Ark. Mat., **40** (2002), 301–321
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A separation theorem and Serre duality for the Dolbeault cohomology

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Abstract. Let X be a complex manifold with finitely many ends such that each end is either q-concave or $(n-q)$-convex. If $q<\frac{1}{2}n$, then we prove that $H^{p,n-q}(X)$ is Hausdorff for all p. This is not true in general if $q\geq\frac{1}{2}n$ (Rossi's example with $n=2$ and $q=1$). If all ends are q-concave, then this is the classical Andreotti–Vesentini separation theorem (and holds also for $q\geq\frac{1}{2}n$). Moreover the result was already known in the case when the q-concave ends can be 'filled in' (again also for $q\geq\frac{1}{2}n$). To prove the result we first have to study Serre duality for the case of more general families of supports (instead of the family of all closed sets and the family of all compact sets) which is the main part of the paper. At the end we give an application to the extensibility of CR-forms of bidegree (p,q) from $(n-q)$-convex boundaries, $q<\frac{1}{2}n$.

1. Introduction

Let X be an n-dimensional complex manifold, $E\to X$ a holomorphic vector bundle and E^* the dual of E.

We use the following standard notation. The space $\mathcal{E}^{p,q}(X,E)$ is the Fréchet space of E-valued C^∞-forms of bidegree (p,q) on X given the topology of uniform convergence of the forms and all their derivatives on compact sets. For each closed $C\subseteq X$, $\mathcal{D}^{p,q}_C(X,E)$ denotes the space of all $f\in\mathcal{E}^{p,q}(X,E)$ with $\operatorname{supp} f\subseteq C$, considered also as a Fréchet space, with the topology induced from $\mathcal{E}^{p,q}(X,E)$. Finally, $\mathcal{D}^{p,q}(X,E)$ is the space of forms with compact support from $\mathcal{E}^{p,q}(X,E)$, given the finest local convex topology such that, for each compact $K\Subset X$, the embedding $\mathcal{D}^{p,q}_K(X,E)\to\mathcal{D}^{p,q}(X,E)$ is continuous. The cohomology groups

$$H^{p,q}(X,E)=\frac{\mathcal{E}^{p,q}(X,E)\cap\operatorname{Ker}\bar\partial}{\bar\partial\mathcal{E}^{p,q-1}(X,E)}\quad\text{and}\quad H^{p,q}_c(X,E)=\frac{\mathcal{D}^{p,q}(X,E)\cap\operatorname{Ker}\bar\partial}{\bar\partial\mathcal{D}^{p,q-1}(X,E)}$$

will be considered as topological vector spaces with the corresponding factor topologies.

This research was partially supported by TMR Research Network ERBFMRXCT 98063.

If X is q-concave in the sense of Andreotti–Grauert and $1 \le q \le n-1$, then $H^{p,n-q}(X,E)$ is Hausdorff (for all p), by the Andreotti–Vesentini separation theorem [AV]. In the present paper we consider the situation, first studied by Ramis [Ra], when X is q-concave only at some of its ends, and q^*-convex at the other ends—for certain appropriate q^*. More precisely, we assume that X is q-concave-q^*-convex in the following sense.

1.1. *Definition.* The manifold X is called *q-concave-q^*-convex*, $1 \le q, q^* \le n$, if it is connected and admits a C^∞ function $\varrho: X \to \mathbf{R}$ without (absolute) minimum and maximum such that the sets $\{z \mid s \le \varrho(z) \le t\}$, $\inf \varrho < s \le t < \sup \varrho$, are compact and the following two conditions are fulfilled:

(1) There exists $s_0 > \inf \varrho$ such that the Levi form of ϱ has at least $n-q+1$ positive eigenvalues on $\{z \mid \inf \varrho < \varrho(z) < s_0\}$.

(2) There exists $t_0 < \sup \varrho$ such that the Levi form of ϱ has at least $n-q^*+1$ positive eigenvalues on $\{z \mid t_0 < \varrho(z) < \sup \varrho\}$.

If $q^* < n-q$, then $H^{p,n-q}(X,E)$ is again Hausdorff. This was first obtained in a more general setting for sheaves on spaces with singularities by J.-P. Ramis [Ra].(¹) Another, direct proof can be found in [LL3].

For $q^* = n-q$ the situation is more complicated. First of all there is the example of Rossi [Ro] of a 2-dimensional 1-concave-1-convex complex manifold such that the "hole" at the 1-concave ends cannot be "filled in". This implies (by similar arguments as in [HL, Section 23]) the existence of 2-dimensional 1-concave-1-convex manifolds X such that $H^{0,1}(X,\mathcal{O})$ is not Hausdorff. For those examples the "hole" at the 1-concave ends cannot be "filled in"—otherwise $H^{0,1}(X,\mathcal{O})$ is Hausdorff. In fact, there is the following theorem (see [HL, Theorem 19.1′]).

1.2. Theorem. *Suppose X is q-concave-$(n-q)$-convex, $1 \le q \le n-1$. Then $H^{p,n-q}(X,E)$ is Hausdorff for all p, provided the following two additional conditions are fulfilled*:

ext(X): *X is an open subset of some larger complex manifold $\widetilde{X}$ such that (with the notation as in Definition 1.1) $\{z \mid \varrho(z) < s_0\} = X \cap \widetilde{D}$ for a certain relatively compact domain $\widetilde{D} \Subset \widetilde{X}$*;

ext(E): *there is a holomorphic vector bundle $\widetilde{E} \to \widetilde{X}$ with $E = \widetilde{E}|_X$.*

The original proof of Theorem 1.2 given in [HL] is not so easy. In [LL1, Theorem 4.1] the following simple proof is given.

(¹) There is a misprint in [Ra, Theorem 2]. In fact, by the formulation given there $H^{0,1}(X,\mathcal{O})$ should be Hausdorff if $n=2$ and X is 1-concave-1-convex, but this is not true, as it follows from the Rossi example which we shall discuss some lines below.

Proof. Set $\mathcal{F}=\mathcal{D}^{n-p,q+1}(X,E^*)\cap\bar\partial\mathcal{D}^{n-p,q}(\widetilde X,\widetilde E^*)$. Then, clearly,

$$\mathcal{F}\supseteq\bar\partial\mathcal{D}^{n-p,q}(X,E^*) \tag{1.1}$$

and it is easy to see that

$$\dim\frac{\mathcal{F}}{\bar\partial\mathcal{D}^{n-p,q}(X,E^*)}\le\dim H^{n-p,q}(\widetilde D,\widetilde E^*). \tag{1.2}$$

Since $\widetilde D$ is q-convex, the right-hand side of (1.2) is finite, by the Andreotti–Grauert theorem [AG]. Since $\widetilde X$ is $(n-q)$-convex, by the same Andreotti–Grauert theorem, also $H^{p,n-q}(\widetilde X,E)$ is finite-dimensional and hence Hausdorff. By Serre duality it follows that $H_c^{n-p,q+1}(\widetilde X,E^*)$ is Hausdorff, i.e. $\bar\partial\mathcal{D}^{n-p,q}(\widetilde X,E^*)$ is topologically closed in $\mathcal{D}^{n-p,q+1}(\widetilde X,E^*)$, which implies that $\mathcal{F}$ is topologically closed in $\mathcal{D}^{n-p,q+1}(X,E^*)$. Hence, by (1.1), the topological closure of $\bar\partial\mathcal{D}^{n-p,q}(X,E^*)$ is contained in $\mathcal{F}$. Since the right-hand side of (1.2) is finite, this yields that $\bar\partial\mathcal{D}^{n-p,q}(X,E^*)$ is of finite codimension in its topological closure and hence equal to this closure, i.e. $H_c^{n-p,q+1}(X,E^*)$ is Hausdorff. By Serre duality it follows that $H^{p,n-q}(X,E)$ is Hausdorff. □

If $q=1$ and $n\ge3$ then the additional condition ext(X) is always satisfied, as was proved by Rossi [Ro]. On the other hand, this is not true for $q=1$ and $n=2$, by the example of Rossi [Ro] mentioned above. In the present paper we prove that if $q<\frac12 n$, then always both extension conditions ext(X) and ext(E) may be omitted, i.e. we prove the following theorem.

1.3. Theorem. *Suppose X is q-concave-$(n-q)$-convex where $1\le q<\frac12 n$ (and hence $n\ge3$). Then $H^{p,n-q}(X,E)$ is Hausdorff for all p.*

As an immediate consequence we obtain a result on the extension of CR-forms of bidegree (p,q) from strictly $(n-q)$-convex boundaries which we explain in Section 4 at the end of the paper.

To prove Therorem 1.3 we follow the scheme of the proof of Theorem 1.2 given above, but there is a difference: Since we do not have the extended manifold $\widetilde X$, we also do not have the family of compact subsets of $\widetilde X$. We only have the trace of this family in X and we have to study *directly* the cohomology supported by this trace without using the compactly supported cohomology of $\widetilde X$. This gives rise to certain functional analytic difficulties which we want to explain now.

1.4. *Definition.* (i) A family Φ of closed subsets of X will be called an *admissible family of supports in* X in each of the following three cases: (1) Φ is the family of *all* closed subsets of X; (2) Φ is the family of all *compact* subsets of X; (3) there exists a C^∞ function $\varrho\colon X\to\mathbf{R}$ without minimum and maximum such that the sets $\{z|s\le\varrho(z)\le t\}$, $\inf\varrho<s\le t<\sup\varrho$, are compact and Φ consists of all closed sets $C\subseteq X$ such that, for some $t\in]\inf\varrho,\sup\varrho[$,

$$C\subseteq\{z\,|\,\varrho(z)\le t\}.$$

Then we may always additionally assume that all critical points of ϱ are nondegenerate.[2] With this additional property (which is sometimes useful by technical reasons), ϱ will be called a *defining function* for Φ.

(ii) If Φ is an admissible family of supports in X, then we denote by $\mathcal{D}^{p,q}_\Phi(X,E)$ the space of forms $f\in\mathcal{E}^{p,q}(X,E)$ with $\operatorname{supp} f\in\Phi$, given the finest local convex topology such that, for each $C\in\Phi$, the embedding $\mathcal{D}^{p,q}_C(X,E)\to\mathcal{D}^{p,q}_\Phi(X,E)$ is continuous (here $\mathcal{D}^{p,q}_C(X,E)$ carries the Fréchet topology introduced above). Further, we then consider the factor space

$$H^{p,q}_\Phi(X,E)=\frac{\mathcal{D}^{p,q}_\Phi(X,E)\cap\operatorname{Ker}\bar\partial}{\bar\partial\mathcal{D}^{p,q-1}_\Phi(X,E)}$$

endowed with the factor topology.

(iii) If Φ is an admissible family of supports in X then we denote by Φ^* the family of all closed subsets C^* of X such that $C^*\cap C$ is compact for all $C\in\Phi$. The family Φ^* is called the *dual family* of Φ.

If Φ is an admissible family of supports defined by the function ϱ, then Φ^* is an admissible family of supports defined by $-\varrho$ and $\Phi^{**}=\Phi$. If Φ is the family of all compact sets, then Φ^* is the family of all closed sets and vice versa.

It is clear that admissible families of supports are families of supports in the sense of Serre [S]. Also it is clear that such families Φ are *cofinal* in the sense of Chirka and Stout [CS], i.e. there exists a sequence $C_j\in\Phi$ such that each $C\in\Phi$ is contained in some C_j. Hence the topology of $\mathcal{D}^{p,q}_\Phi(X,E)$ is the topology of an LF-space (cf., e.g., [T, Chapter 13]) for any admissible family of supports Φ in X.

The problem now is the following. To prove Theorem 1.3 by the scheme of the proof of Theorem 1.2 given above, first we would have to prove that, for certain admissible families of supports Φ in X, the following two conjectures are true.

1.5. *Conjecture.* The space $H^{p,q}_\Phi(X,E)$ is Hausdorff if and only if the space $H^{n-p,n-q+1}_{\Phi^*}(X,E^*)$ is Hausdorff.

([2]) This can always be achieved by small perturbations, as it follows, for example, from Proposition 0.5 in Appendix B of [HL].

1.6. *Conjecture.* The space $H^{p,q}_{\Phi}(X,E)$ is either Hausdorff or

$$\dim \frac{\overline{\bar\partial \mathcal{D}^{p,q-1}_{\Phi}(X,E)}}{\bar\partial \mathcal{D}^{p,q-1}_{\Phi}(X,E)} = \infty, \tag{1.3}$$

where $\overline{\bar\partial \mathcal{D}^{p,q-1}_{\Phi}(X,E)}$ is the topological closure of $\bar\partial \mathcal{D}^{p,q-1}_{\Phi}(X,E)$ in $\mathcal{D}^{p,q}_{\Phi}(X,E)$.

Both conjectures are true if Φ is the family of all closed subsets or the family of all compact subsets. This seems to be well known (at least it is frequently used in the literature). But it is not so easy to find references for explicit proofs of this. Therefore let us say some words about the proofs.

On the proofs of Conjecture 1.5 *and Conjecture* 1.6 *for the families of all closed and compact sets, respectively.* If Φ is the family of all compact sets, then the 'if'-part in Conjecture 1.5 was proved by Serre [S]. In his proof it is used that, by the open mapping theorem in Fréchet spaces, the operator

$$\bar\partial: \mathcal{E}^{p,q-1}(X,E) \longrightarrow \mathcal{E}^{p,q}(X,E) \tag{1.4}$$

is relatively open, if its image is closed. It is not clear whether this is true also for the operator

$$\bar\partial: \mathcal{D}^{p,q-1}(X,E) \longrightarrow \mathcal{D}^{p,q}(X,E). \tag{1.5}$$

Therefore the proof of the 'only-if'-part is more difficult. This part was proved, as it seems, by different authors independently. A complete proof can be found in the work of Laufer [L]. Laufer observes that for Serre's proof only the fact is important that (1.4) sends *weakly* open sets to relatively open sets, and he proves that also (1.5) has this property. (For another proof see [LL1, Section 3]).

Now about Conjecture 1.6. If Φ is the family of all closed sets, then Conjecture 1.6 follows easily from the open mapping theorem in Fréchet spaces. But this is not so easy when Φ is the family of all compact sets. Although $\mathcal{D}^{p,q}(X,E)$ is an LF-space and the open mapping theorem holds for continuous linear surjections between LF-spaces, it is not clear whether the open mapping theorem holds if only the source space is an LF-space and the target space is a closed subspace of an LF-space.

Since we could not find an explicit reference for a proof of Conjecture 1.6, let us sketch a proof here (for another proof see [LL1, Theorem 2.7 and Lemma 2.8]):

Denote by $(\mathcal{E}')^{p,q}(X,E)$ the dual of $\mathcal{E}^{n-p,n-q}(X,E^*)$, i.e. the space of E-valued (p,q)-currents with compact support in X. Then it is well known (Dolbeault isomorphism) that the natural map

$$H^{p,q}_{c}(X,E) \longrightarrow \frac{(\mathcal{E}')^{p,q}(X,E) \cap \operatorname{Ker}\bar\partial}{\bar\partial(\mathcal{E}')^{p,q-1}(X,E)} \tag{1.6}$$

is an algebraic isomorphism.(3) Moreover, by the Hahn–Banach theorem,

(1.7)
$$\overline{\bar\partial\mathcal{D}^{p,q-1}(X,E)}=\left\{\varphi\in\mathcal{D}^{p,q}(X,E)\;\middle|\;\int_X\varphi\wedge\psi=0\text{ for }\psi\in\mathcal{E}^{n-p,n-q}(X,E^*)\cap\operatorname{Ker}\bar\partial\right\}.$$

Further, since $\mathcal{E}^{n-p,n-q}(X,E^*)$ is reflexive, the Hahn–Banach theorem also gives

(1.8)
$$\overline{\bar\partial(\mathcal{E}')^{p,q-1}(X,E)}=\{\varphi\in(\mathcal{E}')^{p,q}(X,E)\mid\langle\varphi,\psi\rangle=0\text{ for }\psi\in\mathcal{E}^{n-p,n-q}(X,E^*)\cap\operatorname{Ker}\bar\partial\},$$

where $\overline{\bar\partial(\mathcal{E}')^{p,q-1}(X,E)}$ is the closure of $\bar\partial(\mathcal{E}')^{p,q-1}(X,E)$ in $(\mathcal{E}')^{p,q}(X,E)$ with respect to the strong topology. From (1.7) and (1.8) it follows that the natural isomorphism (1.6) induces an isomorphism

$$\frac{\overline{\bar\partial\mathcal{D}^{p,q-1}(X,E)}}{\bar\partial\mathcal{D}^{p,q-1}(X,E)}\longrightarrow\frac{\overline{\bar\partial(\mathcal{E}')^{p,q-1}(X,E)}}{\bar\partial(\mathcal{E}')^{p,q-1}(X,E)}. \tag{1.9}$$

Now it remains to observe that $(\mathcal{E}')^{p,q}(X,E)$ with the strong topology is a DFS-space (the strong dual of a Fréchet–Schwartz space) and therefore the Banach open mapping theorem can be applied to prove that the dimension of the space on the right-hand side of (1.9) is always either 0 or ∞. □

Finally we want to note two general functional analytic problems.

1.7. *Problem.* Let E and F be LF-spaces and let $A\colon E\to F$ be a continuous linear operator such that $\operatorname{Im}A$ is topologically closed in F. Is it true that then, for each weakly open $U\subseteq E$, $A(U)$ is open in $\operatorname{Im}A$ with respect to the topology induced from F.

1.8. *Problem.* Let F be an LF-space, $(F_j)_{j=1}^{\infty}$ a defining sequence of Fréchet spaces for F, and H a linear subspace of F such that $H\cap F_j$ is topologically closed for all j. Is it true that then H is topologically closed?

An affirmative answer to Problem 1.7 would imply that Serre's proof proves also Conjecture 1.5 for any admissible family of supports. An affirmative answer to Problem 1.8 would imply a simple proof of Conjecture 1.6 for all admissible families of supports. Note that Problem 1.8 was stated already 50 years ago by Dieudonné and Schwartz [DS, p. 97], but it seems to be still open. For the special case $F=\mathcal{D}^{p,q}(X,E)$ and $H=\bar\partial\mathcal{D}^{p,q-1}(X,E)$ the answer is affirmative (see [LL1]).

(3) Actually (1.6) is even a topological isomorphism if the right-hand side is given the factor topology defined by the strong topology of $(\mathcal{E}')^{p,q}(X,E)$, and the same is true if we use on both sides the weak topologies [L]. But we do not use this here.

We are very grateful to F. Haslinger for the interesting and helpful discussions of these two problems. In particular, we learned from him that Problem 1.8 was posed already in [DS]. Moreover, also for such useful discussions, we want to thank Seán Dineen as well as the participants of the section lead by him during the two complex analysis conferences held in August 1999 in Fukuoka.

2. Serre duality for admissible families of supports

In this section, X is an n-dimensional complex manifold, $E \to X$ a holomorphic vector bundle and Φ an admissible family of supports in X (see Definition 1.4) which is neither the family of all closed, nor the family of all compact sets.([4]) Since we are unable to prove (or disprove) Conjectures 1.5 and 1.6, in the present section, we establish some other results (sufficient for the proof of Theorem 1.3) which we now describe.

2.1. *Definition.* (i) We denote by Φ^0 the family of all open sets of the form $U = \{z \mid \varrho(z) < t\}$ where ϱ is a defining function for Φ and $\inf \varrho < t < \sup \varrho$. For $U \in \Phi^0$, we denote by $\Phi|_U$ the family of all $C \in \Phi$ with $C \subseteq U$.

Note that then $\Phi|_U$ is an admissible family of supports in U—if ϱ is a defining function for Φ with $U = \{z \mid \varrho(z) < t\}$ for some $t < \sup \varrho$, then $\Phi|_U$ is defined by $\varrho|_U$. Observe also that, for any $U^* \in (\Phi^*)^0$, we then have the relation

$$(2.1) \qquad (\Phi^*|_{U^*})^* = \Phi \cap U^* (= \{C \cap U^* \mid C \in \Phi\}).$$

(ii) Let $0 \le p \le n$ and $1 \le q \le n$. We say that $H_\Phi^{p,q}(X,E)$ is *α-Hausdorff* if, for each $C \in \Phi$, the space $\mathcal{D}_C^{p,q}(X,E) \cap \bar{\partial}\mathcal{D}_\Phi^{p,q-1}(X,E)$ is topologically closed in $\mathcal{D}_C^{p,q}(X,E)$ (with respect to the Fréchet topology induced by $\mathcal{E}^{p,q}(X,E)$). We say that $H_\Phi^{p,q}(X,E)$ is *β-Hausdorff* if it is α-Hausdorff and moreover, for each $C^* \in \Phi^*$, there exists $U^* \in (\Phi^*)^0$ with $U^* \supseteq C^*$ such that $H_{\Phi \cap U^*}^{p,q}(U^*,E)$ is α-Hausdorff.

The result which we want to prove in this section, now can be stated as follows.

2.2. Theorem. *Let $0 \le p \le n$ and $1 \le q \le n$. If $H_\Phi^{p,q}(X,E)$ is β-Hausdorff then $H_{\Phi^*}^{n-p,n-q+1}(X,E^*)$ is Hausdorff.*

2.3. *Remarks.* (A) The special case when Φ is the family of all compact sets fits into Theorem 2.2 as follows: Denote by Φ^0 the family of all relatively compact open subsets of X, and set $(\Phi^*)^0 = \{X\}$. Then it is trivial that α-Hausdorffness

([4]) For these two special cases, where the results are stronger, we refer to [LL1] (cf. also (A) in the Remarks 2.3 below).

and β-Hausdorffness coincide. Moreover, in [LL1] it was proved that then also α-Hausdorffness and Hausdorffness coincide. Hence, for this special case, Theorem 2.2 follows from [LL1].

(B) Hausdorffness of $H^{p,q}_\Phi(X,E)$ always yields α-Hausdorffness of $H^{p,q}_\Phi(X,E)$ for any admissible family of supports Φ. This is clear, because $\mathcal{D}^{p,q}_C(X,E)$ is topologically closed in $\mathcal{D}^{p,q}_\Phi(X,E)$ for all $C\in\Phi$. We do not know however whether, in general, α-Hausdorffness of $H^{p,q}_\Phi(X,E)$ yields Hausdorffness of $H^{p,q}_\Phi(X,E)$, i.e. we cannot solve Problem 1.8 for the special case $F=\mathcal{D}^{p,q}_\Phi(X,E)$ and $H=\bar\partial\mathcal{D}^{p,q-1}_\Phi(X,E)$.

(C) We do not know examples when $H^{p,q}_\Phi(X,E)$ is α-Hausdorff but not β-Hausdorff, because in all our examples the *reason* for α-Hausdorffness is the fact that, at the ends of the manifold, certain convexity or concavity conditions are satisfied, and these conditions in all our examples immediately imply also β-Hausdorffness.

For the proof of Theorem 2.2 we need some preparations. Let $(\mathcal{D}')^{p,q}_\Phi(X,E)$ denote the dual of $\mathcal{D}^{n-p,n-q}_{\Phi^*}(X,E^*)$, i.e. the space of E-valued (p,q)-currents on X which extend continuously to $\mathcal{D}^{n-p,n-q}_{\Phi^*}(X,E^*)$. The following lemma shows that (as indicated by the notation) $(\mathcal{D}')^{p,q}_\Phi(X,E)$ is the space of E-valued (p,q)-currents on X with support in Φ.

2.4. Lemma. *Let T be an E-valued (p,q)-current on X, $0\le p,q\le n$. Then $T\in(\mathcal{D}')^{p,q}_\Phi(X,E)$ if and only if* $\operatorname{supp} T\in\Phi$.

Proof. By definition, T is a continuous linear functional on $\mathcal{D}^{n-p,n-q}(X,E^*)$.

First assume that $\operatorname{supp} T\in\Phi$. Take a real C^∞-function χ on X with $\operatorname{supp}\chi\in\Phi$ and $\chi\equiv1$ in a neighborhood of $\operatorname{supp} T$. Then

$$(2.2)\qquad \langle T,\varphi\rangle=\langle T,\chi\varphi\rangle \quad \text{for all } \varphi\in\mathcal{D}^{n-p,n-q}(X,E^*).$$

Moreover, for each $\varphi\in\mathcal{D}^{n-p,n-q}_{\Phi^*}(X,E^*)$, the form $\chi\varphi$ has compact support. Therefore the multiplication by χ defines a continuous map

$$M_\chi:\mathcal{D}^{n-p,n-q}_{\Phi^*}(X,E^*)\longrightarrow\mathcal{D}^{n-p,n-q}(X,E^*)$$

such that, by (2.2), $T=TM_\chi$ on $\mathcal{D}^{n-p,n-q}(X,E^*)$. Hence TM_χ extends T continuously to $\mathcal{E}^{n-p,n-q}_{\Phi^*}(X,E^*)$.

Now we assume that $\operatorname{supp} T\notin\Phi$. Then there is a set $C^*\in\Phi^*$ such that $C^*\cap\operatorname{supp} T$ is not compact and, therefore, we can find a sequence $(\varphi_j)_{j=1}^\infty$ of forms in $\mathcal{D}^{n-p,n-q}(X,E^*)$ with $\operatorname{supp}\varphi_j\subseteq C^*\cap\operatorname{supp} T$ for all j, $\langle T,\varphi_j\rangle=1$ for all j, and $\operatorname{supp}\varphi_j\cap\operatorname{supp}\varphi_k=\emptyset$ if $j\ne k$. Set $\psi_k=\varphi_1+\ldots+\varphi_k$. Then the sequence $(\psi_k)_{k=1}^\infty$ converges in $\mathcal{D}^{n-p,n-q}_{\Phi^*}(X,E^*)$ whereas $\lim_{k\to\infty}\langle T,\psi_k\rangle=\lim_{k\to\infty}k=\infty$. Hence T is not continuous with respect to the topology of $\mathcal{D}^{n-p,n-q}_{\Phi^*}(X,E^*)$. $\square$

2.5. Lemma. *For all p and q with $0 \le p \le n$ and $1 \le q \le n$, we have*

$$\overline{\bar\partial \mathcal{D}_\Phi^{p,q-1}(X,E)} = \left\{ \varphi \in \mathcal{D}_\Phi^{p,q}(X,E) \cap \operatorname{Ker}\bar\partial \,\middle|\, \int_X \varphi \wedge \psi = 0 \right.$$
$$\left. \text{for } \psi \in \mathcal{D}_{\Phi^*}^{n-p,n-q}(X,E^*) \cap \operatorname{Ker}\bar\partial \right\},$$

where $\overline{\bar\partial \mathcal{D}_\Phi^{p,q-1}(X,E)}$ is the topological closure of $\bar\partial \mathcal{D}_\Phi^{p,q-1}(X,E)$ in $\mathcal{D}_\Phi^{p,q}(X,E)$.

Proof. The "$\subseteq$"-part follows from Stokes' formula. To prove the "$\supseteq$"-part, we consider a form $\varphi \in \mathcal{D}_\Phi^{p,q}(X,E) \cap \operatorname{Ker}\bar\partial$ which does not belong to $\overline{\bar\partial \mathcal{D}_\Phi^{p,q-1}(X,E)}$. We have to find $\psi \in \mathcal{D}_{\Phi^*}^{n-p,n-q}(X,E^*) \cap \operatorname{Ker}\bar\partial$ with

$$\int_X \varphi \wedge \psi \neq 0. \tag{2.3}$$

By the Hahn–Banach theorem there is a current $T \in (\mathcal{D}')_{\Phi^*}^{n-p,n-q}(X,E^*)$ with

$$\langle T, \varphi \rangle \neq 0, \tag{2.4}$$

but $\langle T, \bar\partial \psi \rangle = 0$ for all $\psi \in \mathcal{D}_\Phi^{p,q-1}(X,E)$, i.e. $\bar\partial T = 0$. If $q = n$, then, by the regularity of $\bar\partial$ the equation $\bar\partial T = 0$ implies that T is defined by a smooth form ψ. By (2.4) this ψ satisfies (2.3).

Now let $q \le n-1$. Since $\operatorname{supp} T \in \Phi^*$, we can then find a neighborhood U^* of $\operatorname{supp} T$ with $U^* \in (\Phi^*)^0$, and, by regularity of $\bar\partial$, there exist a smooth form ψ and a current S with support in U^* such that $T = \psi + \bar\partial S$. Since $\bar\partial \varphi = 0$ and therefore $\langle \bar\partial S, \varphi \rangle = 0$, (2.3) again follows from (2.4). $\square$

2.6. Lemma. *Let $0 \le p \le n$, $1 \le q \le n$, $C \in \Phi$ and*

$$G := \mathcal{D}_C^{p,q}(X,E) \cap \bar\partial \mathcal{D}_\Phi^{p,q-1}(X,E).$$

Then the following are true:

(i) *If G is topologically closed in $\mathcal{D}_C^{p,q}(X,E)$, then there exists a set $C_0 \in \Phi$ with $G = \mathcal{D}_C^{p,q}(X,E) \cap \bar\partial \mathcal{D}_{C_0}^{p,q-1}(X,E)$.*

(ii) *If there exists a finite dimensional subspace F of $\mathcal{D}_C^{p,q}(X,E)$ such that $F+G$ is topologically closed in $\mathcal{D}_C^{p,q}(X,E)$, then G itself is topologically closed in $\mathcal{D}_C^{p,q}(X,E)$.*

In particular, if G is of finite codimension in $\mathcal{D}_C^{p,q}(X,E) \cap \operatorname{Ker}\bar\partial$, then G is topologically closed in $\mathcal{D}_C^{p,q}(X,E)$.

Proof. Take a sequence $(C_j)_{j=1}^{\infty}$ of sets in Φ such that each set in Φ is contained in some C_j. Denote by D_j the subspace of all $\varphi\in\mathcal{D}_{C_j}^{p,q-1}(X,E)$ with $\bar\partial\varphi\equiv 0$ outside C. Note that each D_j is topologically closed in $\mathcal{D}_{C_j}^{p,q-1}(X,E)$ and hence a Fréchet space. Clearly

$$G=\bigcup_{j=1}^{\infty}\bar\partial D_j. \tag{2.5}$$

To prove (i), we now assume that G is topologically closed in $\mathcal{D}_C^{p,q}(X,E)$. Then it follows from (2.5) that, for certain $j_0\in\mathbf{N}$, $\bar\partial D_{j_0}$ is of second Baire category in G. Since $\bar\partial D_{j_0}$ is the image of a continuous linear operator between Fréchet spaces, by the open mapping theorem, this means that $\bar\partial D_{j_0}=G$. It remains to set $C_0=C_{j_0}$.

Now let F be as in (ii). Without loss of generality, we may assume that moreover

$$F\cap G=\{0\}. \tag{2.6}$$

Using again the Baire category argument and the open mapping theorem, then it follows from (2.5) that, for certain $j_0\in\mathbf{N}$, $F+G=F+\bar\partial D_{j_0}$. By (2.6) this means that $G=\bar\partial D_{j_0}$. Therefore G is the finite codimensional image of a continuous linear operator between Fréchet spaces and hence itself a Fréchet space. □

2.7. *Definition.* Let $0\le p\le n$ and $1\le q\le n$. An ordered pair (U,U_0) of open sets $U,U_0\in\Phi^0$ will be called an $\alpha_\Phi^{p,q}(X,E)$-pair if there exist sets $C,C_0\in\Phi$ such that $\overline{U}$ is contained in the interior of C, $C\subseteq C_0\subseteq U_0$ and

$$\mathcal{D}_C^{p,q}(X,E)\cap\bar\partial\mathcal{D}_\Phi^{p,q-1}(X,E)=\mathcal{D}_C^{p,q}(X,E)\cap\bar\partial\mathcal{D}_{C_0}^{p,q-1}(X,E).$$

From Lemma 2.6 we immediately obtain the following important corollary.

2.8. Corollary. *Let $0\le p\le n$ and $1\le q\le n$ be such that $H_\Phi^{p,q}(X,E)$ is α-Hausdorff. Then for each $U\in\Phi^0$ there exists $U_0\in\Phi^0$ such that (U,U_0) is an $\alpha_\Phi^{p,q}(X,E)$-pair.*

2.9. Lemma. *Let $0\le p\le n$ and $1\le q\le n$ be such that $H_\Phi^{p,q}(X,E)$ is α-Hausdorff. Then, for each pair $U,U_0\in\Phi$ with $\overline{U}\subseteq U_0$, we have the relation*

$$\mathcal{D}_{\overline{U}}^{p,q}(X,E)\cap\bar\partial\mathcal{D}_\Phi^{p,q-1}(X,E)\supseteq\Big\{\varphi\in\mathcal{D}_{\overline{U}}^{p,q}(X,E)\cap\operatorname{Ker}\bar\partial\;\Big|\;\int_{U_0}\psi\wedge\varphi=0 \text{ for } \psi\in\mathcal{D}_{\Phi^*\cap U_0}^{n-p,n-q}(U_0,E^*)\cap\operatorname{Ker}\bar\partial\Big\}. \tag{2.7}$$

Proof. Consider a form $\varphi\in\mathcal{D}^{p,q}_{\overline{U}}(X,E)\cap\operatorname{Ker}\bar\partial$ such that

$$\varphi\notin\bar\partial\mathcal{D}^{p,q-1}_{\Phi}(X,E). \tag{2.8}$$

We have to find a form $\psi\in\mathcal{D}^{n-p,n-q}_{\Phi^*\cap U_0}(U_0,E^*)\cap\operatorname{Ker}\bar\partial$ with

$$\int_{U_0}\psi\wedge\varphi\neq 0. \tag{2.9}$$

Since $H^{p,q}_{\Phi}(X,E)$ is α-Hausdorff and $\overline{U}_0\in\Phi$, $\mathcal{D}^{p,q}_{\overline{U}_0}(X,E)\cap\bar\partial\mathcal{D}^{p,q-1}_{\Phi}(X,E)$ is topologically closed in $\mathcal{D}^{p,q}_{\overline{U}_0}(X,E)$. Therefore, by (2.8) and the Hahn–Banach theorem, we can find a continuous linear form $T:\mathcal{D}^{p,q}_{\overline{U}_0}(X,E)\to\mathbf{C}$ such that

$$\langle T,\varphi\rangle\neq 0 \tag{2.10}$$

and

$$\langle T,\bar\partial\omega\rangle=0\quad\text{for all }\omega\in\mathcal{D}^{p,q-1}_{\Phi}(X,E)\text{ with }\bar\partial\omega\in\mathcal{D}^{p,q}_{\overline{U}_0}(X,E). \tag{2.11}$$

Since T is continuous on $\mathcal{D}^{p,q}_{\overline{U}_0}(X,E)$, it follows (in the same way as in the proof of Lemma 2.4) that there is a compact set $K\subseteq\overline{U}_0$ such that $T\equiv 0$ on $\mathcal{D}^{p,q}_{\overline{U}_0\setminus K}(X,E)$. Denote by $\widetilde{T}$ the E^*-valued $(n-p,n-q)$-current on U_0 defined by T. Then

$$\operatorname{supp}\widetilde{T}\subseteq K\cap U_0 \tag{2.12}$$

and, by (2.11),

$$\bar\partial\widetilde{T}=0. \tag{2.13}$$

If $q=n$, then, by the regularity of $\bar\partial$, the current $\widetilde{T}$ is defined by a smooth form $\psi\in\mathcal{E}^{n-p,0}_{K\cap U_0}(U_0,E^*)$ which satisfies (2.9), by (2.10).

Now let $q\le n-1$. Then we take a set $C^*\in\Phi^*$ which contains K in its interior. Then, by (2.12), (2.13) and the regularity of $\bar\partial$, we can find a smooth form $\psi\in\mathcal{D}^{n-p,n-q}_{C^*\cap U_0}(U_0,E^*)\cap\operatorname{Ker}\bar\partial$ and an E^*-valued $(n-p,n-q-1)$-current S on U_0 with $\operatorname{supp}S\subseteq C^*\cap U_0$ such that

$$\langle\widetilde{T},\eta\rangle=\langle\bar\partial S,\eta\rangle+\int_{U_0}\psi\wedge\eta \tag{2.14}$$

for all $\eta\in\mathcal{D}^{p,q}(U_0,E)$. Since $\overline{U}\cap\operatorname{supp}S\subseteq\overline{U}\cap C^*$ is compact, S extends continuously to $\mathcal{D}^{p,q+1}_{\overline{U}}(U_0,E)$. Hence $\bar\partial S$ extends continuously to $\mathcal{D}^{p,q}_{\overline{U}}(U_0,E)$. Since the same is true for $\widetilde{T}$ and the current defined by ψ, (2.14) holds for all $\eta\in\mathcal{D}^{p,q}_{\overline{U}}(U_0,E)=\mathcal{D}^{p,q}_{\overline{U}}(X,E)$ and hence, in particular, for $\eta=\varphi$. Since $\bar\partial\varphi=0$ now (2.9) again follows from (2.10). □

2.10. *Remark.* It is easy to see that we even have equality in (2.7) if U_0 is so large that (U,U_0) is an $\alpha^{p,q}_{\Phi}(X,E)$-pair (but we do not need this).

2.11. Lemma. *Let $0 \le p \le n$ and $1 \le q \le n$ be such that $H^{p,q}_{\Phi}(X,E)$ is α-Hausdorff and let (U,U_0) be an $\alpha^{p,q}_{\Phi}(X,E)$-pair. Then, for any $U_{00} \in \Phi^0$ with $U_{00} \supseteq U_0$, the space $\mathcal{D}^{n-p,n-q}_{\Phi^* \cap U_{00}}(U_{00},E^*) \cap \operatorname{Ker}\bar\partial$ is dense in $\mathcal{D}^{n-p,n-q}_{\Phi^* \cap U_0}(U_0,E^*) \cap \operatorname{Ker}\bar\partial$ with respect to the topology of $\mathcal{D}^{n-p,n-q}_{\Phi^* \cap U}(U,E^*)$.*

Proof. Let $T : \mathcal{D}^{n-p,n-q}_{\Phi^* \cap U}(U,E^*) \to \mathbf{C}$ be a continuous linear functional such that

$$T(\psi|_U) = 0 \quad \text{for all } \psi \in \mathcal{D}^{n-p,n-q}_{\Phi^* \cap U_{00}}(U_{00},E^*) \cap \operatorname{Ker}\bar\partial. \tag{2.15}$$

By the Hahn–Banach theorem, then we have to prove that

$$T(\psi|_U) = 0 \quad \text{for all } \psi \in \mathcal{D}^{n-p,n-q}_{\Phi^* \cap U_0}(U_0,E^*) \cap \operatorname{Ker}\bar\partial. \tag{2.16}$$

By Lemma 2.4, T is an E-valued (p,q)-current on U with $\operatorname{supp} T \in \Phi|_U = (\Phi^* \cap U)^*$. Take a set $C \in \Phi|_U$ such that $\operatorname{supp} T$ is contained in the interior of C. Since (2.15) in particular means that $\bar\partial T = 0$, then we can find a smooth form $\varphi \in \mathcal{D}^{p,q}_C(U,E) \cap \operatorname{Ker}\bar\partial = \mathcal{D}^{p,q}_C(X,E) \cap \operatorname{Ker}\bar\partial$ and an E-valued $(p,q-1)$-current S on X with $\operatorname{supp} S \subseteq C$ such that

$$\langle T,\eta\rangle = \langle S,\bar\partial\eta\rangle + \int_U \eta \wedge \varphi \tag{2.17}$$

for all $\eta \in \mathcal{D}^{n-p,n-q}(U,E^*)$. By Lemma 2.4, the current S extends continuously to $\mathcal{D}^{n-p,n-q+1}_{\Phi^* \cap U}(U,E^*)$. Since, moreover, $\bar\partial$, T and the current defined by φ are continuous on $\mathcal{D}^{n-p,n-q}_{\Phi^* \cap U}(U,E^*)$, it follows that (2.17) holds for all $\eta \in \mathcal{D}^{n-p,n-q}_{\Phi^* \cap U}(U,E^*)$. Together with (2.15) this implies that

$$\int_{U_{00}} \psi \wedge \varphi = \int_U \psi \wedge \varphi = 0 \quad \text{for all } \psi \in \mathcal{D}^{n-p,n-q}_{\Phi^* \cap U_{00}}(U_{00},E^*) \cap \operatorname{Ker}\bar\partial. \tag{2.18}$$

Since $U_{00} \in \Phi$, it follows from Lemma 2.9 that $\varphi \in \mathcal{D}^{p,q}_{\bar U}(X,E) \cap \bar\partial \mathcal{D}^{p,q-1}_{\Phi}(X,E)$. Since (U,U_0) is an $\alpha^{p,q}_{\Phi}(X,E)$-pair, this implies that φ is of the form $\varphi = \bar\partial\omega$ with $\omega \in \mathcal{D}^{p,q-1}_{\Phi|_{U_0}}(X,E)$. Now let $\psi \in \mathcal{D}^{n-p,n-q}_{\Phi^* \cap U_0}(U_0,E^*) \cap \operatorname{Ker}\bar\partial$. Then (2.17) holds in particular also for $\eta = \psi|_U$, and hence

$$T(\psi|_U) = \int_U \psi \wedge \varphi = \int_{U_0} \psi \wedge \bar\partial\omega = \pm \int_{U_0} \bar\partial\psi \wedge \omega = 0,$$

i.e. (2.16) holds. $\square$

2.12. Lemma. *Let $0 \le p \le n$ and $1 \le q \le n$ be such that $H_{\Phi}^{p,q}(X,E)$ is β-Hausdorff. Then, for each $\psi \in \mathcal{D}_{\Phi^*}^{n-p,n-q+1}(X,E^*) \cap \operatorname{Ker} \bar{\partial}$ satisfying the orthogonality condition*

$$\int_X \psi \wedge \varphi = 0 \quad \textit{for all } \varphi \in \mathcal{D}_{\Phi}^{p,q-1}(X,E) \cap \operatorname{Ker} \bar{\partial}, \tag{2.19}$$

there exists $U^* \in \Phi^*$ such that, for any $U \in \Phi^0$, there exists $\omega_U \in \mathcal{D}_{\bar{U}^* \cap U}^{n-p,n-q}(U,E^*)$ with $\psi|_U = \bar{\partial} \omega_U$.

Proof. Let ψ be as in the lemma. Take $W^* \in (\Phi^*)^0$ with $\operatorname{supp} \psi \subseteq W^*$. Then $W := X \setminus \overline{W}^* \in \Phi$ and, by Corollary 2.8, we can find $V \in \Phi$ such that (W,V) is an $\alpha_{\Phi}^{p,q}(X,E)$-pair. Denote by $D_{\bar{V}}$ the space of all $\varphi \in \mathcal{D}_{\bar{V}}^{p,q-1}(X,E)$ with $\operatorname{supp} \bar{\partial}\varphi \subseteq \overline{W}$. Then $\bar{\partial}\colon D_{\bar{V}} \to \mathcal{D}_{\overline{W}}^{p,q}(X,E) \cap \bar{\partial} \mathcal{D}_{\Phi}^{p,q-1}(X,E)$ is a continuous linear surjection between Fréchet spaces and therefore open. Set(5)

$$\Theta_{\bar{V}} := \Big\{ \varphi \in D_{\bar{V}} \;\Big|\; \max_{z \in \bar{V} \cap \overline{W}^*} \|\varphi(z)\| < 1 \Big\}.$$

Then $\Theta_{\bar{V}} \subset D_{\bar{V}}$ is open and, hence, $\bar{\partial}\Theta_{\bar{V}}$ is open in $\mathcal{D}_{\overline{W}}^{p,q}(X,E) \cap \bar{\partial} \mathcal{D}_{\Phi}^{p,q-1}(X,E)$. Therefore we can find a compact set $K \subseteq \overline{W}$ with

$$\mathcal{D}_{\overline{W \setminus K}}^{p,q}(X,E) \cap \bar{\partial} \mathcal{D}_{\Phi}^{p,q-1}(X,E) \subseteq \bar{\partial}\Theta_{\bar{V}}. \tag{2.20}$$

Since $H_{\Phi}^{p,q}(X,E)$ is β-Hausdorff, we now can find $U^* \in (\Phi^*)^0$ such that $\overline{W}^* \cup K \subseteq U^*$ and $H_{\Phi \cap U^*}^{p,q}(U^*,E)$ is α-Hausdorff. To prove that U^* has the desired property, we consider $U \in \Phi^0$. Since both $H_{\Phi}^{p,q}(X,E)$ and $H_{\Phi \cap U^*}^{p,q}(U^*,E)$ are α-Hausdorff, then, by Corollary 2.8, we can find $U_0 \in \Phi^0$ so large that both (U,U_0) is an $\alpha_{\Phi}^{p,q}(X,E)$-pair and $(U \cap U^*, U_0 \cap U^*)$ is an $\alpha_{\Phi \cap U^*}^{p,q}(U^*,E)$-pair. Moreover we may assume that $\bar{V} \subseteq U_0$.

Denote by $D_{\bar{U}_0 \cap U^*}$ the space of all $\varphi \in \mathcal{D}_{\bar{U}_0 \cap U^*}^{p,q-1}(U^*,E)$ with $\operatorname{supp} \bar{\partial}\varphi \subseteq \bar{U} \cap U^*$. Then $\bar{\partial}\colon D_{\bar{U}_0 \cap U^*} \to \mathcal{D}_{\bar{U} \cap U^*}^{p,q}(U^*,E) \cap \bar{\partial} \mathcal{D}_{\Phi \cap U^*}^{p,q-1}(U^*,E)$ is a continuous linear surjection between Fréchet spaces and therefore open. Set

$$\Theta_{\bar{U}_0 \cap U^*} := \Big\{ \varphi \in D_{\bar{U}_0 \cap U^*} \;\Big|\; \max_{z \in \bar{U}_0 \cap \overline{W}^*} \|\varphi(z)\| < 1 \Big\}.$$

Then $\bar{\partial}\Theta_U^*$ is open in $\mathcal{D}_{\bar{U} \cap U^*}^{p,q}(U^*,E) \cap \bar{\partial} \mathcal{D}_{\Phi \cap U^*}^{p,q-1}(U^*,E)$ and, therefore, we can find a continuous seminorm λ on $\mathcal{D}_{\bar{U} \cap U^*}^{p,q}(U^*,E)$ such that

$$\{ \varphi \in \mathcal{D}_{\bar{U} \cap U^*}^{p,q}(U^*,E) \cap \bar{\partial} \mathcal{D}_{\Phi \cap U^*}^{p,q-1}(U^*,E) \mid \lambda(\varphi) < 1 \} \subseteq \bar{\partial}\Theta_{\bar{U}_0 \cap U^*}. \tag{2.21}$$

(5) Here $\| \cdot \|$ denotes an arbitrary norm on the fibers of the bundle of E-valued forms which depends continuously of the base point.

Now we consider the set

$$\Omega := \left\{ \gamma \in \mathcal{D}^{p,q-1}_{\bar U_0}(X,E) \,\Big|\, \max_{z\in \bar U_0 \cap W^*} \|\gamma(z)\| < 2 \right\}$$

and prove that then

$$\{\varphi \in \mathcal{D}^{p,q}_{\bar U}(X,E) \cap \bar\partial \mathcal{D}^{p,q-1}_{\Phi}(X,E) \,|\, \varphi|_{U^*} \in \bar\partial \Theta_{\bar U_0 \cap U^*}\} \subseteq \bar\partial\Omega. \tag{2.22}$$

In fact, let $\varphi \in \mathcal{D}^{p,q}_{\bar U}(X,E) \cap \bar\partial \mathcal{D}^{p,q-1}_{\Phi}(X,E)$ and $\varphi|_{U^*} = \bar\partial\gamma^*$ for some $\gamma^* \in \Theta_{\bar U_0 \cap U^*}$. Take a C^∞-function $\chi: X \to [0,1]$ such that $\chi \equiv 0$ in a neighborhood of $X \setminus U^*$ and $\chi \equiv 1$ in a neighborhood of $\overline{W}^* \cup K$. Then, after extension by zero, $\chi\gamma^*$ is defined on X and

$$\varphi - \bar\partial(\chi\gamma^*) \in \mathcal{D}^{p,q}_{\overline{W\setminus K}}(X,E) \cap \bar\partial \mathcal{D}^{p,q-1}_{\Phi}(X,E).$$

Hence, by (2.20), $\varphi - \bar\partial(\chi\gamma^*) = \bar\partial\gamma_{\bar V}$ for some $\gamma_{\bar V} \in \Theta_{\bar V}$. Set $\gamma = \chi\gamma^* + \gamma_{\bar V}$. Then $\varphi = \bar\partial\gamma$ and $\gamma \in \Omega$.

Since $\bar\partial$ is surjective as an operator between $\mathcal{D}^{p,q-1}_{\bar U_0}(X,E)$ and $\mathcal{D}^{p,q}_{\bar U}(X,E) \cap \bar\partial \mathcal{D}^{p,q-1}_{\Phi}(X,E)$, then there exists a uniquely defined linear map

$$T: \mathcal{D}^{p,q}_{\bar U}(X,E) \cap \bar\partial \mathcal{D}^{p,q-1}_{\Phi}(X,E) \longrightarrow \mathbf{C}$$

with

$$\langle T, \bar\partial\gamma \rangle = \int_X \psi \wedge \gamma \quad \text{for all } \gamma \in \mathcal{D}^{p,q-1}_{\bar U_0}(X,E). \tag{2.23}$$

Since the embedding $\mathcal{D}^{p,q}_{\bar U}(X,E) \to \mathcal{D}^{p,q}_{\bar U \cap U^*}(U^*,E)$ is continuous, λ defines a continuous seminorm λ_X on $\mathcal{D}^{p,q}_{\bar U}(X,E)$. By (2.21) and (2.22),

$$\{\varphi \in \mathcal{D}^{p,q}_{\bar U}(X,E) \cap \bar\partial \mathcal{D}^{p,q-1}_{\Phi}(X,E) \,|\, \lambda_X(\varphi) < 1\} \subseteq \bar\partial\Omega. \tag{2.24}$$

Since, for all $\gamma \in \Omega$, $\operatorname{supp}(\psi \wedge \gamma) \subseteq \bar U_0 \cap \overline{W}^*$ and $\bar U_0 \cap \overline{W}^*$ is compact, we have the estimate

$$A := \sup_{\gamma\in\Omega} \left| \int_X \psi \wedge \gamma \right| < \infty.$$

By (2.24) and (2.23), this implies that

$$|T(\varphi)| \le A\lambda_X(\varphi) \quad \text{for all } \varphi \in \mathcal{D}^{p,q}_{\bar U}(X,E) \cap \bar\partial \mathcal{D}^{p,q-1}_{\Phi}(X,E).$$

Therefore, by the Hahn–Banach theorem, T admits a continuous linear extension T' to $\mathcal{D}^{p,q}_{\bar U}(X,E)$ with

$$|T'(\varphi)| \le A\lambda_X(\varphi) \quad \text{for all } \varphi \in \mathcal{D}^{p,q}_{\bar U}(X,E). \tag{2.25}$$

Denote by T'' the current on U defined by T'. As λ is continuous on $\mathcal{D}^{p,q}_{\bar{U}\cap U^*}(U^*,E)$, there is a compact set $K^*\Subset U^*$ such that $\lambda_X(\varphi)=0$ for all $\varphi\in\mathcal{D}^{p,q}_{\bar{U}}(X,E)$ with $K^*\cap\operatorname{supp}\varphi=\emptyset$. Therefore it follows from (2.25) that $\operatorname{supp}T''\subseteq K^*\cap U$. Since $U^*\cap U$ is a neighborhood of $K^*\cap U$, therefore, by the regularity of $\bar\partial$, we can find $\omega_U\in\mathcal{D}^{n-p,n-q}_{\bar{U}^*\cap U}(U,E^*)$ such that $T''-\omega_U=\bar\partial S$ for some current S on U with $\operatorname{supp}S\subseteq\bar{U}^*\cap U$. Since, by (2.23), $\psi|_U=\pm\bar\partial T''$ then $\psi|_U=\pm\bar\partial\omega_U$. □

Proof of Theorem 2.2. Consider $\psi\in\mathcal{D}^{n-p,n-q+1}_{\Phi^*}(X,E^*)\cap\operatorname{Ker}\bar\partial$ satisfying the orthogonality condition

$$\int_X\psi\wedge\varphi=0\quad\text{for all }\varphi\in\mathcal{D}^{p,q-1}_{\Phi}(X,E)\cap\operatorname{Ker}\bar\partial.$$

By Lemma 2.5 we have to prove that then there exists $\omega\in\mathcal{D}^{n-p,n-q}_{\Phi^*}(X,E^*)$ with $\psi=\bar\partial\omega$.

By Lemma 2.12, we can find $U^*\in\Phi^*$ with $\operatorname{supp}\psi\subseteq U^*$ as well as sequences $(U_j)_{j=1}^\infty$ and $(\omega_j)_{j=1}^\infty$ of sets $U_j\in\Phi^0$ and of forms $\omega_j\in\mathcal{D}^{n-p,n-q}_{\bar{U}^*\cap U_j}(U_j,E^*)$ such that $\bar{U}_j\subseteq U_{j+1}$, $\bigcup_{j=1}^\infty U_j=X$ and $\psi|_{U_j}=\bar\partial\omega_j$. Moreover we can find $U^{**}\in(\Phi^*)^0$ with $\bar{U}^*\subseteq U^{**}$ such that $H^{p,q}_{\Phi\cap U^{**}}(U^{**},E)$ is α-Hausdorff. We may assume that, for each $j\in\mathbf{N}$, $(U_j\cap U^{**},U_{j+1}\cap U^{**})$ is an $\alpha^{p,q}_{\Phi\cap U^{**}}(U^{**},E)$-pair. In view of the approximation Lemma 2.11 (applied to the manifold U^{**} and the family of supports $\Phi\cap U^{**}$), now we can modify the sequence $(\omega_j)_{j=1}^\infty$ in such a way that (after the modification), for each $j\in\mathbf{N}$, the sequence $(\omega_k)_{k=j}^\infty$ converges in $\mathcal{D}^{n-p,n-q}_{\bar{U}^*\cap U^{**}\cap U_j}(U^{**}\cap U_j,E^*)=\mathcal{D}^{n-p,n-q}_{\bar{U}^*\cap U_j}(U_j,E^*)$ to some $\tilde\omega_j\in\mathcal{D}^{n-p,n-q}_{\bar{U}^*\cap U_j}(U_j,E^*)$. Then $\tilde\omega_j=\tilde\omega_{j+1}$ on U_j and the required form ω can be defined by setting $\omega|_{U_j}:=\tilde\omega_j$. This completes the proof of Theorem 2.2.

3. Proof of Theorem 1.3

In this section, X is an n-dimensional complex manifold which is q-concave-q^*-convex in the sense of Definition 1.1, where $1\le q,q^*\le n$.(6) Moreover we assume that ϱ, s_0 and t_0 have the same meaning as in Definition 1.1, where we additionally assume that all critical points of ϱ are nondegenerate.(7) Furthermore we assume that Φ is the admissible family of supports defined by ϱ (cf. Definition 1.4), Φ^* is the dual family of Φ , E is a holomorphic vector bundle over X, and p is an integer with $1\le p\le n$.

(6) For the proof of Theorem 1.3 one always may assume that $q^*=n-q$ and $q<\frac{1}{2}n$.

(7) This can always be achieved by small perturbations, as it follows, for example, from Proposition 0.5 in Appendix B of [HL].

3.1. Lemma. *If* $\max(q+1, q^*) \le r \le n$, *then* $\dim H^{p,r}_{\Phi^*}(X,E) < \infty$. *In particular*

$$\dim H^{p,n-q}_{\Phi^*}(X,E) < \infty \quad \textit{if } q^* = n-q \textit{ and } q < \tfrac{1}{2}n.$$

Proof. Let some integer r with $\max(q+1, q^*) \le r \le n$ be given. Let $C^{p,l}_{\Phi^*}(X,E)$, $l=r-1, r$, denote the space of E-valued *continuous* (p,l)-forms f on X with $\operatorname{supp} f \in \Phi^*$ and set

$$H^{p,r}_{C,\Phi^*}(X,E) = \frac{C^{p,r}_{\Phi^*}(X,E) \cap \operatorname{Ker} \bar\partial}{C^{p,r}_{\Phi^*}(X,E) \cap \bar\partial C^{p,r-1}_{\Phi^*}(X,E)}.$$

Since the natural map $H^{p,r}_{\Phi^*}(X,E) \to H^{p,r}_{C,\Phi^*}(X,E)$ is an isomorphism (cf. e.g. Corollary 2.15 in [HL]), we have to prove that

$$\dim H^{p,r}_{C,\Phi^*}(X,E) < \infty. \tag{3.1}$$

For s and t with $\inf \varrho < s < s_0$ and $t_0 < t < \sup \varrho$, we denote by $\mathcal{B}^{p,l}_{s,t}$, $l=r-1, r$, the space of all E-valued continuous (p,l)-forms on $\{z \mid \inf \varrho < \varrho(z) \le t\}$ vanishing on $\{z \mid \inf \varrho < \varrho(z) \le s\}$. The spaces $\mathcal{B}^{p,l}_{s,t}$ will be considered as Banach spaces endowed with the topology of uniform convergence on $\{z \mid s \le \varrho(z) \le t\}$. Define the (algebraic) factor spaces

$$H^{p,r}_{s,t} = \frac{\mathcal{B}^{p,r}_{s,t} \cap \operatorname{Ker} \bar\partial}{\mathcal{B}^{p,r}_{s,t} \cap \bar\partial \mathcal{B}^{p,r-1}_{s,t}} \quad \text{for } \inf \varrho < s < s_0 \text{ and } t_0 < t < \sup \varrho$$

and

$$H^{p,r}_{\inf \varrho, t} = \frac{\left(\bigcup_{\inf \varrho < s < s_0} \mathcal{B}^{p,r}_{s,t}\right) \cap \operatorname{Ker} \bar\partial}{\left(\bigcup_{\inf \varrho < s < s_0} \mathcal{B}^{p,r}_{s,t}\right) \cap \bar\partial \left(\bigcup_{\inf \varrho < s < s_0} \mathcal{B}^{p,r-1}_{s,t}\right)} \quad \text{for } t_0 < t < \sup \varrho.$$

To complete the proof of the lemma, it now is sufficient to prove the following three inequalities (which then yield (3.1)):

$$\dim H^{p,r}_{\inf \varrho, t} \le \dim H^{p,r}_{s,t}, \quad \text{if } \inf \varrho < s < s_0 \text{ and } t_0 < t < \sup \varrho; \tag{3.2}$$

$$\dim H^{p,r}_{s,t}(X,E) < \infty, \quad \text{if } \inf \varrho < s < s_0 \text{ and } t_0 < t < \sup \varrho; \tag{3.3}$$

$$\dim H^{p,r}_{C,\Phi^*}(X,E) \le \dim H^{p,r}_{\inf \varrho, t}, \quad \text{if } t_0 < t < \sup \varrho. \tag{3.4}$$

Proof of (3.2). Since $r \ge q+1$ and the Levi form of ϱ has at least $n-q+1$ positive eigenvalues on $\{z \mid \inf \varrho < \varrho(z) \le s_0\}$, it follows from Lemma 1.2(i) in [LL3] that, for all s and t with $\inf \varrho < s < s_0$ and $t_0 < t < \sup \varrho$, the natural map

$$H^{p,r}_{s,t} \longrightarrow H^{p,r}_{\inf \varrho, t}$$

is surjective.

Proof of (3.3). For this we prove the stronger statement,

(3.5) for all s and t with $\inf \varrho < s < s_0$ and $t_0 < t < \sup \varrho$, the space $\mathcal{B}^{p,r}_{s,t} \cap \bar\partial \mathcal{B}^{p,r-1}_{s,t}$ is topologically closed and of finite codimension in $\mathcal{B}^{p,r}_{s,t} \cap \operatorname{Ker} \bar\partial$.

Let s and t with $\inf \varrho < s < s_0$ and $t_0 < t < \sup \varrho$ be given, and let $\mathcal{H}^{p,l}_{s,t}$, $l = r-1, r$, be the subspace of all forms in $\mathcal{B}^{p,l}_{s,t}$ which are Hölder continuous with exponent $\frac{1}{2}$. We consider $\mathcal{H}^{p,l}_{s,t}$ as Banach spaces endowed with the Hölder norm with exponent $\frac{1}{2}$. To prove (3.5), by Ascoli's theorem and Fredholm theory, it now is sufficient to construct continuous linear operators

$$A\colon \mathcal{B}^{p,r}_{s,t} \cap \operatorname{Ker} \bar\partial \longrightarrow \mathcal{H}^{p,r-1}_{s,t},$$
$$K\colon \mathcal{B}^{p,r}_{s,t} \cap \operatorname{Ker} \bar\partial \longrightarrow \mathcal{H}^{p,r}_{s,t}$$

such that

$$\bar\partial A f = f + K f \quad \text{for all } f \in \mathcal{B}^{p,r}_{s,t} \cap \operatorname{Ker} \bar\partial.$$

To do this, we take $\varepsilon > 0$ so small that $s + \varepsilon < s_0$ and $s - \varepsilon > \inf \varrho$. Since $r \geq q+1$ and the Levi form of ϱ has at least $n-q+1$ positive eigenvalues on $\{z \mid \inf \varrho < \varrho(z) \leq s_0\}$, then we obtain from Lemma 1.2(i) in [LL3] a continuous linear operator

$$A_0\colon \mathcal{B}^{p,r}_{s,t} \cap \operatorname{Ker} \bar\partial \longrightarrow \mathcal{H}^{p,r-1}_{s-\varepsilon,t}$$

such that $\bar\partial A_0 f = f$ on $\{z \mid \varrho(z) \leq s + \varepsilon\}$ for all $f \in \mathcal{B}^{p,r}_{s,t} \cap \operatorname{Ker} \bar\partial$. Moreover, since $r \geq q^*$ and the Levi form of ϱ has at least $n - q^* + 1$ positive eigenvalues on $\{z \mid t_0 < \varrho(z) < \sup \varrho\}$, we can apply the local integral operators of Fischer and Lieb to the boundary $\{z \mid \varrho(z) = t\}$ (see [FL], see also Sections 7 and 9 in [HL](8)). In this way we obtain open sets $U_1, \dots, U_N \Subset X$ with

$$\{z \mid s + \varepsilon \leq \varrho(z) \leq t\} \subseteq U_1 \cup \dots \cup U_N \subseteq \{z \mid s < \varrho(z) < \sup \varrho\}$$

as well as continuous linear operators

$$A_j\colon \mathcal{B}^{p,r}_{s,t} \cap \operatorname{Ker} \bar\partial \longrightarrow \mathcal{H}^{p,r-1}_{s-\varepsilon,t}$$

(8) To apply the formulas from [FL] we should assume that the surface $\{z \mid \varrho(z) = t\}$ is smooth, which, of course, we could do for the purpose of this proof. On the other hand, the formulas from [HL] work also in the case when the surface $\{z \mid \varrho(z) = t\}$ is not smooth, but all critical points of ϱ are nondegenerate, which we may assume—cf. the beginning of the present section.

such that $\bar\partial A_j f=f$ on $\{z\in U_j\,|\,\varrho(z)\le t\}$ for all $f\in\mathcal{B}^{p,r}_{s,t}\cap\operatorname{Ker}\bar\partial$, $j=1,\dots,N$. Take real C^∞-functions $\chi_0,\dots,\chi_N$ on X with $\operatorname{supp}\chi_0\Subset\{z\,|\,\varrho(z)<s+\varepsilon\}$, $\operatorname{supp}\chi_j\Subset U_j$ if $1\le j\le N$, and $\chi_0+\dots+\chi_N\equiv 1$ on $\{z\,|\,s-\varepsilon\le\varrho(z)\le t\}$. Then the operators

$$A:=\sum_{j=0}^{N}\chi_j A_j \quad\text{and}\quad K:=\sum_{j=0}^{N}\bar\partial\chi_j\wedge A_j$$

have the required property.

Proof of (3.4). Let t with $t_0<t<\sup\varrho$ be given. It is sufficient to prove that the restriction map

$$H^{p,r}_{\mathcal{C},\Phi^*}(X,E)\longrightarrow H^{p,r}_{\inf\varrho,t}$$

is injective. To do this, we consider $f\in\mathcal{C}^{p,r}_{\Phi^*}(X,E)\cap\operatorname{Ker}\bar\partial$ which defines the zero class in $H^{p,r}_{\inf\varrho,t}$, i.e. $f|_{\{z|\inf\varrho<\varrho(z)\le t\}}=\bar\partial u$ for some $u\in\bigcup_{\inf\varrho<s<s_0}\mathcal{B}^{p,r-1}_{s,t}$. Take $s_u\in\,]\inf\varrho,s_0[$ such that $u\in\mathcal{B}^{p,r-1}_{s_u,t}$. Since $r\ge q^*$ and the Levi form of ϱ has at least $n-q^*+1$ positive eigenvalues on $\{z\,|\,t_0\le\varrho(z)<\sup\varrho\}$ and since, by (3.5), the spaces $\mathcal{B}^{p,r}_{s_u,\tau}\cap\bar\partial\mathcal{B}^{p,r-1}_{s,t}$, $t\le\tau<\sup\varrho$, are topologically closed in $\mathcal{B}^{p,r}_{s,\tau}$, we can use Grauert's bumping method, in the same way as in the proof of Theorem 12.13(ii) in [HL],(9) to obtain an E-valued continuous $(p,r-1)$ form v on all of X solving the equation $f=\bar\partial v$ on X, which, moreover, vanishes on $\{z\,|\inf\varrho<\varrho(z)\le s_u\}$ and hence belongs to $\mathcal{C}^{p,r-1}_{\Phi^*}(X,E)$, i.e. f defines the zero class in $H^{p,r}_{\mathcal{C},\Phi^*}(X,E)$. □

3.2. Lemma. *If* $1\le r\le\min(n-q,n-q^*+1)$, *then* $H^{n-p,r}_{\Phi}(X,E^*)$ *is Hausdorff. In particular, if* $q^*=n-q$ *and* $q<\frac12 n$, *then* $H^{n-p,q+1}_{\Phi}(X,E^*)$ *is Hausdorff.*

Proof. If $1\le r\le\min(n-q,n-q^*+1)$, then $\max(q+1,q^*)\le n-r+1\le n$. Hence, by Lemma 3.1, $\dim H^{p,n-r+1}_{\Phi^*}(X,E)<\infty$. By Lemma 2.6(ii) this implies that $H^{p,n-r+1}_{\Phi^*}(X,E)$ is α-Hausdorff. Moreover it is clear that the same is true if, for sufficiently small $\varepsilon>0$, we replace X by $X_\varepsilon:=\{z\,|\inf\varrho<\varrho(z)<\sup\varrho-\varepsilon\}$ and Φ^* by $\Phi^*\cap X_\varepsilon$, i.e. $H^{p,n-r+1}_{\Phi^*}(X,E)$ is even β-Hausdorff. Therefore, it now follows from Theorem 2.2 that $H^{n-p,r}_{\Phi}(X,E^*)$ is Hausdorff. □

Proof of Theorem 1.3. Since the space $H^{p,n-q}(X,E)$ is Hausdorff if and only if $H^{n-p,q+1}_c(X,E^*)$ is Hausdorff (see e.g. [L] or [LL1]), it is sufficient to prove that $H^{n-p,q+1}_c(X,E^*)$ is Hausdorff. Moreover, since, by Theorem 2.7 in [LL1], for the family of compact sets, Hausdorffness and α-Hausdorffness is the same, we only have to prove that, for each compact set $K\Subset X$, the space $\mathcal{R}_c:=\mathcal{D}^{n-p,q+1}_K(X,E^*)\cap\bar\partial\mathcal{D}^{n-p,q}(X,E^*)$ is topologically closed in $\mathcal{D}^{n-p,q+1}_K(X,E^*)$.

(9) The domain D in Theorem 12.13 of [HL] is assumed to be relatively compact, but in the proof only the consequence is used that then (with the notation of [HL]) $Z^0_{0,r}(\bar D,E)\cap\bar\partial C^0_{0,r-1}(\bar D,E)$ is topologically closed in $Z^0_{0,r}(\bar D,E)$.

Let a compact set $K \Subset X$ be given. By Lemma 3.2, $H_{\Phi}^{n-p,q+1}(X, E^*)$ is Hausdorff. In particular, the space $\mathcal{R}_{\Phi} := \mathcal{D}_K^{n-p,q+1}(X, E^*) \cap \bar{\partial}\mathcal{D}_{\Phi}^{n-p,q}(X, E^*)$ is topologically closed in $\mathcal{D}_K^{n-p,q+1}(X, E^*)$. Therefore, by Lemma 2.6(ii), it is sufficient to prove that

$$\dim \frac{\mathcal{R}_{\Phi}}{\mathcal{R}_c} < \infty.$$

Take s_1 and s_2 with $\inf \varrho < s_1 < s_2 < s_0$ so small that $K \subseteq \{z \mid s_2 < \varrho(z) < \sup \varrho\}$ and set $D = \{z \mid s_1 \leq \varrho(z) \leq s_2\}$. Then it is clear that

$$\dim \frac{\mathcal{R}_{\Phi}}{\mathcal{R}_c} \leq \dim H^{n-p,q}(D, E^*).$$

Now we look at the boundary of D, which consists of the two parts $\partial_1 D = \{z \mid \varrho(z) = s_1\}$ and $\partial_2 D = \{z \mid \varrho(z) = s_2\}$. Since the Levi form of ϱ has at least $n-q+1$ positive eigenvalues on $\overline{D}$, $\partial_1 D$ is q-concave and $\partial_2 D$ is q-convex. Since $q < \frac{1}{2}n$ and hence $q < n-q$, by Andreotti–Grauert theory [AG] (see also Section 22 in [HL]) this implies that $\dim H^{n-p,q}(D, E^*) < \infty$, which completes the proof of Theorem 1.3.

4. Extension of CR-forms

Let X be an n-dimensional complex manifold, E a holomorphic vector bundle over X, and let p and q be integers with $0 \leq p \leq n$ and $1 \leq q \leq n-1$. Furthermore let $D \Subset X$ be a domain whose boundary ∂D is C^∞ and compact, and let f be an E-valued C^∞ smooth CR-form of bidegree (p, q) on ∂D. Consider the two conditions

(i) there exists a $\bar{\partial}$-closed $C^\infty_{p,q}$-form F on $\overline{D}$ with $F|_{\partial D} = f$;

(ii) $\int_{\partial D} f \wedge \psi = 0$ for any E^*-valued $\bar{\partial}$-closed $C^\infty_{n-p,n-q-1}$-form ψ in a neighborhood of $\overline{D}$ such that $\operatorname{supp} \psi \cap \overline{D}$ is compact.

Then the following theorem is known.

4.1. Theorem. *If D is relatively compact and $(n-q)$-convex in the sense of Andreotti–Grauert, then the conditions* (i) *and* (ii) *are equivalent.*

Why this theorem is known (although nowhere explicitly stated), we explain in Remark 4.3 below. Under the stronger hypothesis that D is even strictly $(n-q-1)$-convex (and hence $q \leq n-2$), it was proved in 1965 by Kohn and Rossi [KR].

If $1 \leq q < \frac{1}{2}n$, from Theorem 1.3 one obtains the following generalization to the case when D has some "q-convex holes".

4.2. Theorem. *Suppose X is q-concave-$(n-q)$-convex in the sense of Definition* 1.1. *Let ϱ and t_0 have the same meaning as in Definition* 1.1, *and let $D = \{z \mid \inf \varrho(z) < \varrho < t\}$, where t is some number with $t_0 < t < \sup \varrho$ and $d\varrho(z) \neq 0$ if $\varrho(z) = t$. If now $1 \leq q < \frac{1}{2}n$, then the conditions* (i) *and* (ii) *are equivalent.*

Proof. That (i) implies (ii) follows from Stokes' formula. Assume that (ii) is satisfied. Since f is CR, there is a $C^\infty_{p,q}$ form $\tilde{f}$ on X such that $\tilde{f}|_{\partial D}=f$ and $\bar\partial\tilde{f}$ vanishes to infinite order on ∂D. Since ∂D is compact, we may moreover assume that $\tilde{f}$ has compact support. Setting $\varphi=\bar\partial\tilde{f}$ in D and $\varphi=0$ on $X\setminus D$ we obtain a form $\varphi\in\mathcal{D}^{p,q+1}(X,E)\cap\operatorname{Ker}\bar\partial$. Moreover, by Stokes' formula and condition (ii),

$$\int_X \varphi\wedge\psi=\int_D \bar\partial\tilde{f}\wedge\psi=\pm\int_{\partial D} f\wedge\psi=0 \quad \text{for all } \psi\in\mathcal{D}^{n-p,n-q-1}_{\Phi^*}\cap\operatorname{Ker}\bar\partial.$$

By Lemma 2.5 this means that $\varphi\in\overline{\bar\partial\mathcal{D}^{p,q}(X,E)}$. Moreover, $H^{n-p,n-q}(X,E^*)$ is Hausdorff by Theorem 1.3. By Serre's theorem [S], this implies that $H^{p,q+1}_c(X,E)$ is Hausdorff, i.e. $\overline{\bar\partial D^{p,q}(X,E)}=\bar\partial D^{p,q}(X,E)$. Hence $\varphi=\bar\partial\omega$ for some $\omega\in\mathcal{D}^{p,q}(X,E)$.

Since $\omega|_{X\setminus D}$ is $\bar\partial$-closed and has compact support and $q\geq1$, it follows from Theorem 3.1 in [LL2] that $\omega|_{X\setminus\bar{D}}=\bar\partial\eta|_{X\setminus\bar{D}}$ for some $\eta\in\mathcal{D}^{p,q-1}(X,E)$. Set $\tilde\omega=\omega-\bar\partial\eta$. Then $\tilde\omega\equiv0$ outside D and $\bar\partial\tilde{f}=\phi=\bar\partial\tilde\omega$ on D. Hence $F:=\tilde{f}-\tilde\omega$ is the required extension of f. □

4.3. *Remark.* The proof of Theorem 4.2 becomes a proof of Theorem 4.1 if we replace X by an $(n-q)$-convex neighborhood of $\bar{D}$ and if we use, instead of Theorem 1.3, the fact from classical Andreotti–Grauert theory that $H^{n-p,n-q}(X,E^*)$ is then finite-dimensional and hence Hausdorff.

Note also that if we already know that f extends as a $\bar\partial$-closed form to some neighborhood of ∂D, Theorem 4.1 can be proved without Theorem 3.1 of [LL2], using only the other arguments of the proof given above (see Theorem 20.13 in [HL]).

References

[AG] ANDREOTTI, A. and GRAUERT, H., Théorèmes de finitude pour la cohomologie des espaces complexes, *Bull. Soc. Math. France* **90** (1962), 193–259.

[AV] ANDREOTTI, A. and VESENTINI, E., Carleman estimates for the Laplace–Beltrami equation on complex manifolds, *Inst. Hautes Études Sci. Publ. Math.* **25** (1965), 81–130.

[CS] CHIRKA, E. M. and STOUT, E. L., Removable singularities in the boundary, in *Contributions to Complex Analysis and Analytic Geometry* (Skoda, H. and Trépreau, J.-M., eds.), Aspects of Math. **E26**, pp. 43–104, Viehweg, Braunschweig, 1994.

[DS] DIEUDONNÉ, J. and SCHWARTZ, L., La dualité dans les espaces $(\mathcal{F})$ et $(\mathcal{L}F)$, *Ann. Inst. Fourier (Grenoble)* **1** (1949), 61–101.

[FL] FISCHER, W. and LIEB, I., Lokale Kerne und beschränkte Lösungen für den $\bar\partial$-Operator auf q-konvexen Gebieten, *Math. Ann.* **208** (1974), 249–265.

[HL] HENKIN, G. M. and LEITERER, J., *Andreotti–Grauert Theory by Integral Formulas,* Progress in Math. **74**, Birkhäuser, Basel, 1988.

[KR] KOHN, J. J. and ROSSI, H., On the extension of holomorphic functions from the boundary of a complex manifold, *Ann. of Math.* **81** (1965), 451–472.

[L] LAUFER, H. B., On Serre duality and envelopes of holomorphy, *Trans. Amer. Math. Soc.* **128** (1967), 414–436.

[LL1] LAURENT-THIÉBAUT, C. and LEITERER, J., The Andreotti–Vesentini separation theorem with C^k estimates and extension of CR-forms, in *Several Complex Variables, Proceedings of the Mittag-Leffler Institute, 1987–1988* (Fornæss, J. E., ed.), Mathematical Notes **38**, pp. 416–439, Princeton Univ. Press, Princeton, N. J., 1993.

[LL2] LAURENT-THIÉBAUT, C. and LEITERER, J., The Andreotti–Vesentini separation theorem and global homotopy representation, *Math. Z.* **227** (1998), 711–727.

[LL3] LAURENT-THIÉBAUT, C. and LEITERER, J., On Serre duality, *Bull. Sci. Math.* **124** (2000), 93–106.

[Ra] RAMIS, J.-P., Théorèmes de séparation et de finitude pour l'homologie et la cohomologie des espaces (p,q)-convexes-concaves, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **27** (1973), 933–997.

[Ro] ROSSI, H., Attaching analytic spaces to an analytic space along a pseudoconcave boundary, in *Proceedings of the Conference on Complex Analysis (Minneapolis, 1964)* (Aeppli, A., Calabi, E. and Röhrl, H., eds.), pp. 242–256, Springer-Verlag, Berlin, 1965.

[S] SERRE, J. P., Un théorème de dualité, *Comment. Math. Helv.* **29** (1955), 9–26.

[T] TRÈVES, F., *Topological Vector Spaces, Distributions and Kernels*, Academic Press, New York–London, 1967.

Received May 8, 2000

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