{n=0}^\infty$ as the eigenvalues, although the operator is only conjugate-linear. Since the Friedrichs operator preserves constants, it is clear that $\lambda_0 = 1$ and that e_0 is a constant function normalized to have norm 1. All the other eigenvalues are strictly less than 1. The next result, due to H. S. Shapiro and M. Putinar, shows the relation between the Friedrichs operator and the harmonic Bergman kernel Q_Ω.

Lemma 1.1. *Suppose that Ω is a simply connected domain and that the Friedrichs operator on $A^2(\Omega)$ is compact. Then the harmonic Bergman kernel for $HL^2(\Omega)$ has the following expansion in terms of the eigenfunctions and eigenvalues to the Friedrichs operator*

(1.4)

$$Q_\Omega(z,\zeta) = \frac{1}{|\Omega|_\Sigma} + 2\,\mathrm{Re}\left(\sum_{n=1}^\infty \frac{1}{1-\lambda_n^2} e_n(z)\overline{e_n(\zeta)}\right) - 2\,\mathrm{Re}\left(\sum_{n=1}^\infty \frac{\lambda_n}{1-\lambda_n^2} e_n(z) e_n(\zeta)\right),$$

where

$$|\Omega|_\Sigma = \int_\Omega d\Sigma.$$

We remark that if Ω is not simply connected but the Friedrichs operator is still compact, then (1.4) gives the reproducing kernel for the subspace of harmonic functions with well-defined harmonic conjugates on Ω. Since $\{e_n\}_{n=0}^{\infty}$ is a complete orthonormal basis for $A^2(\Omega)$, we have the expansion

$$K_\Omega(z,\zeta) = \frac{1}{|\Omega|_\Sigma} + \sum_{n=1}^{\infty} e_n(z)\overline{e_n(\zeta)}$$

for the analytic Bergman kernel, and thus

$$\begin{aligned} Q_\Omega(z,\zeta) + \frac{1}{|\Omega|_\Sigma} - 2\operatorname{Re} K_\Omega(z,\zeta) &= 2\operatorname{Re}\biggl(\sum_{n=1}^{\infty} \frac{\lambda_n^2}{1-\lambda_n^2} e_n(z)\overline{e_n(\zeta)}\biggr) \\ &\quad - 2\operatorname{Re}\biggl(\sum_{n=1}^{\infty} \frac{\lambda_n}{1-\lambda_n^2} e_n(z) e_n(\zeta)\biggr). \end{aligned} \tag{1.5}$$

The idea to get control over the harmonic Bergman kernel is to estimate the left-hand side of this identity. In order to do the calculations, it is convenient to shift to disk coordinates. Let ϕ be a conformal map of the unit disk $\mathbf{D}$ onto Ω and define the kernel functions

$$K_\omega(z,\zeta) = K_\Omega(\phi(z),\phi(\zeta)) \quad \text{and} \quad Q_\omega(z,\zeta) = Q_\Omega(\phi(z),\phi(\zeta)). \tag{1.6}$$

It is a simple exercise to show that K_ω and Q_ω are the analytic and harmonic Bergman kernels for the weighted space $L^2(\mathbf{D},\omega)$ in $\mathbf{D}$ for the weight $\omega = |\phi'|^2$. Moreover, the weighted Friedrichs operator F_ω associated to K_ω defined by

$$F_\omega f(z) = \int_{\mathbf{D}} K_\omega(z,\zeta)\bar{f}(\zeta)\omega(\zeta)\,d\Sigma(\zeta), \quad z\in\mathbf{D},$$

has the same eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$ as F_Ω, and the eigenfunctions $\{f_n\}_{n=0}^{\infty}$ of F_ω are related to the ones of F_Ω by $f_n = e_n \circ \phi$. For our calculations, we need the following explicit representation of the Bergman kernel K_ω at hand

$$K_\omega(z,\zeta) = \frac{1}{\phi'(z)\overline{\phi'(\zeta)}} \frac{1}{(1-z\bar{\zeta})^2}, \quad z,\zeta\in\mathbf{D}. \tag{1.7}$$

If Ω is a disk, then it is known that all the eigenvalues λ_n are zero for $n\geq 1$, and it is obvious that they are invariant under translation, dilation and rotation of the domain Ω. Therefore, we have chosen to work within the class of domains which are the image of the unit disk under a function in the class S, that is, $\Omega = \phi(\mathbf{D})$ for

$\phi \in S$. We recall that an analytic function ϕ on $\mathbf{D}$ is in the class S if it is univalent on $\mathbf{D}$ and $\phi'(0)=0$ and $\phi'(0)=1$. The derivative of such a function ϕ is

$$\phi'(z) = 1+\psi(z), \quad z \in \mathbf{D}, \tag{1.8}$$

where ψ is analytic in $\mathbf{D}$ and has $\psi(0)=0$. The normalized area of the domain Ω, $|\Omega|_\Sigma$, is then given by

$$|\Omega|_\Sigma = \int_{\mathbf{D}} |\phi'|^2 \, d\Sigma = 1 + \int_{\mathbf{D}} |\psi|^2 \, d\Sigma.$$

Loosely speaking, we can say that the function ψ measures how much $\Omega=\phi(\mathbf{D})$ deviates from being the unit disk, and we claim that it is possible to estimate the right-hand side of (1.5) in terms of the norm of ψ in some suitable function space. To formulate our main result, we introduce the family of Dirichlet type spaces $\mathcal{D}_s(\mathbf{D})$ on the disk for $s>-1$. An analytic function f in $\mathbf{D}$ with power series expansion $f(z)=\sum_{n=0}^\infty \hat{f}(n) z^n$ belongs to $\mathcal{D}_s(\mathbf{D})$, $s>-1$, if and only if

$$\|f\|^2_{\mathcal{D}_s(\mathbf{D})} = \sum_{n=0}^\infty \frac{\Gamma(n+2+s)}{n!\,\Gamma(2+s)} |\hat{f}(n)|^2 < \infty, \tag{1.9}$$

where

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt$$

is the Gamma function. When $s=0$, $\mathcal{D}_s(\mathbf{D})$ coincides with the usual Dirichlet space but with a different norm. The reason for choosing this normalization is that $\mathcal{D}_s(\mathbf{D})$ is isometrically dual to the weighted Bergman space $A^2_s(\mathbf{D})$ for the weight $(1-|z|^2)^s$ of $\mathcal{D}_s(\mathbf{D})$ under the Cauchy pairing

$$\langle f, g \rangle_{\mathbf{T}} = \sum_{n=0}^\infty \hat{f}(n) \overline{\hat{g}(n)}, \quad f \in \mathcal{D}_s(\mathbf{D}),\ g \in A^2_s(\mathbf{D}),$$

where $\hat{f}(n)$ and $\hat{g}(n)$ are the Fourier coefficients for f and g, respectively. By appealing to the Cauchy–Schwarz inequality, it is easy to show that $\mathcal{D}_s(\mathbf{D})$ is continuously embedded in $H^\infty(\mathbf{D})$ for $s>0$ and

$$\|\psi\|_\infty \le \mathfrak{I}_s \|\psi\|_{\mathcal{D}_s(\mathbf{D})},$$

where

$$\mathfrak{I}_s = \left(\sum_{n=1}^\infty \frac{n!\,\Gamma(2+s)}{\Gamma(n+2+s)} \right)^{1/2}.$$

Since $\phi'=1+\psi$ is in the denominator in the expression for the analytic Bergman kernel we must require that ψ does not attain the value -1, and therefore we choose the norm condition on ψ in $\mathcal{D}_s(\mathbf{D})$ to be such that the sup-norm of ψ is strictly less than 1. The main result of the paper is the following theorem.

Theorem 1.2. *Suppose that $s>0$ and $R<\mathfrak{I}_s^{-1}$. Then there exists a constant $C_{R,s}$ such that for all $\psi\in\mathcal{D}_s(\mathbf{D})$ with $\|\psi\|_{\mathcal{D}_s(\mathbf{D})}<R$, the harmonic Bergman kernel Q_ω for the weight $\omega=|1+\psi|^2$ in $\mathbf{D}$ satisfies the estimate*

$$\left|Q_\omega(z,\zeta)+\frac{1}{|\Omega|_\Sigma}-2\operatorname{Re}K_\omega(z,\zeta)\right|\le\begin{cases}\dfrac{C_{R,s}}{|1-z\bar{\zeta}|^{2-s/2}}, & 0<s<4,\\ C_{R,s}\sqrt{1+\log\dfrac{1}{|1-z\bar{\zeta}|}}, & s=4,\\ C_{R,s}, & s>4.\end{cases}\tag{1.10}$$

Moreover, the constants $C_{R,s}$ tend to zero, as $R\to 0$.

The constants $C_{R,s}$ depend in a quite complicated way on R, but they can be estimated in terms of certain integrals, as will be clear from the proof in Section 3.

The main motivation for estimating harmonic Bergman kernels is the application to biharmonic Green functions. If G_Ω is the harmonic Green function for a domain Ω, then the biharmonic Green function Γ_Ω has the representation

$$\Gamma_\Omega(z,\zeta)=\int_\Omega G_\Omega(z,\eta)G_\Omega(\eta,\zeta)\,d\Sigma(\eta)-\int_\Omega\int_\Omega G_\Omega(z,\eta)Q_\Omega(\eta,\xi)G_\Omega(\xi,\zeta)\,d\Sigma(\eta)\,d\Sigma(\xi).$$

By using this formula, P. R. Garabedian showed that if the biharmonic Green function for a domain Ω is positive then we must have $Q_\Omega(z,\zeta)\le 0$ for $z,\zeta\in\partial\Omega$, $z\ne\zeta$ [2]. On the other hand, if Ω is starshaped and the harmonic Bergman kernel is sufficiently negative on the boundary, then it can be shown that the Green function is positive too [5, Chapter III]. To specify what sufficiently negative means, we change to the disk coordinates given by the conformal map ϕ. If the the weighted harmonic Bergman kernel Q_ω on the disk for $\omega=|\phi'|^2$ satisfies

$$Q_\omega(z,\zeta)\le-\frac{1}{|\phi'(z)|^2}\frac{\operatorname{Re}[\phi(z)\overline{z\phi'(z)}]}{\operatorname{Re}[\phi(\zeta)\overline{\zeta\phi'(\zeta)}]}\frac{1}{|z-\zeta|^2},\tag{1.11}$$

then the biharmonic Green function Γ_Ω is positive. This condition implies that a semigroup of operators acting on $HL^2(\Omega)$ preserves the cone of positive harmonic function which, in turn, implies that the biharmonic Green function is positive. We refer to [5, Chapter III] for more details. We obtain the following corollary.

Corollary 1.3. *For every $s>\frac{1}{2}$, there is a constant C_s such that if $\|\psi\|_{\mathcal{D}_s(\mathbf{D})}\le C_s$, then the biharmonic Green function for $\Omega=\phi(\mathbf{D})$ is positive.*

As for the theorem, it is possible to estimate the constants C_s in terms of certain integrals, but we avoid this as the expressions are quite complicated.

Finally, the choice of the spaces $\mathcal{D}_s(\mathbf{D})$ is a bit arbitrary and can certainly be replaced by other spaces. One could also consider other types of condition on the domain or the weight in order to obtain estimates for the harmonic Bergman kernel.

1.1. An outline of the proof of the main theorem

For the proof of the main theorem, we need a result on orthogonal projections in a Hilbert space due to A. Lenard [6]. Let E and F be two closed subspaces of a Hilbert space H such that $\|P_E P_F\|<1$ in operator norm, where P_E and P_F are the orthogonal projections onto E and F, respectively. Then $E+F$ is a closed subspace of H, and whence equal to $E\vee F$ which, by definition, is the smallest closed subspace which includes both E and F. The orthogonal projection onto $E\vee F$ is

$$P_{E\vee F}=(\mathrm{Id}-P_F)(\mathrm{Id}-P_E P_F)^{-1}P_F+(\mathrm{Id}-P_E)(\mathrm{Id}-P_F P_E)^{-1}P_E. \tag{1.12}$$

(see also Problem 122 in [3] for a discussion of a related result). For our application, we choose E and F to be the Bergman spaces $A_0^2(\mathbf{D},\omega)$ and $\overline{A_0^2(\mathbf{D},\omega)}$, where the subscript 0 indicates that these are the subspaces of $A^2(\mathbf{D},\omega)$ and $\overline{A^2(\mathbf{D},\omega)}$ which consist of those functions f with integral mean zero

$$\int_{\mathbf{D}} f(z)\omega(z)\,d\Sigma(z)=0.$$

In this case, the condition $\|P_E P_F\|<1$ is equivalent to that the Friedrichs inequality holds with a constant strictly less than 1. The idea of the proof is to combine the representation of the harmonic Bergman kernel given by Lenard's formula and the representation in Lemma 1.1. The lemma is actually a consequence of Lenard's result if the Friedrichs operator is compact. A series expansion of (1.12) gives terms which are powers of the Friedrichs operator and it converges in operator norm if $\|F_\omega\|<1$, but we need pointwise convergence in $\bar{\mathbf{D}}\times\bar{\mathbf{D}}\setminus\triangle(\mathbf{T})$, where $\mathbf{T}$ is the unit circle and $\triangle(\mathbf{T})=\{(z,z):z\in\mathbf{T}\}$. To prove the pointwise convergence of this series, we use the duality between the Dirichlet type spaces $\mathcal{D}_s(\mathbf{D})$ and the weighted Bergman spaces $A_s^2(\mathbf{D})$. We show that all but a finite number of the integral kernels are bounded functions on $\mathbf{D}\times\mathbf{D}$, and that the tail of the series can be controlled with the aid of the first singular value λ_1 and the sup-norm for the integral kernel of an even power of the Friedrichs operator. The number of unbounded integral kernels depends on s, and if $s>4$, then all terms, except for the Bergman kernel itself, are bounded.

To carry out this program, we need some technical results which we present in Section 2. We introduce an integral operator which annihilates anti-analytic functions and enable us to convert certain integrals on the disk to integrals on the circle. Finally, we present some estimates of integrals which depend on several parameters and variables. These estimates are crucial in order to get control over the series expansion of the harmonic Bergman kernel.

2. Bergman and Dirichlet spaces of analytic functions

The purpose of this section is to prove some technical results which we need for the proof of Theorem 1.2. These results are related to the general theory of weighted Bergman spaces, as presented by the books [4], [10]. We begin by introducing the weighted Bergman spaces $A^p_\alpha(\mathbf{D})$ and show that they are dual to the spaces $\mathcal{D}_s(\mathbf{D})$ for $p=2$ and $\alpha=s$. We also present two families of integrals which depend on several parameters and variables and show how they can be estimated. Finally, we prove some properties of an integral operator $P^*_+ : L^1(\mathbf{D}) \to \mathcal{O}(\mathbf{D})$ which we need later. Here, $\mathcal{O}(\mathbf{D})$ is the space of all analytic functions on $\mathbf{D}$.

The weighted Bergman space $A^p_\alpha(\mathbf{D})$, $0<p<\infty$, $\alpha>-1$, is the class of all analytic functions f in $\mathbf{D}$ such that

$$\|f\|^p_{A^p_\alpha(\mathbf{D})} = (\alpha+1) \int_{\mathbf{D}} |f(z)|^p (1-|z|^2)^\alpha \, d\Sigma(z) < \infty.$$

Here we are only concerned with the case $p=2$, which is the Hilbert space case. For $p=2$, the norm of a function f with power series expansion $f(z)=\sum_{n=0}^\infty \hat{f}(n) z^n$ is

$$\|f\|^2_{A^2_\alpha(\mathbf{D})} = \sum_{n=0}^\infty \frac{n!\,\Gamma(2+\alpha)}{\Gamma(n+2+\alpha)} |\hat{f}(n)|^2. \tag{2.1}$$

A comparison with the norm for the Dirichlet type spaces shows immediately that $\mathcal{D}_s(\mathbf{D})$ and $A^2_s(\mathbf{D})$ are isometrically dual to each other under the Cauchy pairing as claimed. We will also use the integral version of the Cauchy pairing which looks like

$$\langle f, g\rangle_{\mathbf{T}} = \lim_{r\to 1_-} \int_{\mathbf{T}} f(rz)\overline{g(rz)} \, d\sigma(z),$$

where $d\sigma(z)$ is the Lebesgue measure on the unit circle $\mathbf{T}$ normalized so that $\mathbf{T}$ has mass 1. In the sequel, we will adopt the common practice of omitting the limit in the Cauchy pairing.

Define the linear operator $P^*_+ : L^1(\mathbf{D}) \to \mathcal{O}(\mathbf{D})$ by

$$P^*_+[F](z) = \int_{\mathbf{D}} \frac{z\bar{\zeta}}{1-z\bar{\zeta}} F(\zeta) \, d\Sigma(\zeta), \quad z \in \mathbf{D}. \tag{2.2}$$

This operator has the property that if f is an analytic function with $f(0)=0$, then

$$\int_{\mathbf{D}} f(\zeta)\overline{F(\zeta)} \, d\Sigma(\zeta) = \int_{\mathbf{T}} f(\zeta)\overline{P^*_+[F](\zeta)} \, d\sigma(\zeta) \tag{2.3}$$

whenever both sides are defined. The right-hand side should be interpreted in the sense of the Cauchy pairing. It is clear from the definition that P_+^* annihilates anti-analytic functions, and by direct integration one shows that for all positive integers k and l,

$$P_+^*[\zeta^k\bar{\zeta}^l](z)=\frac{z^{k-l}}{k+1}$$

if $k>l$, and that the left-hand side is zero otherwise. For our application, F will be the weight ω and f an analytic function of several variables. To apply the duality between Dirichlet type spaces $\mathcal{D}_s(\mathbf{D})$ and the Bergman spaces $A^2_\alpha(\mathbf{D})$, we need some technical results.

Proposition 2.1. *Suppose that $s>0$ and $f\in\mathcal{D}_s(\mathbf{D})$, then $P_+^*[f]$ and $P_+^*[|f|^2]$ both belong to $\mathcal{D}_{s+2}(\mathbf{D})$, and if $f(0)=0$, then*

$$\|P_+^*[f]\|_{\mathcal{D}_{s+2}(\mathbf{D})}\leq\|f\|_{\mathcal{D}_s(\mathbf{D})},$$
$$\|P_+^*[|f|^2]\|_{\mathcal{D}_{s+2}(\mathbf{D})}\leq\mathcal{I}_s\|f\|^2_{\mathcal{D}_s(\mathbf{D})}.$$

Proof. We start with $P_+^*[f]$. By expanding f in a power series and integrating, we obtain

$$P_+^*[f](z)=\sum_{n=1}^{\infty}\frac{\hat{f}(n)z^n}{n+1},$$

where $f(z)=\sum_{n=1}^{\infty}\hat{f}(n)z^n$, and therefore

$$\|P_+^*[f]\|^2_{\mathcal{D}_{s+2}(\mathbf{D})}=\sum_{n=1}^{\infty}\frac{\Gamma(4+s+n)}{n!\Gamma(4+s)(n+1)^2}|\hat{f}(n)|^2.$$

The first inequality now follows from well-known properties of the Gamma function. A similar calculation for the other term gives

$$P_+^*[|f|^2](z)=\sum_{k=1}^{\infty}\sum_{m=0}^{\infty}\frac{\hat{f}(m+k)\overline{\hat{f}(m)}z^k}{m+k+1}$$

and

$$\|P_+^*[|f|^2]\|^2_{\mathcal{D}_{s+2}(\mathbf{D})}\leq\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|\hat{f}(m)\hat{f}(n)|$$
$$\times\sum_{k=1}^{\infty}\frac{\Gamma(4+s+k)}{k!\Gamma(4+s)(1+m+k)(1+n+k)}|\hat{f}(m+k)\hat{f}(n+k)|.$$

From the properties of the Gamma function, it follows that

$$\sum_{k=1}^{\infty} \frac{\Gamma(4+s+k)}{k!\Gamma(4+s)(1+n+k)^2} |\hat{f}(n+k)|^2 \le \|f\|^2_{\mathcal{D}_s(\mathbf{D})}$$

and therefore, by Cauchy–Schwarz inequality, we have

$$\|P_+^*[|f|^2]\|^2_{\mathcal{D}_{s+2}(\mathbf{D})} \le \|f\|^2_{\mathcal{D}_s(\mathbf{D})} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |\hat{f}(m)\hat{f}(n)|.$$

Another application of the Cauchy–Schwarz inequality then yields

$$\|P_+^*[|f|^2]\|^2_{\mathcal{D}_{s+2}(\mathbf{D})} \le \mathcal{I}_s^2 \|f\|^4_{\mathcal{D}_s(\mathbf{D})}.$$

The proof is complete. □

The next result is a classical integral estimate which has many applications in the theory of Bergman spaces [10], [4]. It is often used to treat duality questions for different types of spaces of analytic functions in **D** and this is the way we intend to use it as well.

Theorem 2.2. *For any real β and $\alpha > -1$ let*

$$I_{\alpha,\beta}(z) = \int_{\mathbf{D}} \frac{(1-|w|^2)^\alpha}{|1-z\bar{w}|^{2+\alpha+\beta}}\, d\Sigma(w), \quad z \in \mathbf{D}.$$

Then we have

$$I_{\alpha,\beta}(z) \asymp \begin{cases} 1, & \beta < 0, \\ \log \dfrac{1}{1-|z|^2}, & \beta = 0, \\ \dfrac{1}{(1-|z|^2)^\beta}, & \beta > 0, \end{cases}$$

as $|z| \to 1_-$.

By the symbol $\asymp$ we mean that both sides are comparable in size asymptotically. We shall also use the symbol $\lesssim$, which should be interpreted as the left-hand side is asymptotically less than a constant times the right-hand side. We will need the following lemma which we state without proof (see [5, pp. 124–125] for a proof).

Lemma 2.3. *For α, β and γ which satisfy $2+\alpha > \max(\beta,\gamma)$ and $\min(\beta,\gamma) > 0$, define the function*

$$J_{\alpha,\beta,\gamma}(z,\zeta) = \int_{\mathbf{D}} \frac{(1-|w|^2)^\alpha}{|(1-z\bar{w})^\beta (1-\zeta\bar{w})^\gamma|}\, d\Sigma(w), \quad (z,\zeta) \in \bar{\mathbf{D}} \times \bar{\mathbf{D}} \setminus \triangle(\mathbf{T}).$$

Then we have

$$J_{\alpha,\beta,\gamma}(z,\zeta) \lesssim \begin{cases} 1, & \beta+\gamma<2+\alpha, \\ \log \dfrac{1}{|1-z\bar{\zeta}|}, & \beta+\gamma=2+\alpha, \\ \dfrac{1}{|1-z\bar{\zeta}|^{\beta+\gamma-2-\alpha}}, & \beta+\gamma>2+\alpha, \end{cases}$$

as $|1-z\bar{\zeta}|\to 0$.

3. Expansion of the harmonic Bergman kernel

The formula (1.6) shows that a Bergman kernel for a simply connected domain corresponds to a Bergman kernel for a weighted Bergman space in the unit disk. This follows from the fact that if $\phi\colon \mathbf{D}\to\Omega$ is conformal, then the linear mapping $L^2(\Omega)\ni h\mapsto h\circ\phi\in L^2(\mathbf{D},\omega)$, $\omega=|\phi'|^2$, is an isometry which preserves analytic and harmonic functions. Similarly, the Friedrichs operator for Ω corresponds to the weighted Friedrichs operator F_ω on $\mathbf{D}$.

To prove the main theorem, we apply Lenard's formula (1.12) for the Bergman spaces $A_0^2(\mathbf{D},\omega)$ and $\overline{A_0^2(\mathbf{D},\omega)}$. The reproducing kernel or the Bergman kernel $K_{\omega,0}$ for $A_0^2(\mathbf{D},\omega)$ is related to the Bergman kernel K_ω for $A^2(\mathbf{D},\omega)$ by

$$K_{\omega,0}=K_\omega-\frac{1}{|\Omega|_\Sigma},$$

where $\Omega=\phi(\mathbf{D})$ and

$$|\Omega|_\Sigma=\int_\Omega d\Sigma=\int_{\mathbf{D}}\omega\, d\Sigma=1+\|\psi\|^2_{A_0^2(\mathbf{D})}.$$

(Recall that $\phi'=1+\psi$ and $\psi(0)=0$.)

We now define two linear operators on $L^2(\mathbf{D},\omega)$:

$$T_{\omega,0}f(z)=\int_{\mathbf{D}} T_{\omega,0}(z,\zeta)f(\zeta)\omega(\zeta)\, d\Sigma(\zeta)$$

and

$$S_{\omega,0}f(z)=\int_{\mathbf{D}} S_{\omega,0}(z,\zeta)f(\zeta)\omega(\zeta)\, d\Sigma(\zeta)$$

which have the integral kernels

$$T_{\omega,0}(z,\zeta)=\int_{\mathbf{D}} K_{\omega,0}(z,\eta)K_{\omega,0}(\zeta,\eta)\omega(\eta)\, d\Sigma(\eta)$$

and

$$S_{\omega,0}(z,\zeta)=\int_{\mathbf{D}}\int_{\mathbf{D}} K_{\omega,0}(z,\eta)K_{\omega,0}(\xi,\eta)K_{\omega,0}(\xi,\zeta)\omega(\xi)\omega(\eta)\, d\Sigma(\xi)\, d\Sigma(\eta).$$

Note that the integral kernels are defined with respect to the weighted measure $\omega\, d\Sigma$. The operators $T_{\omega,0}$ and $S_{\omega,0}$ play the role of the composition operators $P_E P_F$ and $P_E P_F P_E$ when $E = A_0^2(\mathbf{D},\omega)$ and $F = \overline{A_0^2(\mathbf{D},\omega)}$. Restricted to the subspace $A_0^2(\mathbf{D},\omega)$, we have the following relations to the Friedrichs operator: If C is the conjugation operator ($Cf = \bar{f}$), then $F_{\omega,0} = T_{\omega,0} C$ and $S_{\omega,0} = F_{\omega,0}^2$. The operator $S_{\omega,0}$ is, in contrast to F_ω, linear and self-adjoint, and it satisfies the operator inequality $0 \le S_{\omega,0} \le \mathrm{Id}$ (to be interpreted in the sense that $0 \le \langle S_{\omega,0} f, f\rangle_\omega \le \|f\|_\omega^2$ for all $f \in A^2(\mathbf{D},\omega)$). The last property follows from the equality

$$\langle S_{\omega,0} f, f\rangle_\omega = \|F_{\omega,0} f\|_\omega^2$$

and the fact that $\|F_{\omega,0}\| \le 1$ in operator norm. We point out that everything we have said so far also holds for the Friedrichs operator F_ω on $A^2(\mathbf{D},\omega)$. The only difference between F_ω and $F_{\omega,0}$ is that the former preserves constants while the latter maps them to zero. To estimate the integral kernels for $T_{\omega,0}$ and $S_{\omega,0}$, we need the P_+^* transform of the weight. Since P_+^* annihilates anti-analytic functions we have

$$P_+^*[\omega](z) = P_+^*[\psi](z) + P_+^*[|\psi|^2](z).$$

So if $s>0$, we have by Proposition 2.1

$$\|P_+^*[\omega]\|_{\mathcal{D}_{s+2}(\mathbf{D})} \le \|\psi\|_{\mathcal{D}_s(\mathbf{D})} + \mathcal{I}_s \|\psi\|_{\mathcal{D}_s(\mathbf{D})}^2. \tag{3.1}$$

For the rest of this section, we assume that $\|\psi\|_{\mathcal{D}_s(\mathbf{D})} < \mathcal{I}_s^{-1}$. We want to prove that the kernels for $S_{\omega,0}$ and $T_{\omega,0}$ are smaller and more regular than $K_{\omega,0}$ in some sense. To do this, it is convenient to split $K_{\omega,0}$ into two parts, one which is unbounded but contains the factor $z\bar{\zeta}$ and the other which can be estimated in sup-norm by the norm of ψ in the Dirichlet space. Specifically, we have

$$K_{\omega,0}(z,\zeta) = A_\omega(z,\zeta) + B_\omega(z,\zeta),$$

where

$$A_\omega(z,\zeta) = K_\omega(z,\zeta) - K_\omega(z,0) - K_\omega(0,\zeta) + K_\omega(0,0)$$

and

$$B_\omega(z,\zeta) = K_\omega(z,0) + K_\omega(0,\zeta) - K_\omega(0,0) - \frac{1}{|\Omega|_\Sigma}.$$

An explicit computation of A_ω yields

$$A_\omega(z,\zeta) = z\bar{\zeta}\,\frac{(2 - z\bar{\zeta}) + (\psi(z)/z)\overline{(\psi(\zeta)/\zeta)}(1 - z\bar{\zeta})^2}{\phi'(z)\overline{\phi'(\zeta)}(1 - z\bar{\zeta})^2},$$

and since $|\phi'(z)|\geq(1-\mathcal{I}_s\|\psi\|_{\mathcal{D}_s(\mathbf{D})})$, A_ω obeys the estimate

$$|A_\omega(z,\zeta)|\leq\frac{|z\zeta|}{(1-\mathfrak{I}_s\|\psi\|_{\mathcal{D}_s(\mathbf{D})})^2}\frac{7}{|1-z\bar{\zeta}|^2}. \tag{3.2}$$

Since $|\Omega|_\Sigma=1+\|\psi\|^2_{A_0^2(\mathbf{D})}$, the term B_ω equals

$$B_\omega(z,\zeta)=\frac{1}{(1+\|\psi\|^2_{A^2(\mathbf{D})})}\left(\|\psi\|^2_{A^2(\mathbf{D})}\frac{1-\psi(z)\overline{\psi(\zeta)}}{\phi'(z)\overline{\phi'(\zeta)}}-\frac{\psi(z)+\overline{\psi(\zeta)}+\psi(z)\overline{\psi(\zeta)}}{\phi'(z)\overline{\phi'(\zeta)}}\right),$$

and therefore,

$$|B_\omega(z,\zeta)|\leq\frac{4}{(1-\mathfrak{I}_s\|\psi\|_{\mathcal{D}_s(\mathbf{D})})^2}, \tag{3.3}$$

where we have used that $\|\psi\|_{A_0^2(\mathbf{D})}\leq\frac{1}{2}$. By property (2.3) of the operator P_+^*, the integral kernel $T_{\omega,0}(z,\zeta)$ can be split into a sum of the four terms

$$\int_{\mathbf{T}}A_\omega(z,\eta)A_\omega(\zeta,\eta)P_+^*[\omega](\eta)\,d\sigma(\eta),\qquad\int_{\mathbf{T}}B_\omega(z,\eta)A_\omega(\zeta,\eta)P_+^*[\omega](\eta)\,d\sigma(\eta),$$
$$\int_{\mathbf{T}}A_\omega(z,\eta)B_\omega(\zeta,\eta)P_+^*[\omega](\eta)\,d\sigma(\eta),\qquad\int_{\mathbf{T}}B_\omega(z,\eta)B_\omega(\zeta,\eta)\omega(\eta)\,d\Sigma(\eta).$$

Proposition 3.1. *Suppose that $s>0$. Then there exists a constant D_s such that for all $\psi\in\mathcal{D}_s(\mathbf{D})$, $\|\psi\|_{\mathcal{D}_s(\mathbf{D})}<\mathcal{I}_s$, we have*

$$|T_{\omega,0}(z,\zeta)|\leq D_s\frac{\|\psi\|^{1/2}_{\mathcal{D}_s(\mathbf{D})}}{(1-\mathcal{I}_s\|\psi\|_{\mathcal{D}_s(\mathbf{D})})^4}\times\begin{cases}\dfrac{1}{|1-z\bar{\zeta}|^{2-s/2}}, & 0<s<4,\\[2ex] \sqrt{1+\log\dfrac{1}{|1-z\bar{\zeta}|}}, & s=4,\\[2ex] 1, & s>4,\end{cases} \tag{3.4}$$

for the weight $\omega=|1+\psi|^2$.

Proof. We denote the four terms which constitute $T_{\omega,0}$ above by T_1, T_2, T_3, and T_4.

By the duality between $A^2_{s+2}(\mathbf{D})$ and $\mathcal{D}_{s+2}(\mathbf{D})$, we have

$$|T_1(z,\zeta)|\leq\|A_\omega(z,\cdot\,)A_\omega(\zeta,\cdot\,)\|_{A^2_{s+2}(\mathbf{D})}\|P_+^*[\omega]\|_{\mathcal{D}_{s+2}(\mathbf{D})}.$$

Since $A_\omega(z,\eta)$ satisfies (3.2), the first factor on the right-hand side satisfies

$$\|A_\omega(z,\cdot\,)A_\omega(\zeta,\cdot\,)\|^2_{A^2_{s+2}(\mathbf{D})} \lesssim \frac{1}{(1-\mathcal{I}_s\|\psi\|_{\mathcal{D}_s(\mathbf{D})})^8}\int_{\mathbf{D}} \frac{(1-|w|^2)^{s+2}}{|(1-z\bar{w})(1-\zeta\bar{w})|^4}\,d\Sigma(w).$$

Lemma 2.3 together with (3.1) shows that the kernel T_1 satisfies the right type of estimate. Similarly, by using Theorem 2.2 we obtain

$$\|T_k\|_\infty \le C_k \frac{\|\psi\|^{3/2}_{\mathcal{D}_s(\mathbf{D})}}{(1-\mathcal{I}_s\|\psi\|_{\mathcal{D}_s(\mathbf{D})})^4},$$

for $k=2,3$, and finally for $k=4$,

$$\|T_4\|_\infty \le C_4 \frac{\|\psi\|^{2}_{\mathcal{D}_s(\mathbf{D})}}{(1-\mathcal{I}_s\|\psi\|_{\mathcal{D}_s(\mathbf{D})})^4}.$$

This proves the inequality. □

The operator $S^n_{\omega,0}$, where n is a positive integer, is defined as the composition of $S_{\omega,0}$ with itself n-times, and $S^n_{\omega,0}T_{\omega,0}$ is the composition of this operator with $T_{\omega,0}$. These operators arise when we expand Lenard's formula (1.12) in a Neumann series. The next result show that their integral kernels can be estimated. Recall that we consider the integral kernels with respect to the measure $\omega\, d\Sigma$.

Corollary 3.2. *Suppose that $s>0$. Then there exists constants $B_{n,s}$ and $D_{n,s}$ such that the kernel functions for $S^n_{\omega,0}$ and $S^n_{\omega,0}T_{\omega,0}$ can be estimated by*

$$(3.5)\qquad |S^n_{\omega,0}(z,\zeta)| \le B_{n,s}\frac{\|\psi\|^{n}_{\mathcal{D}_s(\mathbf{D})}}{(1-\mathcal{I}_s\|\psi\|_{\mathcal{D}_s(\mathbf{D})})^{8n}} \times \begin{cases} \dfrac{1}{|1-z\bar{\zeta}|^{2-ns}}, & 2>ns,\\ \sqrt{1+\log\dfrac{1}{|1-z\bar{\zeta}|}}, & 2=ns,\\ 1, & 2<ns,\end{cases}$$

and

$$(3.6)\qquad |S^n_{\omega,0}T_{\omega,0}(z,\zeta)| \le D_{n,s}\frac{\|\psi\|^{n+1/2}_{\mathcal{D}_s(\mathbf{D})}}{(1-\mathcal{I}_s\|\psi\|_{\mathcal{D}_s(\mathbf{D})})^{8n+4}} \times \begin{cases} \dfrac{1}{|1-z\bar{\zeta}|^{2-(2n+1)s/2}}, & 2>\frac12(2n+1)s,\\ \sqrt{1+\log\dfrac{1}{|1-z\bar{\zeta}|}}, & 2=\frac12(2n+1)s,\\ 1, & 2<\frac12(2n+1)s,\end{cases}$$

for $n=1,2,3,\ldots$.

Proof. The result follows from the previous proposition and Lemma 2.3 by applying the same techniques. □

In terms of the eigenvalues and eigenfunctions, the integral kernels $T_{\omega,0}$ and $S_{\omega,0}$ are given by

$$(3.7)\qquad T_{\omega,0}(z,\zeta)=\sum_{n=1}^{\infty}\lambda_n f_n(z)f_n(\zeta) \quad\text{and}\quad S_{\omega,0}(z,\zeta)=\sum_{n=1}^{\infty}\lambda_n^2 f_n(z)\overline{f_n(\zeta)},$$

and the powers $S^k_{\omega,0}$ and $S^k_{\omega,0}T_{\omega,0}$, have similar expansions:

$$(3.8)\quad S^k_{\omega,0}(z,\zeta)=\sum_{n=1}^{\infty}\lambda_n^{2k} f_n(z)\overline{f_n(\zeta)} \quad\text{and}\quad S^k_{\omega,0}T_{\omega,0}(z,\zeta)=\sum_{n=1}^{\infty}\lambda_n^{2k+1} f_n(z)f_n(\zeta).$$

Friedrichs showed in his original paper [1] that the inequality which bears his name is equivalent to the existence of a constant $\gamma>0$ such that

$$(3.9)\qquad \int_{\mathbf{D}} u^2(z)\omega(z)\,d\Sigma(z)\le\gamma\int_{\mathbf{D}} v^2(z)\omega(z)\,d\Sigma(z),$$

provided u is the harmonic conjugate to the harmonic function v, and u satisfies

$$\int_{\mathbf{D}} u(z)\omega(z)\,d\Sigma(z)=0.$$

The constant θ in the Friedrichs inequality is related to γ by

$$(3.10)\qquad \theta=\frac{\gamma-1}{\gamma+1}.$$

The inequality (3.9) can also be stated as the operation of taking harmonic conjugates being continuous in $HL^2(\mathbf{D},\omega)$. As this operation is continuous on $HL^2(\mathbf{D})$ with constant 1, it follows that the Friedrichs inequality holds for $A^2(\mathbf{D},\omega)$ if the norm is comparable to the unweighted norm. We use these facts to estimate the first eigenvalue λ_1 of the Friedrichs operator.

Proposition 3.3. *Suppose that $\psi\in\mathcal{D}_s(\mathbf{D})$, and $\|\psi\|_{\mathcal{D}_s(\mathbf{D})}\le\mathcal{I}_s^{-1}$. Then the first eigenvalue λ_1 for the Friedrichs operator F_ω, $\omega=|1+\psi|^2$, satisfies*

$$\lambda_1\le\frac{2\mathcal{I}_s\|\psi\|_{\mathcal{D}_s(\mathbf{D})}}{1+\mathcal{I}_s^2\|\psi\|^2_{\mathcal{D}_s(\mathbf{D})}}.$$

Proof. Under the hypotheses of the proposition, the weight ω satisfies the following bounds

$$\inf_{z\in\mathbf{D}} \omega(z) \geq (1-\mathcal{I}_s\|\psi\|_{\mathcal{D}_s(\mathbf{D})})^2 \quad \text{and} \quad \sup_{z\in\mathbf{D}} \omega(z) \leq (1+\mathcal{I}_s\|\psi\|_{\mathcal{D}_s(\mathbf{D})})^2.$$

If u is the harmonic conjugate to $v\in HL^2(\mathbf{D},\omega)$, and $\int_{\mathbf{D}} u\omega\, d\Sigma=0$, the argument before the proposition shows that

$$\frac{\|u\|_\omega^2}{\|v\|_\omega^2} \leq \frac{\sup_{z\in\mathbf{D}} \omega(z)}{\inf_{z\in\mathbf{D}} \omega(z)} \frac{\|u\|^2}{\|v\|^2} \leq \frac{\sup_{z\in\mathbf{D}} \omega(z)}{\inf_{z\in\mathbf{D}} \omega(z)},$$

where we have used that

$$\inf_{z\in\mathbf{D}} \omega(z)\|f\|^2 \leq \|f\|_\omega^2 \leq \sup_{z\in\mathbf{D}} \omega(z)\|f\|^2$$

for all $f\in L^2(\mathbf{D})$. Therefore the constant γ can be chosen so that

$$\gamma \leq \frac{\sup_{z\in\mathbf{D}} \omega(z)}{\inf_{z\in\mathbf{D}} \omega(z)} \leq \frac{(1+\mathcal{I}_s\|\psi\|_{\mathcal{D}_s(\mathbf{D})})^2}{(1-\mathcal{I}_s\|\psi\|_{\mathcal{D}_s(\mathbf{D})})^2}.$$

Since λ_1 is the best constant in the Friedrichs inequality, the result now follows from (3.10) together with the bounds on the weight. □

We are now in a position to prove the main result.

Proof of Theorem 1.2. Let k be the smallest positive integer which is larger than $2/s$. It follows from Lemma 1.1 together with the formulas (3.7) and (3.8), that the identity

$$\begin{aligned} Q_\omega(z,\zeta) - \frac{1}{|\Omega|_\Sigma} - 2\operatorname{Re} K_{\omega,0}(z,\zeta) &= 2\operatorname{Re}\sum_{j=1}^{k-1} S_{\omega,0}^j(z,\zeta) - 2\operatorname{Re}\sum_{j=1}^{k-1} S_{\omega,0}^j T_{\omega,0}(z,\zeta) \\ &\quad + 2\operatorname{Re}\biggl(\sum_{n=1}^\infty \frac{\lambda_n^{2k}}{1-\lambda_n^2} f_n(z)\overline{f_n(\zeta)} - \sum_{n=1}^\infty \frac{\lambda_n^{2k+1}}{1-\lambda_n^2} f_n(z) f_n(\zeta)\biggr) \end{aligned}$$

holds. To prove the theorem, we have to estimate the modulus of the left-hand side. According to Corollary 3.2, the integral kernel for $S_{\omega,0}^k$ is bounded on $\overline{\mathbf{D}}\times\overline{\mathbf{D}}$, and since λ_1 is the largest characteristic value, the modulus of the infinite series can be estimated by

$$\begin{aligned} 2\biggl|\sum_{n=1}^\infty \frac{\lambda_n^{2k}}{1-\lambda_n^2} f_n(z)\overline{f_n(\zeta)} - \sum_{n=1}^\infty \frac{\lambda_n^{2k+1}}{1-\lambda_n^2} f_n(z) f_n(\zeta)\biggr| &\leq \frac{2}{1-\lambda_1^2}(1+\lambda_1)\sum_{n=1}^\infty \lambda_n^{2k}|f_n(z)f_n(\zeta)| \\ &\leq \frac{2}{1-\lambda_1}\|S_{\omega,0}^k\|_\infty. \end{aligned}$$

Here, we have also used the Cauchy–Schwarz inequality in the last step. The theorem now follows from Proposition 3.1 and Corollary 3.2. □

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