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A non-existence theorem of lacunas for hyperbolic differential operators with constant coefficients

Motoo Uchida

Abstract. We prove a theorem on non-existence of lacunas of the fundamental solution for hyperbolic differential operators with constant coefficients.

Introduction

In [ABG], Atiyah, Bott and Gårding clarified and even generalized the Petrovskii theory [P] of lacunas for hyperbolic differential operators. Their theory allows us to draw some conclusions on (non)-existence of lacunas of fundamental solutions. For example, it is proved in [ABG] (Theorem 7.7 of Part II) that the fundamental solution of a hyperbolic operator $P(D)$ in n variables has no strong lacunas if $n \leq 3$. (We say that the fundamental solution has no strong lacunas if its support is equal to the propagation cone.) It follows also from the theory of [ABG] (by Theorem 8.9) that, for any fixed strictly hyperbolic homogeneous operator $P_m(D)$ of order m, there is an open dense subset U (which is in fact Zariski open) of the complex affine space of polynomials in D of order less than m such that, if $Q(D) \in U$, then the fundamental solution of a hyperbolic operator $P_m + Q$ has no strong lacunas. However, this open set U is not given in any explicit way.

It seems up to now that any explicit theorem on (non)-existence of strong lacunas for hyperbolic operators in n variables, $n \geq 4$, is not known. (The local lacunas for hyperbolic operators are studied in several papers. For a recent study, see [V] and the references cited there.)

In this short paper, based on the Atiyah–Bott–Gårding theory, we shall give a criterion of non-existence of lacunas for hyperbolic differential operators in n variables, $n \geq 4$, in a simple explicit form (even in weakly hyperbolic cases). The main result is Theorem 1.1 in the next section. The proof is very short and even an application of the existing theory, but this result will be the first that referred to an explicit relation between strong lacunas of the fundamental solutions and the total symbols of hyperbolic operators. (Note that only the principal symbols of operators are concerned in the theory of Petrovskii [P] and Atiyah–Bott–Gårding [ABG].) For example, by this theorem, we can finally understand a result of Mathisson [M] as a part of general theory. In fact, for any homogeneous hyperbolic operator $P(D)$ and a non-zero constant C, the absence of strong lacunas for $P(D)+C$ follows from Theorem 1.1. (See the note referring to Mathisson's result on p. 175 of [ABG, Part II].)

We also establish, in Theorem 1.1, an equality in the general inclusion $\mathrm{WF}(E)\subset \mathcal{W}$ (see Section 1 for the notation) on the singularities of fundamental solutions. The singularities of fundamental solutions are studied in full detail by Hörmander [H2] in the case of at most double characteristics.

1. Main result

Let n be a positive integer, and let D denote $(D_1, \dots, D_n)$, where D_ν denotes the symbol of partial differential $-i\partial/\partial x_\nu$ on $\mathbf{R}^n$.

Let $\theta\in\mathbf{R}^n\setminus\{0\}$. Let $P(D)$ be a differential operator on $\mathbf{R}^n$ with constant coefficients (i.e., a polynomial in D), and assume $P(D)$ to be hyperbolic in the direction θ in the sense of Gårding. Let K be the propagation cone of P with respect to θ (see (3.55) of [ABG, Part I]), and let

$$\mathcal{W}=\{(x,\xi)\in T^*\mathbf{R}^n \mid x\in K_\xi,\ \xi\neq 0\},$$

where K_ξ denotes the local propagation cone of P at ξ (i.e., the propagation cone of the localization P_ξ of P) with respect to θ (see (3.61) of [ABG, Part I]). Let E be the fundamental solution of $P(D)$ with support in $\{x|x\theta\geq 0\}$. It is known in general that

$$\operatorname{supp} E\subset K \quad \text{and} \quad \mathrm{WF}(E)\subset \mathrm{WF}_A(E)\subset \mathcal{W},$$

where $\mathrm{WF}(E)$ (resp. $\mathrm{WF}_A(E)$) denotes the C^∞ (resp. analytic) wave front set of E. (For the latter inclusion, refer also to Theorem 12.6.2 of [H1]. See also Example 3.2.3 of [SKK, Chapter 1].)

Let $P(z)$ be the polynomial in z, $z=(z_1, \dots, z_n)$, that corresponds to $P(D)$ by $z\mapsto D$. Let $V(P)$ denote the closed algebraic set in $\mathbf{C}^n$ defined by $P(z)=0$.

Theorem 1.1. *Assume that $V(P)$ is irreducible and everywhere non-singular. Then we have*

$$\operatorname{supp} E = K \tag{1.1}$$

and

$$\mathrm{WF}(E) = \mathrm{WF}_A(E) = \mathcal{W}. \tag{1.2}$$

This theorem is a special case of the following more general result.

Theorem 1.2. *Let $b(s)$ be the Bernstein–Sato polynomial of $P(z)$. If $b(s)$ has no integer roots except $s=-1$, we have* (1.1) *and* (1.2).

Since $b(s)=(s+1)(s+n/2)$ if P is homogeneous of degree 2 (n being the rank of the quadratic form P), we have (1.1) and (1.2) for such P if n is odd or $n=2$. This explains, in a clear way, the absence of strong lacunas for the d'Alembert operator

$$P(D) = D_1^2 + \ldots + D_{n-1}^2 - D_n^2$$

in even space dimension (i.e., in the case where $n-1$ is even).

2. Proof of Theorem 1.2

Theorem 1.2 follows from the main result of [ABG] (Theorem 8.9 of Part I or Lemma 9.5 of Part II) and the existence of the Bernstein–Sato polynomials. For the latter, see [B1], [B2]. We follow the notation of Section 1.

Let us first recall Lemma 9.5 of [ABG, Part II].

Lemma 2.1. ([ABG]) *Let $P(D)$ be a hyperbolic differential operator with constant coefficients. For $k \in \mathbf{N}$, let E_k denote the forward fundamental solution of $P(D)^k$ with respect to θ. For $k \gg 1$, we have*

$$\operatorname{supp} E_k = K \quad \textit{and} \quad \mathrm{WF}(E_k) = \mathcal{W}. \tag{2.1}$$

The fundamental solution E_k is given by

$$E_k(x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{ix(\xi - it\theta)} P(\xi - it\theta)^{-k} \, d\xi \tag{2.2}$$

with $t \gg 0$. The right-hand side of (2.2) is regarded as the (inverse) Fourier transform of $P(\xi - it\theta)^{-k}$ as a tempered distribution on $\mathbf{R}^n$. On the other hand, by the theory of Bernstein–Sato polynomials, we can find a polynomial $Q(s;z,D_z)$ in s with coefficients in $\mathbf{C}[z,D_z]$ (i.e. the ring of differential operators with polynomial coefficients in z), and a non-zero polynomial $b(s)$ so that

$$Q(s;z,D_z)P(z)^{s+1}=b(s)P(z)^s.$$

(The monic polynomial of the minimum degree of such $b(s)$ is called the Bernstein–Sato polynomial.) Hence, by (2.2), we have

$$b(-k)E_k(x)=Q^\vee(-k;x,D_x)E_{k-1}(x), \tag{2.3}$$

where $Q^\vee(s;x,D_x)$ denotes the formal Fourier transform of $Q(s;z,D_z)$:

$$Q^\vee(s;x,D_x)=\sum_{k,\alpha,\beta} s^k C_{k\alpha\beta}D_x^\alpha(-x)^\beta, \quad \text{if } Q(s;z,D_z)=\sum_{k,\alpha,\beta} s^k C_{k\alpha\beta}z^\alpha D_z^\beta.$$

By (2.3) a lacuna of E_{k-1} is also that of E_k (and the wave front set of E_{k-1} contains that of E_k) provided that $b(-k)\neq 0$. Hence Theorem 1.2 follows from (2.1) and (2.3). □

Remark (*added in the revised version*). For a differential operator $P(D)$ with hyperbolic principal part, even if it is not hyperbolic in the sense of Gårding, we are able to construct its forward fundamental solution E as a hyperfunction by defining

$$E(x)=\int_C \frac{e^{ix\zeta}}{P(\zeta)}\,d\zeta,$$

where C is the cycle given by $\zeta=\xi-iR(1+|\xi|)^a\theta$, with parameter $\xi\in\mathbf{R}^n$, for some positive number $a<1$ and $R\gg 1$. The right-hand side defines a hyperfunction on $\mathbf{R}^n$ supported in the propagation cone K of $P_m(D)$ and satisfies

$$P(D)E(x)=\delta(x).$$

For this hyperfunction fundamental solution, we are also able to prove Theorem 1.2 for the support equality (1.1). For the proof, we need an analogue of the support part of Lemma 2.1 for hyperfunction fundamental solutions. In order to prove this generalization, we have only to notice that the hyperfunction fundamental solution defined above belongs in fact to a space of Gevrey ultra-distributions $\mathcal{D}^{(\gamma)\prime}(\mathbf{R}^n)$ for $\gamma=1/a$ (see [K]). Then we can use the topology of ultra-distributions and reduce the support equality for E_k, $k\gg 1$, in Lemma 2.1 (for differential operators with hyperbolic principal part) to the case of homogeneous hyperbolic operators by the following lemma which can be easily proved.

Lemma 2.2. *Let $P_m(D)$ be the principal part of $P(D)$ (which is hyperbolic with respect to θ by assumption), $m=\operatorname{ord}P$. Let $F(x)$ be the forward fundamental solution to $P_m(D)$. Then*

$$t^{n-m}E(tx)\to F(x), \quad \textit{as } t\to 0,$$

in $\mathcal{D}^{(\gamma)\prime}(\mathbf{R}^n)$.

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Motoo Uchida
Department of Mathematics
Graduate School of Science
Osaka University
Machikaneyama-Cho 1-16
Toyonaka, Osaka 560-0043
Japan
email: uchida@math.wani.osaka-u.ac.jp