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L^p-norms of Hermite polynomials and an extremal problem on Wiener chaos

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Abstract. We establish sharp asymptotics for the L^p-norm of Hermite polynomials and prove convergence in distribution of suitably normalized Wick powers. The results are combined with numerical integration to study an extremal problem on Wiener chaos.

1. Introduction

Hermite polynomials arise quite naturally whenever Gaussian variables are involved. They also play an important role as eigenfunctions for the quantum-mechanical harmonic oscillator. Their L^p-norms should therefore be of general interest. The main object of this paper is to derive precise asymptotics for these norms with p fixed (Theorem 2.1, Remark 3.2). This is done by direct calculations based on a classical asymptotic expansion by Plancherel and Rotach. As an application we give a partial negative answer to an extremal question posed by Janson (Proposition 5.1). This matter is then further analyzed, chiefly using numerical methods. A short intermezzo treats weak convergence of suitably normalized Wick powers of a Gaussian variable (Theorem 4.3).

This paper is a shortened version of [L], where further information can be found. The author wishes to thank his advisor Svante Janson for his help and support.

1.1. Notation

We shall take the Hermite polynomials $\{h_n\}_{n=0}^{\infty}$ to be monic and orthogonal with respect to the Gaussian measure $d\gamma(x)=(2\pi)^{-1/2}e^{-x^2/2}\,dx$. The L^p-norms will be taken with respect to this measure, so that $\|f\|_p=\left(\int_{\mathbf{R}}|f|^p\,d\gamma\right)^{1/p}$ for measurable functions f. The Hermite polynomials are oscillating up to the largest zero, which is $N-O(n^{-1/6})$, with $N=\sqrt{4n+2}$ [S]. We shall let N keep this meaning throughout.

Indicator (characteristic) functions are denoted by $\mathbf{1}$, and c will denote positive finite constants, not necessarily the same on each occurrence.

2. Main result

Our main result is the following theorem, to be proved in Section 3 below. Recall that, with our normalization, $\|h_n\|_2=\sqrt{n!}$.

Theorem 2.1. *The following holds as* $n\to\infty$:

(a) *If* $0<p<2$ *then*

$$\|h_n\|_p=\frac{c(p)}{n^{1/4}}\sqrt{n!}\left(1+O\left(\frac{1}{n}\right)\right). \tag{2.1}$$

(b) *If* $2<p<\infty$ *then*

$$\|h_n\|_p=\frac{c(p)}{n^{1/4}}\sqrt{n!}\,(p-1)^{n/2}\left(1+O\left(\frac{1}{n}\right)\right). \tag{2.2}$$

The constants $c(p)$ *are given by*

$$c(p)=\left(\frac{2}{\pi}\right)^{1/4}\mu_p\left(\frac{2}{2-p}\right)^{1/2p}, \qquad p<2,$$

$$c(p)=\left(\frac{2}{\pi}\right)^{1/4}\left(\frac{p-1}{2(p-2)}\right)^{(p-1)/2p}, \quad p>2,$$

where μ_p *is defined by* (3.1) *below.*

Remark 2.2. This is an improvement over Theorem 5.19 of [J], who uses combinatorial properties of Wick products together with a hypercontractivity argument as a main tool to give upper and lower bounds of our type, but with various powers of n. The best power $n^{-1/4}$ is established for $p\geq 4$, though.

Remark 2.3. Theorem 2.1 can be sharpened, cf. Remark 3.2 below.

Remark 2.4. If $p<q$, then $\|h_n\|_p=o(\|h_n\|_q)$ if and only if $q\geq 2$.

Remark 2.5. The peculiar dependence on p, with one sharply marked threshold value ($p=2$) has been noticed before. Aptekarev–Buyarov–Dehesa [ABD] have investigated Jacobi polynomials, where this threshold can take any value $2<p<\infty$.

Remark 2.6. For $p<2$ one value can be given explicitly: $c(1)=2^{7/4}/\pi^{5/4}$. It is also easy to see that $c(p)$ is increasing on $(0,2)$ (of course) and decreasing on $(2,\infty)$ with finite limits $c(0)=(e/8\pi)^{1/4}$ and $c(\infty)=(2\pi)^{-1/4}$. Moreover, $c(p)\sim(\pi|p-2|)^{-1/4}$ as $p\to 2$ from either direction.

Remark 2.7. Let $p=1$. The fact that $\int h_n(x)e^{-x^2/2}\,dx=-h_{n-1}(x)e^{-x^2/2}$ combined with Remarks 2.6 and 3.2 (below) yields the following asymptotics for the values of a Hermite polynomial at the zeros of its successor:

$$\sum_{k=1}^{n}|h_{n-1}(x_k)|e^{-x_k^2/2}=\frac{2^{5/4}}{\pi^{3/4}n^{1/4}}\sqrt{n!}\,\left(1+O\left(\frac{1}{n^2}\right)\right),$$

where $x_1,\dots,x_n$ are the zeros of h_n.

Remark 2.8. A natural question is where the main contributions to the norms come from. It follows from the proof in Section 3.1 that if $p<2$, then the region $|x|\le c\sqrt{\log n}$, dominates in the sense of the error bounds above. If $p>2$, however, a slight sharpening of the argument in Section 3.2 shows that the important parts are the intervals $|x\mp p\sqrt{n}/\sqrt{p-1}\,|\le c\sqrt{\log n}$. Thus, the dominating part is (a small part of) the oscillating region for $p<2$ and the non-oscillating region for $p>2$.

If $p=2$, then $\|h_n\|_p=\sqrt{n!}$. Using Theorem 2.9 below, it is not hard to see that the (entire) oscillating part is again the important one. More precisely,

$$\left(\int_{|x|\ge N-\varepsilon n^{-1/6}}|h_n|^2\,d\gamma\right)^{1/2}=O\left(\frac{\sqrt{n!}}{n^{1/6}}\right)$$

for any $\varepsilon>0$.

Our main tool in proving Theorem 2.1 is the following asymptotic expansion of Hermite polynomials due to Plancherel and Rotach [PR], see also [S], §8.22, proven by generating functions and the method of steepest descent. Recall that $N=\sqrt{4n+2}$.

Theorem 2.9. (a) *Let* $x=N\sin\varphi$, $|\varphi|<\frac{1}{2}\pi$. *Then*

$$e^{-x^2/4}h_n(x)=\frac{a_n}{\sqrt{\cos\varphi}}\left(\sin\left(\frac{N^2}{8}(2\varphi+\sin 2\varphi)-\frac{(n-1)\pi}{2}\right)+O\left(\frac{1}{n\cos^3\varphi}\right)\right) \tag{2.3}$$

with $a_n=(2/\pi)^{1/4}n^{-1/4}\sqrt{n!}$.

(b) *Let* $x=N\cosh\phi$, $0<\phi<\infty$. *Then*

$$e^{-x^2/4}h_n(x)=\frac{b_n}{\sqrt{\sinh\phi}}\exp\left(\frac{N^2}{8}(2\phi-\sinh 2\phi)\right)\left(1+O\left(\frac{1}{n(e^{-\phi}\sinh\phi)^3}\right)\right) \tag{2.4}$$

with $b_n=(8\pi)^{-1/4}n^{-1/4}\sqrt{n!}$.

(c) *Let* $x=N-3^{-1/3}n^{-1/6}t$, *t bounded. Then*

$$e^{-x^2/4}h_n(x)=d_n\left(A(t)+O\left(\frac{1}{n^{2/3}}\right)\right), \tag{2.5}$$

where $d_n=3^{1/3}(2/\pi^3)^{1/4}n^{-1/12}\sqrt{n!}$ *and* $A(t)$ *is the Airy function of* [S, §1.81].

3. Proof of Theorem 2.1

We turn to the proof of Theorem 2.1. The notation of Theorem 2.9 will be used throughout this section.

3.1. The case $p<2$

We start with a simple, but useful lemma.

Lemma 3.1. *Let* g *be a non-negative periodic function with period* 1, *and let* r *be a non-negative integer. Then*

$$\int_{\mathbf{R}} g(x)x^{2r}e^{-x^2/\omega}\,dx=\omega^{r+1/2}\Gamma\left(r+\tfrac{1}{2}\right)\int_0^1 g(x)\,dx(1+O(e^{-\omega})),$$

as $\omega\to\infty$.

Proof. We have

$$\begin{aligned}\int_{\mathbf{R}} g(x)x^{2r}e^{-x^2/\omega}\,dx&=\sum_{k\in\mathbf{Z}}\int_k^{k+1} g(x)x^{2r}e^{-x^2/\omega}\,dx\\&=\int_0^1 g(x)\left(\sum_{k\in\mathbf{Z}}(x+k)^{2r}e^{-(k+x)^2/\omega}\right)dx.\end{aligned}$$

But the sum in the parentheses is $\omega^{r+1/2}\Gamma\left(r+\frac{1}{2}\right)(1+O(\omega^r e^{-\pi^2\omega}))$ uniformly in x, as is easily seen by Poisson's summation formula. □

We shall only be concerned with the case $g(x)=|\sin\pi x|^p$ (or $|\cos\pi x|^p$). Hence, we define the following quantity, which may be looked upon as "the L^p mean of a harmonic oscillation":

$$\mu_p=\left(\int_0^1|\sin\pi x|^p\,dx\right)^{1/p}=\left(\frac{\Gamma\left(\frac{1}{2}(p+1)\right)}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}(p+2)\right)}\right)^{1/p}; \tag{3.1}$$

the last identity follows from the substitution $t=\sin^2\pi x$.

Now, fix p with $0<p<2$. Let $\varepsilon_n=\lambda\sqrt{\log n/n}$, where λ is a large constant, and put $\alpha_n=N\sin\varepsilon_n\sim 2\lambda\sqrt{\log n}$. We shall see that the main contribution to the L^p-norm comes from the region $|x|\le\alpha_n$, i.e. $|\varphi|\le\varepsilon_n$. Namely, let $h_n=h_n^{(1)}+h_n^{(2)}$ with $h_n^{(1)}(x)=h_n(x)\mathbf{1}_{\{x:|x|\le\alpha_n\}}$. Furthermore, let $\tilde{h}_n$ arise from $h_n^{(1)}$ by dropping the O-term in (2.3). For simplicity, suppose that n is odd. Put $f(\varphi)=\frac{1}{4}(2\varphi+\sin 2\varphi)=\varphi+O(\varphi^3)$ and $\beta=\frac{1}{2}(2-p)>0$. Changing the variable of integration from x over φ to $y=N^2f(\varphi)/2\pi$ and noting that $f'(\varphi)=\cos^2\varphi$, we obtain

$$(3.2)\qquad \begin{aligned}\|\tilde{h}_n\|_p^p&=\frac{a_n^p}{\sqrt{2\pi}}\int_{-\alpha_n}^{\alpha_n}\left|\sin\frac{N^2f(\varphi)}{2}\right|^p(\cos\varphi)^{-p/2}e^{-\beta x^2/2}\,dx\\&=\frac{a_n^p\sqrt{2\pi}}{N}\int_{-N^2f(\varepsilon_n)/2\pi}^{N^2f(\varepsilon_n)/2\pi}|\sin\pi y|^p\left(\cos f^{-1}\left(\frac{2\pi y}{N^2}\right)\right)^{\beta-2}\\&\quad\times\exp\left(-\frac{1}{2}\beta N^2\sin^2 f^{-1}\left(\frac{2\pi y}{N^2}\right)\right)dy\\&=\frac{a_n^p\sqrt{2\pi}}{N}\int_{-N^2f(\varepsilon_n)/2\pi}^{N^2f(\varepsilon_n)/2\pi}|\sin\pi y|^pe^{-2\pi^2\beta y^2/N^2}\,dy\left(1+O\left(\frac{1}{n}\right)\right),\end{aligned}$$

where the last step follows by Taylor expansions. Now, $N^2f(\varepsilon_n)/2\pi\sim c\lambda\sqrt{n\log n}$. The standard estimate of a Gaussian tail shows that the domain of integration may be changed to the entire real line with an error $O(n^{-s})$ for any s if $\lambda=\lambda(s)$ is large enough. But then Lemma 3.1 (with $r=0$) applies, and we conclude that

$$(3.3)\qquad \|\tilde{h}_n\|_p^p=\frac{\sqrt{2\pi}}{N}\sqrt{\frac{N^2}{2\pi^2\beta}}\sqrt{\pi}\,(a_n\mu_p)^p\left(1+O\left(\frac{1}{n}\right)\right)=\frac{(a_n\mu_p)^p}{\sqrt{\beta}}\left(1+O\left(\frac{1}{n}\right)\right),$$

which is (2.1) with $\tilde{h}_n$ in the place of h_n. Since the sine term in (2.3) contributes to this with the non-zero factor μ_p^p, (3.3) holds with $\tilde{h}_n$ replaced by $h_n^{(1)}$.

It remains to take care of $h_n^{(2)}$. We use Lyapounov's inequality, which, for a function f on a finite measure space with total mass M and $0<p\le q<\infty$, may be written

$$\|f\|_p\le M^{1/p-1/q}\|f\|_q.$$

Take $q=2$ and put $\varepsilon=1/p-1/2>0$. Then, for large n,

$$\|h_n^{(2)}\|_p\le\{2\gamma[\alpha_n,\infty)\}^{\varepsilon}\|h_n^{(2)}\|_2\le e^{-\varepsilon\alpha_n^2/2}\|h_n\|_2\le e^{-\varepsilon\lambda^2\log n}\sqrt{n!}\,,$$

which is $O(a_n/n^s)$ if $\lambda(s)$ is large. This establishes (2.1).

3.2. The case $p>2$

Now, fix $p>2$. Let ε and ω be positive numbers, ε small and ω large enough in a sense to be specified later. Put $h_n = h_n^{(1)} + h_n^{(2)} + h_n^{(3)}$ with

$$h_n^{(1)}(x) = h_n(x)\mathbf{1}_{\{x:|x|\geq N+n^{-1/6}\}},$$
$$h_n^{(2)}(x) = h_n(x)\mathbf{1}_{\{x:|x|\leq N-n^{-1/6}\}},$$

and let $\tilde{h}_n$ arise from $h_n^{(1)}$ by dropping the O-term in (2.4) and restricting x to $N\cosh\varepsilon \leq |x| \leq N\cosh\omega$.

We treat $\tilde{h}_n$ first. Changing variables from x to ϕ, we write $\|\tilde{h}_n\|_p^p$ as

$$\frac{2b_n^p}{\sqrt{2\pi}} N \int_\varepsilon^\omega (\sinh\phi)^{1-p/2} \exp\biggl(N^2\biggl(\frac{p}{8}(2\phi - \sinh 2\phi) + \frac{p-2}{4}\cosh^2\phi\biggr)\biggr)\,d\phi$$
$$=: \frac{2b_n^p}{\sqrt{2\pi}} N \int_\varepsilon^\omega G(\phi) e^{N^2 g(\phi)}\,d\phi.$$

Elementary calculus shows that g, defined as above for $\phi\geq 0$, has a strict global maximum at $\phi_0 = \frac{1}{2}\log(p-1)$ with $g(\phi_0) = \frac{1}{8}p\log(p-1)$, $g''(\phi_0) = -\frac{1}{2}(p-2)$, and $\sinh\phi_0 = (p-2)/2\sqrt{p-1}$. If $\varepsilon < \phi_0 < \omega$ the Laplace method (e.g. [Br]) gives

$$\|\tilde{h}_n\|_p^p = \frac{2b_n^p}{\sqrt{2\pi}} N G(\phi_0) e^{N^2 g(\phi_0)} \sqrt{\frac{2\pi}{N^2(-g''(\phi_0))}} \biggl(1 + O\biggl(\frac{1}{N^2}\biggr)\biggr)$$
$$= (2b_n)^p \biggl(\frac{p-1}{2(p-2)}\biggr)^{(p-1)/2} (p-1)^{np/2} \biggl(1 + O\biggl(\frac{1}{n}\biggr)\biggr),$$

which, after taking the pth root, is (2.2) with $\tilde{h}_n$ instead of h_n. It is clear that the O-term in (2.4) is bounded for $\phi \geq N + n^{-1/6}$. Hence, we may replace $\tilde{h}_n$ by $h_n^{(1)}$, in fact with an exponentially small difference (the contribution from G close to $\phi=0$ is only a power of n).

We complete the proof of (2.2) by claiming that the contributions from $h_n^{(2)}$ and $h_n^{(3)}$ are also exponentially smaller than that of $h_n^{(1)}$, proving this for $h_n^{(2)}$ only. Let $\gamma(p)$ denote constants depending on p, not necessarily the same each time. Note that the O-term in (2.3) is bounded in the relevant region. Hence, $|h_n(x)| \leq cn^{\gamma(p)}\sqrt{n!}\,e^{x^2/4}$ for $|x| \leq N - n^{-1/6}$, and

$$\|h_n^{(2)}\|_p^p \leq cn^{\gamma(p)}(n!)^{p/2} \int_0^N e^{(p/4-1/2)x^2}\,dx \leq cn^{\gamma(p)}(n!)^{p/2} e^{(p-2)n}.$$

Thus,

$$\|h_n^{(2)}\|_p \leq cn^{\gamma(p)}\sqrt{n!}\,e^{(p-2)n/p},$$

and we need only notice that $(p-2)/p < \frac{1}{2}\log(p-1)$ for $p>2$, as is easily seen. This completes the proof of Theorem 2.1.

Remark 3.2. Theorem 2.1 may be extended to asymptotic expansions. Thus, for $p<2$ one has, for any k,

$$\|h_n\|_p = \frac{c(p)}{n^{1/4}}\sqrt{n!}\left(1+\frac{c_1(p)}{n}+\ldots+\frac{c_k(p)}{n^k}+O\left(\frac{1}{n^{k+1}}\right)\right), \tag{3.4}$$

and similarly for $p>2$. The main reason for this is that the asymptotics of Theorem 2.9 can be continued to any order [PR]. For $p>2$ one then merely inserts these terms into the correction terms that arise from the Laplace method.

For $p<2$ the situation is a little more complicated, since the expansion (2.3) starts with a sine expression rather than with 1, making it less obvious how to take the pth power close to its zeros. The problem can be resolved by modifying the substitution leading to (3.2) and applying Lemma 3.1 with various values of r; cf. [L].

The paper [L] also contains a calculation of the first correction term. Some rather tedious work yields

$$c_1(p) = \frac{p-1}{8(2-p)}, \qquad p<2,$$
$$c_1(p) = -\frac{p^2-4p+6}{24(p-2)^2}, \quad p>2.$$

Thus, we can sharpen Theorem 2.1 to

$$\|h_n\|_p = \frac{c(p)}{n^{1/4}}\sqrt{n!}\left(1+\frac{p-1}{8(2-p)n}+O\left(\frac{1}{n^2}\right)\right), \qquad p<2,$$
$$\|h_n\|_p = \frac{c(p)}{n^{1/4}}\sqrt{n!}\,(p-1)^{n/2}\left(1-\frac{p^2-4p+6}{24(p-2)^2 n}+O\left(\frac{1}{n^2}\right)\right), \quad p>2.$$

4. Convergence in distribution of Wick powers

In the light of Theorem 2.1 one may suspect that if ξ is a standard Gaussian variable, then $h_n(\xi)$ converges in distribution when normalized by $n^{-1/4}\sqrt{n!}$. We shall see that this is indeed the case, which will give us a new proof of Theorem 2.1(a).

To this end we make use of (2.3), letting a_n keep its meaning from there. By disregarding large values of x it is easily seen that, for odd n, $h_n(\xi)/a_n$ converges in distribution if and only if $e^{\xi^2/4}\sin(\sqrt{n}\,\xi)$ does, the limits being the same (for even n, sin should be replaced by cos). We shall prove a slightly more general statement, based on the following reformulation of "Fejér's lemma" [K]. For the notion of Rényi mixing, see [R].

Lemma 4.1. *Let X be an absolutely continuous random variable, and let g be a periodic function with period T. Then $g(\omega X)$ is Rényi mixing, as $\omega \to \infty$. More precisely,*

$$P(g(\omega X) \in A; E) \to P(g(U) \in A)P(E),$$

as $\omega \to \infty$, for any event E and Borel set $A \subset \mathbf{R}$, where U is uniformly distributed on $[0, T]$.

A combination of the above lemma with Theorem 4.5 of [Bi] yields

Proposition 4.2. *Let X be absolutely continuous. Then, as $\omega \to \infty$, both $(X, \sin \omega X)$ and $(X, \cos \omega X)$ converge in distribution to $(X, \sin U)$, where U is uniformly distributed on $[0, 2\pi]$ and independent of X.*

Letting $X = \xi$ be standard Gaussian, an application of the continuous mapping theorem to Proposition 4.2 together with the remarks at the beginning of this section establishes the desired result. Recall that the nth Wick power of ξ satisfies $:\xi^n: = h_n(\xi)$ so that $\|:\xi^n:\|_p = \|h_n\|_p$, cf. [J].

Theorem 4.3. *Let ξ be standard Gaussian. Then, as $n \to \infty$,*

$$\frac{:\xi^n:}{n^{-1/4}\sqrt{n!}} \xrightarrow{\mathrm{d}} \left(\frac{2}{\pi}\right)^{1/4} e^{\xi^2/4} \sin U,$$

where U is uniform on $[0, 2\pi]$ and independent of ξ.

Remark 4.4. Together with the (easily established) fact that $\|:\xi^n:/a_n\|_p$ is bounded if $p < 2$, this offers a simple probabilistic proof of Theorem 2.1(a), except for the error bound $O(n^{-1})$. One merely notes that $\|e^{\xi^2/4}\|_p = [2/(2-p)]^{1/2p}$ for $p < 2$, and that $\|\sin U\|_p = \mu_p$.

5. An extremal problem on Wiener chaos

We shall use the above results to give a partial solution to the following extremal problem. Let H be a Gaussian Hilbert space and consider $H^{:n:}$, the homogeneous Wiener chaos of order n (e.g. [J]). Using multiplicative properties of the Skorohod integral, [J] shows in Remark 7.37 that when p is an even integer

$$(5.1) \qquad \begin{cases} \text{the functional } \|X\|_p / \|X\|_2 \text{ is maximized for } X \in H^{:n:} \\ \text{by letting } X \text{ be a Wick power } :\xi^n: . \end{cases}$$

He also asks whether this holds for other values of p. We shall see that the answer is largely negative if $p < 2$.

Proposition 5.1. *Let H be an infinite-dimensional Gaussian Hilbert space, and $0<p<2$. Then* (5.1) *fails for all sufficiently large n.*

Proof. Let ξ and $\{\xi_i\}_{i=1}^{\infty}$ be orthonormal elements of H. Suppose that (5.1) holds for a certain $n\geq 1$, so that

$$\frac{\|:\xi^n:\|_p}{\|:\xi^n:\|_2} \geq \frac{\|X\|_p}{\|X\|_2}$$

for all $X\in H^{:n:}$. Take $X=X_k=\sum_{i=1}^{k} :\xi_i^n:$. By the central limit theorem $X_k/\|X_k\|_2$ converges in distribution and with all moments to a standard Gaussian variable, i.e. to ξ. Hence,

$$\frac{\|:\xi^n:\|_p}{\|:\xi^n:\|_2} \geq \left\| \frac{X_k}{\|X_k\|_2} \right\|_p \to \|\xi\|_p =: \varkappa(p).$$

But $\|:\xi^n:\|_p=\|h_n\|_p$. Thus,

$$\|h_n\|_p \geq \varkappa(p)\sqrt{n!}\,. \tag{5.2}$$

But this fails for large n by Remark 2.4. □

We believe that more is true; that (5.2) is false for all $n\geq 2$ and $0<p<2$, so that the phrase "sufficiently large" can be removed from Proposition 5.1, and that a counterexample is furnished by summing sufficiently many Wick powers. As an illustration we give a proof for $n=2$ based on numerical integration. Here one only needs two Wick powers. (This seems not to be the case for $n>2$. Instead, numerical evidence suggests that the number of Wick powers then required increases indefinitely as $p\to 0$.)

The integrals below have been calculated to nine decimal places using the computer algebra program Maple, cf. [L] for details. This means that the proof is not completely rigorous, but can, no doubt, be made so at wish by tracking the errors of the integrals more precisely. As a compensation, there is an extra factor of $\frac{3}{4}$ in (5.4) below.

Proposition 5.2. *Suppose that* $\dim H\geq 2$. *Then* (5.1) *fails in $H^{:2:}$ for $p<2$.*

Proof. Let ξ and η be independent standard Gaussian variables in H. We claim that

$$\frac{\|:\xi^2:\|_p}{\|:\xi^2:\|_2} < \frac{\|:\xi^2:+:\eta^2:\|_p}{\|:\xi^2:+:\eta^2:\|_2}, \quad 0<p<2.$$

By elementary calculus this is equivalent to

$$f(p) := \int_0^{\infty} \left(\frac{2^{p/2}}{\sqrt{\pi}} \frac{\left|x-\frac{1}{2}\right|^p}{\sqrt{x}} - |x-1|^p \right) e^{-x}\,dx < 0 \tag{5.3}$$

for $0<p<2$. Trivially, $f(0)=f(2)=0$. One can express f in terms of confluent hypergeometric functions, which offers a simple way to calculate it to great accuracy. Differentiating under the integral, one obtains expressions for f' and f'' similar to (5.3). Simple estimates and numerical integration then show that $|f''|\le A=4$ on $[0,2]$.

Now, given $a\in[0,2]$ with $f(a)\le 0$, we have $f(p)\le f(a)+f'(a)(p-a)+\frac{1}{2}A(p-a)^2$ so that, starting at a and moving in either direction, f cannot reach zero before $p=a+\Delta p$ with $\Delta p=\big(-f'(a)\pm\sqrt{f'(a)^2-2Af(a)}\,\big)/A$. The following iterations thus guarantee that $f(p)<0$ for $p_k<p<2$:

$$(5.4)\qquad \begin{cases} p_0=2, \\ p_{k+1}=p_k+\dfrac{3}{4}\Delta p_k=p_k-\dfrac{3}{4}\dfrac{f'(p_k)+\sqrt{f'(p_k)^2-2Af(p_k)}}{A}, \end{cases}$$

where the extra factor $\frac{3}{4}$ has been added for safety. Note that we are moving to the left so that $\Delta p<0$. The numbers $f'(p_k)$ are calculated by numerical integration, and $f(p_k)$ by the hypergeometric representation mentioned above. The results are shown in Table 1. Since $p_9<1$, we conclude that $f<0$ on $[1,2)$.

Table 1. Results of the iterations (5.4). The values are actually calculated to nine decimal places.

k	p_k	$f(p_k)$	$f'(p_k)$	Δp_k
0	2	0	0.1812	−0.0906
1	1.9320	−0.0113	0.1532	−0.1228
2	1.8399	−0.0239	0.1205	−0.1435
3	1.7323	−0.0351	0.0888	−0.1566
4	1.6149	−0.0438	0.0606	−0.1640
5	1.4919	−0.0498	0.0367	−0.1672
6	1.3665	−0.0531	0.0170	−0.1672
7	1.2411	−0.0542	0.0009	−0.1648
8	1.1174	−0.0534	−0.0123	−0.1604
9	0.9971	−0.0513	−0.0232	−0.1545

Starting a similar iteration at $p_0=0$, one also reaches $p=1$ after a few iterations, and so $f<0$ on $(0,2)$. □

Remark 5.3. For $p=1$ we can give the value $f(1)=2\big(\sqrt{e}-\sqrt{\pi}\,\big)/e\sqrt{\pi}<0$. By continuity, $f<0$ in a neighbourhood of $p=1$ without appealing to numerics.

We close with a brief discussion of possible generalizations of (5.1). Fix a Gaussian Hilbert space H and let $J_n(p,q)$ be the statement (5.1) with $\|X\|_2$ replaced by $\|X\|_q$. The argument of Proposition 5.1 shows that if $\dim H=\infty$ then $J_n(p,q)$ fails for large n whenever $p<q$ and $q\geq 2$. If $0<p,q<2$, the same holds provided that $g(p)<g(q)$, where $g(p)=c(p)/\varkappa(p)$ with $c(p)$ as in Theorem 2.1. By Remark 2.6 this is true at least for p fixed and q close to 2.

One may be tempted to conjecture that $J_n(p,q)$ holds whenever $p>q$. This is false, however. Namely, since $g(p)=(2\pi)^{-1/4}\Gamma\left(\frac{1}{2}(p+2)\right)^{-1/p}(2/(2-p))^{1/2p}$, straightforward calculations show that

$$\lim_{p\to 0} g'(p) = -\frac{1}{48(2\pi)^{1/4}}(\pi^2-3)e^{\gamma/2+1/4}<0,$$

where γ is Euler's constant. Hence $g(p)<g(q)$ for small $0<q<p$.

We have performed further numerical integration using the NAG software package. The cases $\dim H\leq 3$ and $n=2,3,4$ and 9 have been studied in some detail. The results indicate that $J_n(p,q)$ is in general false whenever $p<q$ or $p\leq 2$. We still believe that $J_n(p,q)$ holds at least for $p>q\geq 2$. A proof of this seems to require new ideas, however.

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