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Infinitely many solutions of a symmetric semilinear elliptic equation on an unbounded domain

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Abstract. We study a semilinear elliptic equation of the form

$$-\Delta u+u=f(x,u), \quad u\in H_0^1(\Omega),$$

where f is continuous, odd in u and satisfies some (subcritical) growth conditions. The domain $\Omega\subset\mathbf{R}^N$ is supposed to be an unbounded domain ($N\geq 3$). We introduce a class of domains, called strongly asymptotically contractive, and show that for such domains Ω, the equation has infinitely many solutions.

1. Introduction

The aim of this paper is to study subcritical semilinear elliptic equations in unbounded domains. As an example, let $N\geq 3$, $p\in(2,2^*)$, where $2^*=2N/(N-2)$, and consider the equation

$$-\Delta u+u=|u|^{p-2}u, \quad u\in H_0^1(\Omega), \tag{1.1}$$

where Ω is a domain in $\mathbf{R}^N$. When Ω is bounded, the equation has infinitely many solutions, and the corresponding functional

$$\varphi(u)=\frac{1}{2}\int_\Omega (u^2+|\nabla u|^2)\,dx-\frac{1}{p}\int_\Omega |u|^p\,dx$$

has an unbounded sequence of critical values. There are several proofs of this result, the first of which was made by Ambrosetti and Rabinowitz [1] (see also [10]).

Let c be a real number. A $(\text{P–S})_c$ sequence is a sequence $u_j\in H_0^1(\Omega)$ such that

$$\varphi(u_j)\to c \quad\text{and}\quad \varphi'(u_j)\to 0.$$

The functional φ is said to satisfy the $(\text{P–S})_c$ condition if every $(\text{P–S})_c$ sequence has a convergent subsequence.

When this condition is satisfied for all $c>0$, there are known methods of obtaining an unbounded sequence of critical values of φ (see e.g. [8]). In the case when Ω is bounded, the $(\text{P–S})_c$ condition follows from the compactness of the embedding of $H_0^1(\Omega)$ into $L^p(\Omega)$.

In this paper we prove that in many cases, φ satisfies the $(\text{P–S})_c$ condition even when Ω is unbounded. A typical example of a domain of this kind is the tube

$$\Omega=\{(x,y)\in\mathbf{R}^n\times\mathbf{R}^{N-n}:|y|<g(x)\},$$

where $1\le n\le N-1$ and $g:\mathbf{R}^n\to\mathbf{R}$ is continuous, positive and such that the limit

$$g_\infty=\lim_{|x|\to\infty}g(x)$$

exists and $g(x)>g_\infty$ for all $x\in\mathbf{R}^n$.

2. Preliminaries and formulation of the problem

Let $N\ge3$ and let $\Omega\subset\mathbf{R}^N$ be a domain. Consider the equation

$$-\Delta u(x)+u(x)=f(x,u(x)),\quad u\in H_0^1(\Omega), \tag{2.1}$$

where $f\in C(\overline{\Omega}\times\mathbf{R},\mathbf{R})$ satisfies the following conditions:

(f_1) there are constants $2<p\le q<2^*$ and $C>0$ such that for any $x\in\Omega$ and $s\in\mathbf{R}$,

$$|f(x,s)|\le C(|s|^{p-1}+|s|^{q-1});$$

(f_2) there are constants $\mu>2$, $\nu>2$ and $D>0$ such that for any $x\in\Omega$ and $s\in\mathbf{R}\setminus\{0\}$,

$$sf(x,s)\ge\mu F(x,s)\equiv\mu\int_0^s f(x,\sigma)\,d\sigma>0$$

and

$$\liminf_{s\to0}\frac{F(x,s)}{|s|^\nu}\ge D;$$

(f_3) for any $s\in\mathbf{R}$,

$$f(x,s)=-f(x,-s);$$

(f_4) there exists a function $f_\infty\in C(\mathbf{R},\mathbf{R})$ such that

$$\lim_{R\to\infty}\sup_{\substack{x\in\Omega\setminus B_R(0)\\ s\in\mathbf{R}}}|f(x,s)-f_\infty(s)|=0.$$

We use the standard norm

$$\|u\|_{H^1}=\left(\int_\Omega (|u|^2+|\nabla u|^2)\,dx\right)^{1/2}$$

and the corresponding inner product

$$(u,v)_{H^1}=\int_\Omega (uv+\nabla u\cdot\nabla v)\,dx$$

on $H_0^1(\Omega)=W_0^{1,2}(\Omega)$, where $\cdot$ denotes the usual scalar product in $\mathbf{R}^N$. The functional corresponding to (2.1) is

$$\begin{aligned}\varphi(u)&=\frac{1}{2}\int_\Omega (u(x)^2+|\nabla u(x)|^2)\,dx-\int_\Omega F(x,u(x))\,dx\\&=\frac{1}{2}\|u\|_{H^1}^2-\int_\Omega F(x,u(x))\,dx.\end{aligned}\tag{2.2}$$

Then $\varphi\in C^1(H_0^1(\Omega),\mathbf{R})$, and the critical points of φ are the weak solutions of (2.1).

Definition 1. We will say that the domain Ω is *strongly asymptotically contractive* if $\Omega\neq\mathbf{R}^N$ and for any sequence $\alpha_j\in\mathbf{R}^N$ such that $|\alpha_j|\to\infty$, there exists a subsequence α_{j_l} and a point $\beta\in\mathbf{R}^N$ such that for any $R>0$ there exists an open set $M_R\Subset\Omega+\beta$, a closed set Z of measure 0 and an integer $l_R>0$ such that

$$(\Omega+\alpha_{j_l})\cap B_R(0)\subset M_R\cup Z\quad\text{for any } l\geq l_R.$$

Note that every bounded domain is strongly asymptotically contractive. The following examples show that there are a lot of other domains satisfying this condition. Our first two examples of domains were also studied by del Pino and Felmer in [5], where the existence of least energy solutions of (2.1) was proved.

Example 1. Let $1\leq n\leq N-1$ and let $g:\mathbf{R}^n\to\mathbf{R}$ be continuous and positive. Suppose that the limit

$$g_\infty=\lim_{|x|\to\infty} g(x)$$

exists and that $g(x)>g_\infty$ for all $x\in\mathbf{R}^n$. Let $\Omega\subset\mathbf{R}^N$ be the domain defined by

$$\Omega=\{(x,y)\in\mathbf{R}^n\times\mathbf{R}^{N-n}:|y|<g(x)\}.$$

Then the domain Ω is strongly asymptotically contractive. To see this, let $\alpha_j=(\gamma_j,\delta_j)\in\mathbf{R}^n\times\mathbf{R}^{N-n}$ be such that $|\alpha_j|\to\infty$, as $j\to\infty$. If α_j has a subsequence α_{j_l} on which δ_{j_l} is unbounded, then for any $R>0$ and any l sufficiently large,

$$(\Omega+\alpha_{j_l})\cap B_R(0)=\emptyset.$$

Hence we can restrict ourselves to sequences α_j with bounded δ_j-components. Such a sequence δ_j has a convergent subsequence $\delta_{j_l}\to\delta$. Let $\beta=(0,\delta)$ and let

$$M_R=\{(x,y)\in\mathbf{R}^n\times\mathbf{R}^{N-n};|y-\delta|<g_\infty+\tfrac{1}{2}(g(x)-g_\infty)\}\cap B_R(0).$$

Let $R>0$ and let $\varepsilon_R>0$ be such that

$$\varepsilon_R<\min_{x\in\overline{B_R(0)}}\frac{1}{2}(g(x)-g_\infty).$$

Then $M_R\subset\Omega+\beta$ and there exists an integer $l_R>0$ such that for $l\geq l_R$,

$$(\Omega+\alpha_{j_l})\cap B_R(0)\subset\{(x,y)\in\mathbf{R}^n\times\mathbf{R}^{N-n}:|y-\delta|<g_\infty+\varepsilon_R\}\cap B_R(0)\subset M_R.$$

Example 2. Let $g\in C(\mathbf{R}^{N-1},\mathbf{R})$ be positive, and suppose that the limit

$$g_\infty=\lim_{|x|\to\infty}g(x)$$

exists and that $g(x)>g_\infty$ for all $x\in\mathbf{R}^{N-1}$. Let $\Omega\subset\mathbf{R}^N$ be the domain defined by

$$\Omega=\{(x,y)\in\mathbf{R}^{N-1}\times\mathbf{R};0<y<g(x)\}.$$

Then Ω is strongly asymptotically contractive. The proof is similar to the proof of Example 1, and therefore is omitted.

Example 3. As a third example, note that a finite union of intersecting domains as in Example 1 is strongly asymptotically contractive.

Note that $\mathbf{R}^N$ is not strongly asymptotically contractive, and neither is the straight cylinder

$$\Omega=\{(x,y)\in\mathbf{R}^n\times\mathbf{R}^{N-n}:|y|<\alpha\},$$

where $1\leq n\leq N-1$ and $\alpha>0$.

Theorem 1. *Let Ω be a strongly asymptotically contractive, and let $f\in C(\overline{\Omega}\times\mathbf{R},\mathbf{R})$ satisfy (f_1), (f_2), (f_3) and (f_4). Then equation (2.1) has infinitely many solutions and the corresponding functional φ has infinitely many critical values.*

The rest of this paper concerns the proof of this theorem.

3. The (P–S)$_c$ condition

In this section, we prove that φ satisfies the (P–S)$_c$ condition for every $c>0$ if Ω is a strongly asymptotically contractive domain.

Lemma 1. *Let φ be given by (2.2), where $f\in C(\bar{\Omega}\times\mathbf{R},\mathbf{R})$ satisfies condition (f_2). Let u_j be a (P–S)$_c$-sequence, i.e. a sequence such that*

(i) $\varphi(u_j)\to c$;

(ii) $\varphi'(u_j)\to 0$.

Then $c\geq 0$. Moreover, $\|u_j\|_{H^1}$ is bounded and

$$\limsup_{j\to\infty}\|u_j\|_{H^1}^2\leq\frac{c}{\dfrac{1}{2}-\dfrac{1}{\mu}}.$$

Proof. By (2.2),

$$\langle\varphi'(u_j),u_j\rangle=\|u_j\|_{H^1}^2-\int_\Omega f(x,u_j(x))u_j(x)\,dx.$$

Let $\varepsilon>0$ be given. Then by (2.2), (f_2), (i) and (ii), for j large enough,

$$\begin{aligned}
c+\varepsilon+\varepsilon\|u_j\|_{H^1}&\geq\varphi(u_j)-\frac{1}{\mu}\langle\varphi'(u_j),u_j\rangle\\
&=\left(\frac{1}{2}-\frac{1}{\mu}\right)\|u_j\|_{H^1}^2+\frac{1}{\mu}\int_\Omega(f(x,u_j(x))u_j(x)-\mu F(x,u_j(x)))\,dx\\
&\geq\left(\frac{1}{2}-\frac{1}{\mu}\right)\|u_j\|_{H^1}^2.
\end{aligned}$$

Hence $\|u_j\|_{H^1}$ is bounded and $c\geq 0$. Moreover,

$$\|u_j\|\leq\frac{\varepsilon}{2\left(\dfrac{1}{2}-\dfrac{1}{\mu}\right)}+\sqrt{\frac{\varepsilon^2}{4\left(\dfrac{1}{2}-\dfrac{1}{\mu}\right)^2}+\frac{c+\varepsilon}{\dfrac{1}{2}-\dfrac{1}{\mu}}}.$$

Since $\varepsilon>0$ was arbitrary, the result follows. □

The following version of the concentration-compactness principle (see [3], [4]) is from Schindler and Tintarev [7]. (See also [6] and [9].)

Lemma 2. *Let u_j be a bounded sequence in $H^1(\mathbf{R}^N)$. Then there exist $w^{(1)}$, $w^{(2)},\dots\in H^1(\mathbf{R}^N)$ and $\alpha_j^{(n)}\in\mathbf{R}^N$, $j,n\in\mathbf{Z}_+$, such that on a renumbered subsequence,*

$$
\begin{aligned}
&u_j(\,\cdot\,-\alpha_j^{(n)})\rightharpoonup w^{(n)}, && \\
&u_j-\sum_{n=1}^{\infty} w^{(n)}(\,\cdot\,+\alpha_j^{(n)})\to 0 && \text{in } L^p(\mathbf{R}^N),\ \text{where } p\in(2,2^*),\\
&|\alpha_j^{(n)}-\alpha_j^{(m)}|\to\infty, && \text{if } m\neq n, \\
&\sum_{n=1}^{\infty}\|w^{(n)}\|_{H^1}^2\le \lim_{j\to\infty}\|u_j\|_{H^1}^2, && \\
&\sum_{n=1}^{\infty}\|w^{(n)}\|_{L^p}^p= \lim_{j\to\infty}\|u_j\|_{L^p}^p, && \text{where } p\in(2,2^*).
\end{aligned}
\tag{3.1}
$$

Lemma 3. *Let $\Omega\subset\mathbf{R}^N$ be a strongly asymptotically contractive domain, and let φ be the functional defined in* (2.2), *where $f\in C(\bar\Omega\times\mathbf{R},\mathbf{R})$ satisfies (f_1), (f_2), (f_3) and (f_4). Then φ satisfies the* $(\mathrm{P\text{-}S})_c$ *condition for any $c>0$.*

Proof. Let $c>0$, and let $u_j\in H_0^1(\Omega)$ be a $(\mathrm{P\text{-}S})_c$ sequence for φ. We start by observing that by Lemma 1, the sequence u_j is bounded. Hence we can apply Lemma 2, and rewrite u_j as a sum

$$
u_j=w^{(0)}+\sum_{n=1}^{\infty} w^{(n)}(\,\cdot\,+\alpha_j^{(n)})+r_j, \tag{3.2}
$$

where $r_j\to 0$ in $L^p(\mathbf{R}^N)$, $\alpha_j^{(n)}\in\mathbf{R}^N$, $|\alpha_j^{(n)}|\to\infty$ and $w^{(n)}\in H^1(\mathbf{R}^N)$ are as given in Lemma 2.

Since Ω is a strongly asymptotically contractive domain, for any $n\ge 1$ there exists a subsequence $\alpha_{j_l}^{(n)}$ and a number $\beta^{(n)}\in\mathbf{R}^N$ such that for any $R>0$, there exists an open set $M_R^{(n)}\Subset\Omega+\beta^{(n)}$, a closed set $Z^{(n)}$ of zero measure and an integer $l_R>0$ such that for any $l\ge l_R$

$$
(\Omega+\alpha_{j_l}^{(n)})\cap B_R(0)\subset M_R^{(n)}\cup Z^{(n)}.
$$

By taking a subsequence, we can assume that this relation holds for every $j\ge j_R$.

Hence

$$
\begin{aligned}
\operatorname{supp} w^{(n)}\cap B_R(0) &\subset \overline{\bigcup_{j=j_R}^{\infty}\operatorname{supp} u_j(\,\cdot\,-\alpha_j^{(n)})\cap B_R(0)}\\
&\subset \overline{M_R^{(n)}}\cup Z^{(n)}\Subset(\Omega+\beta^{(n)})\cup Z^{(n)}.
\end{aligned}
\tag{3.3}
$$

and thus

$$\operatorname{supp} w^{(n)} \subset \Omega + \beta^{(n)}$$

modulo a set of measure zero. Let $\widetilde{w}^{(n)} = w^{(n)}(\cdot - \beta^{(n)})$ and let $\widetilde{\alpha}_j^{(n)} = \alpha_j^{(n)} + \beta^{(n)}$. Note that by (3.3), there exists an open set $U \subset \Omega$ such that $\widetilde{w}^{(n)} \equiv 0$ in U. This fact will be used when applying the maximum principle below.

Let $v \in C_0^\infty(\Omega)$ be arbitrary. Then

$$\begin{aligned}
&\int_\Omega (\nabla \widetilde{w}^{(n)}(x) \cdot \nabla v(x) + \widetilde{w}^{(n)}(x) v(x))\, dx \\
&\qquad = \lim_{j \to \infty} \int_\Omega (\nabla u_j(x - \widetilde{\alpha}_j^{(n)}) \cdot \nabla v(x) + u_j(x - \widetilde{\alpha}_j^{(n)}) v(x))\, dx \\
&\qquad = \lim_{j \to \infty} \int_{\Omega + \widetilde{\alpha}_j^{(n)}} f(x - \widetilde{\alpha}_j^{(n)}, u_j(x - \widetilde{\alpha}_j^{(n)})) v(x)\, dx \\
&\qquad = \lim_{j \to \infty} \int_{\Omega + \widetilde{\alpha}_j^{(n)}} f(x - \widetilde{\alpha}_j^{(n)}, \widetilde{w}^{(n)}(x)) v(x)\, dx \\
&\qquad = \int_\Omega f_\infty(\widetilde{w}^{(n)}(x)) v(x)\, dx.
\end{aligned}$$

which means that for $n \geq 1$, $\widetilde{w}^{(n)}$ is a weak solution of the equation

$$-\Delta \widetilde{w}^{(n)} + \widetilde{w}^{(n)} = f_\infty(\widetilde{w}^{(n)}).$$

By regularity theory, $\widetilde{w}^{(n)}$ is continuous in Ω. We divide the domain Ω into three parts:

$$\begin{aligned}
\Omega_+ &= \{x \in \Omega : \widetilde{w}^{(n)}(x) > 0\}, \\
\Omega_- &= \{x \in \Omega : \widetilde{w}^{(n)}(x) < 0\}, \\
\Omega_0 &= \{x \in \Omega : \widetilde{w}^{(n)}(x) = 0\}.
\end{aligned}$$

By the previous paragraph, Ω_0 has a nonempty interior. Note that by (f_2),

$$\begin{aligned}
-\Delta \widetilde{w}^{(n)} + \widetilde{w}^{(n)} &> 0 \quad \text{in } \Omega_+, \\
-\Delta \widetilde{w}^{(n)} + \widetilde{w}^{(n)} &< 0 \quad \text{in } \Omega_-.
\end{aligned}$$

We claim that $\widetilde{w}^{(n)} \equiv 0$ in Ω, i.e. $\Omega_0 = \Omega$. Suppose to the contrary that $\Omega_+ \cup \Omega_-$ is nonempty. Then either Ω_+ or Ω_- has a component whose boundary is intersecting the boundary of Ω_0°. Suppose first that Ω_+ has a component $\widetilde{\Omega}_+$ such that $\partial \widetilde{\Omega}_+ \cap \partial \Omega_0^\circ \neq \emptyset$. By applying the strong maximum principle (see [2], Theorem 8.19,

p. 198) on $(\widetilde{\Omega}_+ \cup \Omega_0)^\circ$, $\widetilde{w}^{(n)} \equiv 0$ in this domain, and so $\widetilde{\Omega}_+ = \emptyset$. The same argument shows that Ω_- cannot have a component adjacent to Ω_0°. This shows that $\widetilde{w}^{(n)} \equiv 0$ in Ω.

Let p and q be as in condition (f_1). We have proved that $u_j \to w^{(0)}$ in $L^p(\mathbf{R}^N) \cap L^q(\mathbf{R}^N)$, and since $u_j \in H_0^1(\Omega) \subset L^p(\Omega) \cap L^q(\Omega)$, $u_j \to w^{(0)}$ in $L^p(\Omega) \cap L^q(\Omega)$. By the continuity of the superposition operator

$$L^p(\Omega) \cap L^q(\Omega) \ni u \mapsto f(x,u) \in L^{p/(p-1)}(\Omega) + L^{q/(q-1)}(\Omega)$$

(see e.g. Theorem A.4., p. 134, in [10]),

$$f(x,u_j) \to f(x, w^{(0)}) \quad \text{in } L^{p/(p-1)}(\Omega) + L^{q/(q-1)}(\Omega).$$

We write $f = f_1 + f_2$, where $f_1 \in L^{p/(p-1)}(\Omega)$ and $f_2 \in L^{q/(q-1)}(\Omega)$. Observe that

$$\begin{aligned} \|u_j - w^{(0)}\|_{H^1}^2 &= \langle \varphi'(u_j) - \varphi'(w^{(0)}), u_j - w^{(0)} \rangle \\ &\quad + \int_\Omega (f(x, u_j(x)) - f(x, w^{(0)}(x)))(u_j(x) - w^{(0)}(x))\, dx. \end{aligned}$$

Obviously,

$$\langle \varphi'(u_j) - \varphi'(w^{(0)}), u_j - w^{(0)} \rangle \to 0,$$

and by the Hölder inequality,

$$\begin{aligned} &\left| \int_\Omega (f(x,u_j(x)) - f(x, w^{(0)}(x)))(u_j(x) - w^{(0)}(x))\, dx \right| \\ &\qquad \le \left| \int_\Omega (f_1(x,u_j(x)) - f_1(x, w^{(0)}(x)))(u_j(x) - w^{(0)}(x))\, dx \right| \\ &\qquad\quad + \left| \int_\Omega (f_2(x,u_j(x)) - f_2(x, w^{(0)}(x)))(u_j(x) - w^{(0)}(x))\, dx \right| \\ &\qquad \le \|f_1(\cdot, u_j) - f_1(\cdot, w^{(0)})\|_{L^{p/(p-1)}} \|u_j - w^{(0)}\|_{L^p} \\ &\qquad\quad + \|f_2(\cdot, u_j) - f_2(\cdot, w^{(0)})\|_{L^{q/(q-1)}} \|u_j - w^{(0)}\|_{L^q} \\ &\qquad \to 0. \end{aligned}$$

By taking the infimum over the functions $f_1 \in L^{p/(p-1)}(\Omega)$ and $f_2 \in L^{q/(q-1)}(\Omega)$ such that $f = f_1 + f_2$, we obtain $u_j \to w^{(0)}$ in $H_0^1(\Omega)$. □

4. Infinitely many solutions

We obtain an infinite sequence of critical values from the following theorem (see e.g. Theorem 6.5 of [8]).

Theorem 2. *Suppose that V is an infinite-dimensional Banach space and suppose $\varphi \in C^1(V, \mathbf{R})$ satisfies $(\text{P–S})_c$ for every $c>0$, $\varphi(u)=\varphi(-u)$ for all u, and assume the following conditions*:

(i) *there exist $\alpha>0$ and $\varrho>0$ such that if $\|u\|=\varrho$ and $u \in V$, then $\varphi(u) \geq \alpha$;*

(ii) *for any finite-dimensional subspace $W \subset V$ there exists $R=R(W)$ such that $\varphi(u) \leq 0$ for $u \in W$, $\|u\| \geq R$.*

Then φ possesses an unbounded sequence of critical values.

Proof of Theorem 1. We apply Theorem 2 with $V=H_0^1(\Omega)$. It is clear that $\varphi \in C^1(H_0^1(\Omega), \mathbf{R})$ is even. By Lemma 3, the $(\text{P–S})_c$ condition is satisfied for every $c>0$. We only need to check conditions (i) and (ii).

Integrating (f_1), there is a constant $C_1>0$ such that for all $x \in \Omega$ and $s \in \mathbf{R}$,

$$|F(x,s)| \leq C_1(|s|^p+|s|^q).$$

By the Sobolev embedding theorem, we have the estimate

$$\varphi(u) \geq \frac{1}{2}\|u\|_{H^1}^2 - C_1 \int_\Omega (|u(x)|^p+|u(x)|^q)\,dx \geq \frac{1}{2}\|u\|_{H^1}^2 - C_2\|u\|_{H^1}^p - C_2\|u\|_{H^1}^q.$$

Let $\|u\|=\varrho$, where ϱ is free for the moment. Then

$$\varphi(u) \geq \tfrac{1}{2}\varrho^2 - C_2\varrho^p - C_2\varrho^q,$$

and we would like to choose ϱ such that $\varphi(u)$ is as large as possible when $\|u\|=\varrho$. For such ϱ, we have

$$\varrho - C_2 p \varrho^{p-1} - C_2 q \varrho^{q-1} = 0.$$

Since the left-hand side is positive for small values of $\varrho>0$ and negative for large ϱ, by the intermediate value theorem, this equation has a solution. We choose ϱ to be this solution. Then

$$C_2\varrho^p = \frac{1}{p}\varrho^2 - C_2\frac{q}{p}\varrho^q.$$

and so for $\|u\|=\varrho$,

$$\varphi(u) \geq \left(\frac{1}{2}-\frac{1}{p}\right)\varrho^2 + C_2\left(\frac{q}{p}-1\right)\varrho^q \geq \left(\frac{1}{2}-\frac{1}{p}\right)\varrho^2.$$

Thus condition (i) is fulfilled with $\alpha=\left(\frac{1}{2}-1/p\right)\varrho^2$.

By (f_2), there is a constant $C_3>0$ such that for every $x \in \Omega$ and $s \in \mathbf{R}$, $|F(x,s)| \geq C_3|s|^{\mu_1}$, where $\mu_1=\min\{\mu, \nu\}$. Indeed, let $\varepsilon>0$ be given. By integration of the first identity of (f_2), we have for $|s|>\varepsilon$ and $x \in \Omega$,

$$F(x,s) \geq \frac{F(x,\varepsilon)}{\varepsilon^\mu}|s|^\mu.$$

By letting $\varepsilon\to 0$ and using the second identity of (f_2), the claim follows. Let W be a finite-dimensional subspace of $H_0^1(\Omega)$. Since all norms are equivalent of W, and since

$$\varphi(u)\leq\tfrac{1}{2}\|u\|_{H^1}^2-C_3\|u\|_{L^{\mu_1}}^{\mu_1},$$

condition (ii) follows. □

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