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A property of strictly singular one-to-one operators

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Abstract. We prove that if T is a strictly singular one-to-one operator defined on an infinite dimensional Banach space X, then for every infinite dimensional subspace Y of X there exists an infinite dimensional subspace Z of X such that $Z \cap Y$ is infinite dimensional, Z contains orbits of T of every finite length and the restriction of T to Z is a compact operator.

1. Introduction

An operator on an infinite dimensional Banach space is called *strictly singular* if it fails to be an isomorphism when it is restricted to any infinite dimensional subspace (by "operator" we will always mean a "continuous linear map"). It is easy to see that an operator T on an infinite dimensional Banach space X is strictly singular if and only if for every infinite dimensional subspace Y of X there exists an infinite dimensional subspace Z of Y such that the restriction of T to Z, $T|_Z: Z \to X$, is a compact operator. Moreover, Z can be assumed to have a basis. Compact operators are special examples of strictly singular operators. If $1 \le p < q \le \infty$ then the inclusion map $i_{p,q}: l_p \to l_q$ is a strictly singular (non-compact) operator. A *hereditarily indecomposable* Banach space is an infinite dimensional space such that no subspace can be written as a topological sum of two infinite dimensional subspaces. W. T. Gowers and B. Maurey constructed the first example of a hereditarily indecomposable space [9]. It is also proved in [9] that every operator on a complex hereditarily indecomposable space can be written as a strictly singular perturbation of a multiple of the identity. If X is a complex hereditarily indecomposable space and T is a strictly singular operator on X then the spectrum of T resembles the spectrum of a compact operator on a complex Banach space: it is either the singleton $\{0\}$ (i.e. T is quasi-nilpotent), or a sequence $\{\lambda_n : n = 1, 2, ...\} \cup \{0\}$, where λ_n is an eigenvalue

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of T with finite multiplicity for all n, and $(\lambda_n)_n$ converges to 0, if it is an infinite sequence. It was asked whether there exists a hereditarily indecomposable space X which gives a positive solution to the "identity plus compact" problem, namely, every operator on X is a compact perturbation of a multiple of the identity. This question was answered in negative in [2] for the hereditarily indecomposable space constructed in [9], (for related results see [7], [8], and [1]). By [3], (or the more general beautiful theorem of V. Lomonosov [10]), if a Banach space gives a positive solution to the "identity plus compact" problem, it also gives a positive solution to the famous invariant subspace problem. The invariant subspace problem asks whether there exists a separable infinite dimensional Banach space on which every operator has a non-trivial invariant subspace, (by "non-trivial" we mean "different than $\{0\}$ and the whole space"). It remains unknown whether l_2 is a positive solution to the invariant subspace problem. Several negative solutions to the invariant subspace problem are known [4], [5], [11], [12], [13]. In particular, there exists a strictly singular operator with no non-trivial invariant subspace [14]. It is unknown whether every strictly singular operator on a super-reflexive Banach space has a non-trivial invariant subspace. Our main result (Theorem 1) states that if T is a strictly singular one-to-one operator on an infinite dimensional Banach space X, then for every infinite dimensional Banach space Y of X there exists an infinite dimensional Banach space Z of X such that $Z\cap Y$ is infinite dimensional, the restriction of T to Z, $T|_Z: Z\to X$, is compact, and Z contains orbits of T of every finite length (i.e. for every $n\in\mathbf{N}$ there exists $z_n\in Z$ such that $\{z_n, Tz_n, T^2z_n, \dots, T^nz_n\}\subset Z$). We raise the following question.

Question. *Let T be a quasi-nilpotent operator on a super-reflexive Banach space X, such that for every infinite dimensional subspace Y of X there exists an infinite dimensional subspace Z of X such that $Z\cap Y$ is infinite dimensional, $T|_Z: Z\to X$ is compact and Z contains orbits of T of every finite length. Does T have a non-trivial invariant subspace?*

By our main result, an affirmative answer to the above question would give that every strictly singular, one-to-one, quasi-nilpotent operator on a super-reflexive Banach space has a non-trivial invariant subspace; in particular, we would obtain that every operator on the super-reflexive hereditarily indecomposable space constructed by V. Ferenczi [6] has a non-trivial invariant subspace, and thus the invariant subspace problem would be answered in affirmative.

2. The main result

Our main result is the following theorem.

Theorem 1. *Let T be a strictly singular one-to-one operator on an infinite dimensional Banach space X. Then, for every infinite dimensional subspace Y of X there exists an infinite dimensional subspace Z of X, such that $Z\cap Y$ is infinite dimensional, Z contains orbits of T of every finite length, and the restriction of T to Z, $T|_Z\colon Z\to X$, is a compact operator.*

The proof of Theorem 1 is based on Theorem 3. We first need to define the biorthogonal constant of a finite set of normalized vectors of a Banach space.

Definition 2. Let X be a Banach space, $n\in\mathbf{N}$, and $x_1, x_2, \dots, x_n$ be normalized elements of X. We define the biorthogonal constant of $x_1, \dots, x_n$ to be

$$\mathrm{bc}\{x_1, \dots, x_n\} := \sup\left\{\max\{|\alpha_1|, \dots, |\alpha_n|\} : \left\|\sum_{i=1}^{n} \alpha_i x_i\right\| = 1\right\}.$$

Notice that

$$\frac{1}{\mathrm{bc}\{x_1, \dots, x_n\}} = \inf\left\{\left\|\sum_{i=1}^{n} \beta_i x_i\right\| : \max_{1\le i\le n} |\beta_i| = 1\right\},$$

and that $\mathrm{bc}\{x_1, \dots, x_n\}<\infty$ if and only if $x_1, \dots, x_n$ are linearly independent.

Before stating Theorem 3 recall that if T is a quasi-nilpotent operator on a Banach space X, then for every $x\in X$ and $\eta>0$ there exists an increasing sequence $(i_n)_{n=1}^{\infty}$ in $\mathbf{N}$ such that $\|T^{i_n}x\|\le\eta\|T^{i_n-1}x\|$. Theorem 3 asserts that if T is a strictly singular one-to-one operator on a Banach space X then for arbitrarily small $\eta>0$ and $k\in\mathbf{N}$ there exists $x\in X$, $\|x\|=1$, such that $\|T^i x\|\le\eta\|T^{i-1}x\|$ for $i=1, 2, \dots, k+1$, and moreover, the biorthogonal constant of x, $Tx/\|Tx\|, \dots, T^k x/\|T^k x\|$ does not exceed $1/\sqrt{\eta}$.

Theorem 3. *Let T be a strictly singular one-to-one operator on a Banach space X. Let Y be an infinite dimensional subspace of X, F be a finite codimensional subspace of X and $k\in\mathbf{N}$. Then there exists $\eta_0\in(0,1)$ such that for every $0<\eta\le\eta_0$ there exists $x\in Y$, $\|x\|=1$, satisfying*

(a) *$T^{i-1}x\in F$ and $\|T^i x\|\le\eta\|T^{i-1}x\|$ for $i=1, 2, \dots, k+1$;*

(b)

$$\mathrm{bc}\left\{x, \frac{Tx}{\|Tx\|}, \dots, \frac{T^k x}{\|T^k x\|}\right\} \le \frac{1}{\sqrt{\eta}},$$

(where T^0 denotes the identity operator on X).

We postpone the proof of Theorem 3.

Proof of Theorem 1. Let T be a strictly singular one-to-one operator on an infinite dimensional Banach space X, and Y be an infinite dimensional subspace of X. Inductively we construct a normalized sequence $(z_n)_{n\in\mathbf{N}}\subset Y$, an increasing sequence of finite families $(z_j^*)_{j\in J_n}$ of normalized functionals on X (i.e. $(J_n)_{n\in\mathbf{N}}$ is an increasing sequence of finite index sets), and a sequence $(\eta_n)_{n\in\mathbf{N}}\subset(0,1)$, as follows:

For $n=1$ apply Theorem 3 for $F=X$ (set $J_1=\emptyset$), $k=1$, to obtain $\eta_1<1/2^6$ and $z_1\in Y$, $\|z_1\|=1$, such that

$$\|T^i z_1\| < \eta_1 \|T^{i-1} z_1\| \quad \text{for } i=1,2, \tag{1}$$

and

$$\mathrm{bc}\left\{ z_1, \frac{Tz_1}{\|Tz_1\|} \right\} < \frac{1}{\sqrt{\eta_1}}. \tag{2}$$

For the inductive step, assume that for $n\geq 2$, $(z_i)_{i=1}^{n-1}\subset Y$, $(z_j^*)_{j\in J_i}$ $(i=1,\dots,n-1)$, and $(\eta_i)_{i=1}^{n-1}$ have been constructed. Let J_n be a finite index set with $J_{n-1}\subseteq J_n$ and $(x_j^*)_{j\in J_n}$ be a set of normalized functionals on X such that

$$\begin{aligned} &\text{for every } x\in \mathrm{span}\{T^i z_j : 1\leq j\leq n-1,\ 0\leq i\leq j\} \\ &\text{there exists } j_0\in J_n \text{ such that } |x_{j_0}^*(x)|\geq \tfrac{1}{2}\|x\|. \end{aligned} \tag{3}$$

Apply Theorem 3 for $F=\bigcap_{j\in J_n}\ker x_j^*$, and $k=n$, to obtain $\eta_n<1/n^2 2^{2n+4}$ and $z_n\in Y$, $\|z_n\|=1$, such that

$$T^{i-1} z_n \in F \text{ and } \|T^i z_n\| < \eta_n \|T^{i-1} z_n\| \quad \text{for } i=1,2,\dots,n+1, \tag{4}$$

and

$$\mathrm{bc}\left\{ z_n, \frac{Tz_n}{\|Tz_n\|}, \dots, \frac{T^n z_n}{\|T^n z_n\|} \right\} < \frac{1}{\sqrt{\eta_n}}. \tag{5}$$

This finishes the induction.

Let $\widetilde{Z}=\mathrm{span}\{T^i z_n : n\in\mathbf{N},\ 0\leq i\leq n\}$, and for $n\in\mathbf{N}$, let $Z_n=\mathrm{span}\{T^i z_n : 0\leq i\leq n\}$. Let $x\in\widetilde{Z}$ with $\|x\|=1$ and write $x=\sum_{n=1}^\infty x_n$, where $x_n\in Z_n$ for all $n\in\mathbf{N}$. We claim that

$$\|Tx_n\| < \frac{1}{2^n} \quad \text{for all } n\in\mathbf{N}. \tag{6}$$

Indeed, write

$$x=\sum_{n=1}^{\infty}\sum_{i=0}^{n}a_{i,n}\frac{T^i z_n}{\|T^i z_n\|}\quad\text{and}\quad x_n=\sum_{i=0}^{n}a_{i,n}\frac{T^i z_n}{\|T^i z_n\|}\ \text{for } n\in\mathbf{N}.$$

Fix $n\in\mathbf{N}$ and set $\tilde{x}_n=x_1+x_2+...+x_n$. Let $j_0\in J_{n+1}$ such that

$$\|\tilde{x}_n\|\leq 2|x_{j_0}^*(\tilde{x}_n)|=2|x_{j_0}^*(x)|\leq 2\|x_{j_0}^*\|\,\|x\|=2,$$

by (3), and since for $n+1\leq m$, $J_{n+1}\subseteq J_m$ and thus by (4), $x_m\in\ker x_{j_0}^*$. Thus $\|x_n\|=\|\tilde{x}_n-\tilde{x}_{n-1}\|\leq\|\tilde{x}_n\|+\|\tilde{x}_{n-1}\|\leq 4$ (where $\tilde{x}_0=0$). Hence, by (2) and (5) we obtain that

$$(7)\qquad |a_{i,n}|\leq 4\,\mathrm{bc}\left\{z_n,\frac{Tz_n}{\|Tz_n\|},...,\frac{T^n z_n}{\|T^n z_n\|}\right\}\leq\frac{4}{\sqrt{\eta_n}}\quad\text{for } i=0,...,n.$$

Therefore, using (1), (4), (7) and the choice of η_n,

$$\|Tx_n\|=\left\|\sum_{i=0}^{n}a_{i,n}\frac{T^{i+1}z_n}{\|T^i z_n\|}\right\|\leq\sum_{i=0}^{n}|a_{i,n}|\frac{\|T^{i+1}z_n\|}{\|T^i z_n\|}\leq\sum_{i=0}^{n}\frac{4}{\sqrt{\eta_n}}\eta_n=4n\sqrt{\eta_n}<\frac{1}{2^n},$$

which finishes the proof of (6). Let Z to be the closure of $\widetilde{Z}$. Notice that $Z\cap Y\supset\{z_n:n\in\mathbf{N}\}$, and thus $Z\cap Y$ is infinite dimensional. We claim that $T|_Z:Z\to X$ is a compact operator, which will finish the proof of Theorem 1. Indeed, let $(y_m)_{m\in\mathbf{N}}\subset\widetilde{Z}$, where for all $m\in\mathbf{N}$ we have $\|y_m\|=1$, and write $y_m=\sum_{n=1}^{\infty}y_{m,n}$, where $y_{m,n}\in Z_n$ for all $n\in\mathbf{N}$. It suffices to prove that $(Ty_m)_{m\in\mathbf{N}}$ has a Cauchy subsequence. Indeed, since Z_n is finite dimensional for all $n\in\mathbf{N}$, there exists $(y_m^1)_{m\in\mathbf{N}}$ a subsequence of $(y_m)_{m\in\mathbf{N}}$ such that $(Ty_{m,1}^1)_{m\in\mathbf{N}}$ is Cauchy (with the obvious notation that if $y_m^1=y_p$ for some p, then $y_{m,n}^1=y_{p,n}$). Let $(y_m^2)_{m\in\mathbf{N}}$ be a subsequence of $(y_m^1)_{m\in\mathbf{N}}$ such that $(Ty_{m,2}^2)_{m\in\mathbf{N}}$ is Cauchy (with the obvious notation that if $y_{m,n}^2=y_p$ for some p, then $y_{m,n}^2=y_{p,n}$). Continue similarly, and let $\tilde{y}_m=y_m^m$ and $\tilde{y}_{m,n}=y_{m,n}^m$ for all $m,n\in\mathbf{N}$. Then for $m\in\mathbf{N}$ we have $\tilde{y}_m=\sum_{n=1}^{\infty}\tilde{y}_{m,n}$, where $\tilde{y}_{m,n}\in Z_n$ for all $n\in\mathbf{N}$. Also, for all $n,m\in\mathbf{N}$ with $n\leq m$, $(\tilde{y}_t)_{t\geq m}$ and $(\tilde{y}_{t,n})_{t\geq m}$ are subsequences of $(y_t^m)_{t\in\mathbf{N}}$ and $(y_{t,n}^m)_{t\in\mathbf{N}}$, respectively. Thus for all $n\in\mathbf{N}$, $(T\tilde{y}_{t,n})_{t\in\mathbf{N}}$ is a Cauchy sequence. We claim that $(T\tilde{y}_m)_{t\in\mathbf{N}}$ is a Cauchy sequence. Indeed, for $\varepsilon>0$ let $m_0\in\mathbf{N}$ be such that $1/2^{m_0-1}<\varepsilon$ and let $m_1\in\mathbf{N}$ be such that

$$(8)\qquad \|T\tilde{y}_{s,n}-T\tilde{y}_{t,n}\|<\frac{\varepsilon}{2m_0}\quad\text{for all } s,t\geq m_1 \text{ and } n=1,2,...,m_0.$$

Thus for $s,t\geq m_1$ we have, using (6), (8) and the choice of m_0,

$$\begin{aligned}\|T\tilde{y}_s-T\tilde{y}_t\| &= \left\|\sum_{n=1}^{\infty} T\tilde{y}_{s,n}-T\tilde{y}_{t,n}\right\| \\ &\leq \sum_{n=1}^{m_0} \|T\tilde{y}_{s,n}-T\tilde{y}_{t,n}\| + \sum_{n=m_0+1}^{\infty} \|T\tilde{y}_{s,n}\| + \sum_{n=m_0+1}^{\infty} \|T\tilde{y}_{t,n}\| \\ &< m_0\frac{\varepsilon}{2m_0}+2\sum_{n=m_0+1}^{\infty}\frac{1}{2^n} \\ &= \frac{\varepsilon}{2}+\frac{2}{2^{m_0}} \\ &< \varepsilon\end{aligned}$$

which proves that $(T\tilde{y}_m)_{m\in\mathbf{N}}$ is a Cauchy sequence and finishes the proof of Theorem 1. □

For the proof of Theorem 3 we need the next two results.

Lemma 4. *Let T be a strictly singular one-to-one operator on an infinite dimensional Banach space X. Let $k\in\mathbf{N}$ and $\eta>0$. Then for every infinite dimensional subspace Y of X there exists an infinite dimensional subspace Z of Y such that for all $z\in Z$ and for all $i=1,\dots,k$ we have that*

$$\|T^i z\| \leq \eta\|T^{i-1}z\|$$

(where T^0 denotes the identity operator on X).

Proof. Let T be a strictly singular one-to-one operator on an infinite dimensional Banach space X, $k\in\mathbf{N}$ and $\eta>0$. We first prove the following claim.

Claim. *For every infinite dimensional linear submanifold (which is not assumed closed) W of X there exists an infinite dimensional linear submanifold Z of W such that $\|Tz\|\leq\eta\|z\|$ for all $z\in Z$.*

Indeed, since W is infinite dimensional there exists a normalized basic sequence $(z_i)_{i\in\mathbf{N}}$ in W having biorthogonal constant at most equal to 2, such that $\|Tz_i\|\leq \eta/2^{i+2}$ for all $i\in\mathbf{N}$. Let $Z=\operatorname{span}\{z_i:i\in\mathbf{N}\}$ be the linear span of the z_i's. Then Z is an infinite dimensional linear submanifold of W. We now show that Z satisfies the conclusion of the claim. Let $z\in Z$ and write z in the form $z=\Sigma_{i=1}^{\infty}\lambda_i z_i$ for some scalars $(\lambda_i)_{i\in\mathbf{N}}$ such that at most finitely many λ_i's are non-zero. Since the biorthogonal constant of $(z_i)_{i\in\mathbf{N}}$ is at most equal to 2, we have that $|\lambda_i|\leq 4\|z\|$ for all i. Thus

$$\|Tz\| = \left\|\sum_{i=1}^{\infty}\lambda_i T z_i\right\| \leq \sum_{i=1}^{\infty}|\lambda_i|\,\|Tz_i\| \leq \sum_{i=1}^{\infty} 4\|z\|\frac{\eta}{2^{i+2}} = \eta\|z\|$$

which finishes the proof of the claim.

Let Y be an infinite dimensional subspace of X. Inductively for $i=0,1,\dots,k$, we define Z_i, a linear submanifold of X, such that

(a) Z_0 is an infinite dimensional linear submanifold of Y and Z_i is an infinite dimensional linear ubmanifold of $T(Z_{i-1})$ for $i\geq 1$;

(b) $\|Tz\|\leq\eta\|z\|$ for all $z\in Z_i$ and for all $i\geq 0$.

Indeed, since Y is infinite dimensional, we obtain Z_0 by applying the above claim for $W=Y$. Obviously (a) and (b) are satisfied for $i=0$. Assume that for some $i_0\in\{0,1,\dots,k-1\}$, a linear submanifold Z_{i_0} of X has been constructed satisfying (a) and (b) for $i=i_0$. Since T is one-to-one and Z_{i_0} is infinite dimensional we have that $T(Z_{i_0})$ is an infinite dimensional linear submanifold of X and we obtain Z_{i_0+1} by applying the above claim for $W=T(Z_{i_0})$. Obviously (a) and (b) are satisfied for $i=i_0+1$. This finishes the inductive construction of the Z_i's. By (a) we obtain that Z_k is an infinite dimensional linear submanifold of $T^k(Y)$. Let $W=T^{-k}(Z_k)$. Then W is an infinite dimensional linear submanifold of X. Since $Z_k\subseteq T^k(Y)$ and T is one-to-one, we have that $W\subseteq Y$. By (a) we obtain that for $i=0,1,\dots,k$ we have $Z_k\subseteq T^{k-i}Z_i$, hence

$$T^iW=T^iT^{-k}Z_k=T^{-(k-i)}Z_k\subseteq T^{-(k-i)}T^{k-i}Z_i=Z_i$$

(since T is one-to-one). Thus by (b) we obtain that $\|T^iz\|\leq\eta\|T^{i-1}z\|$ for all $z\in W$ and $i=1,2,\dots,k$. Obviously, if Z is the closure of W then Z satisfies the statement of the lemma. □

Corollary 5. *Let T be a strictly singular one-to-one operator on an infinite dimensional Banach space X. Let $k\in\mathbf{N}$, $\eta>0$ and F be a finite codimensional subspace of X. Then for every infinite dimensional subspace Y of X there exists an infinite dimensional subspace Z of Y such that for all $z\in Z$ and for all $i=1,\dots,k+1$,*

$$T^{i-1}z\in F \quad \textit{and} \quad \|T^iz\|\leq\eta\|T^{i-1}z\|$$

(where T^0 denotes the identity operator on X).

Proof. For any linear submanifold W of X and for any finite codimensional subspace F of X we have that

$$\dim(W/(F\cap W))\leq\dim(X/F)<\infty. \tag{9}$$

Indeed for any $n>\dim X/F$ and for any linear independent vectors $x_1,\dots,x_n$ in $W\setminus(F\cap W)$ we have that there exist scalars $\lambda_1,\dots,\lambda_n$ with $(\lambda_1,\dots,\lambda_n)\neq(0,\dots,0)$ and $\sum_{i=1}^n\lambda_ix_i\in F$ (since $n>\dim(X/F)$). Thus $\sum_{i=1}^n\lambda_ix_i\in F\cap W$ which implies (9).

Let $R(T)$ denote the range of T. Apply (9) for $W=R(T)$ to obtain

$$\dim(R(T)/(R(T)\cap F))\le\dim(X/F)<\infty. \tag{10}$$

Since T is one-to-one we have that

$$\dim(X/T^{-1}(F))\le\dim(R(T)/(R(T)\cap F)). \tag{11}$$

Indeed, for any $n>\dim(R(T)/(R(T)\cap F))$ and for any linear independent vectors $x_1,\dots,x_n$ of $X\backslash T^{-1}(F)$, we have that $Tx_1,\dots,Tx_n$ are linear independent vectors of $R(T)\backslash T(T^{-1}(F))=R(T)\backslash F$ (since T is one-to-one). Thus $Tx_1,\dots,Tx_n\in R(T)\backslash(R(T)\cap F)$ and as $n>\dim(R(T)/(R(T)\cap F))$, there are scalars $\lambda_1,\dots,\lambda_n$ with $(\lambda_1,\dots,\lambda_n)\neq(0,\dots,0)$ such that $\sum_{i=1}^{n}\lambda_i Tx_i\in R(T)\cap F$. Therefore $T(\sum_{i=1}^{n}\lambda_i x_i)\in F$, and hence $\sum_{i=1}^{n}\lambda_i x_i\in T^{-1}(F)$, which proves (11). By combining (10) and (11) we obtain

$$\dim(X/T^{-1}(F))<\infty. \tag{12}$$

By (12) we have that

$$\dim(X/T^{-i}(F))<\infty \quad \text{for } i=1,2,\dots,k. \tag{13}$$

Thus $\dim(X/W_1)<\infty$, where $W_1=F\cap T^{-1}(F)\cap\dots\cap T^{-k}(F)$. Therefore if we apply (9) for $W=Y$ and $F=W_1$ we obtain

$$\dim(Y/(Y\cap W_1))\le\dim(X/W_1)<\infty, \tag{14}$$

and therefore $Y\cap W_1$ is infinite dimensional.

Now use Lemma 4, replacing Y by $Y\cap W_1$, to obtain an infinite dimensional subspace Z of $Y\cap W_1$ such that

$$\|T^i z\|\le\eta\|T^{i-1}z\| \quad \text{for all } z\in Z \text{ and } i=1,\dots,k+1.$$

Notice that for $z\in Z$ and $i=1,\dots,k$ we have that $z\in W_1$, and thus $T^{i-1}z\in F$. □

Now we are ready to prove Theorem 3.

Proof of Theorem 3. We prove by induction on k that for every infinite dimensional subspace Y of X, finite codimensional subspace F of X, $k\in\mathbf{N}$, function $f:(0,1)\to(0,1)$ such that $f(\eta)\searrow 0$ as $\eta\searrow 0$, and for $i_0\in\{0\}\cup\mathbf{N}$, there exists $\eta_0>0$ such that for every $0<\eta\le\eta_0$ there exists $x\in Y$, $\|x\|=1$, satisfying

(a′) $T^{i-1}x\in F$ and $\|T^i x\|\le\eta\|T^{i-1}x\|$ for $i=1,2,\dots,i_0+k+1$;

(b′)

$$\mathrm{bc}\left\{\frac{T^{i_0}x}{\|T^{i_0}x\|},\frac{T^{i_0+1}x}{\|T^{i_0+1}x\|},\dots,\frac{T^{i_0+k}x}{\|T^{i_0+k}x\|}\right\}\le\frac{1}{f(\eta)}.$$

For $k=1$ let Y, F, f, and i_0 be as above, and let $\eta_0\in(0,1)$ satisfy

$$f(\eta_0)<\tfrac{1}{62}. \tag{15}$$

Let $0<\eta\leq\eta_0$. Apply Corollary 5 for k and η replaced by i_0+1 and $\frac{1}{4}\eta$, respectively, to obtain an infinite dimensional subspace Z_1 of Y such that for all $z\in Z_1$ and for $i=1,2,\dots,i_0+2$,

$$T^{i-1}z\in F \quad\text{and}\quad \|T^iz\|\leq\tfrac{1}{4}\eta\|T^{i-1}z\|. \tag{16}$$

Let $x_1\in Z_1$ with $\|x_1\|=1$. If $\operatorname{bc}\{T^{i_0}x_1/\|T^{i_0}x_1\|,\ T^{i_0+1}x_1/\|T^{i_0+1}x_1\|\}\leq 1/f(\eta)$ then x_1 satisfies (a′) and (b′) for $k=1$, thus we may assume that

$$\operatorname{bc}\left\{\frac{T^{i_0}x_1}{\|T^{i_0}x_1\|},\frac{T^{i_0+1}x_1}{\|T^{i_0+1}x_1\|}\right\}>\frac{1}{f(\eta)}. \tag{17}$$

Let

$$0<\eta_2\leq\frac{\eta}{4}\wedge\min_{1\leq i\leq i_0}\frac{\|T^{i_0}x_1\|}{2\|T^ix_1\|}\wedge\min_{i_0<i\leq i_0+2}\frac{\|T^ix_1\|}{2\|T^{i_0}x_1\|}f(\eta). \tag{18}$$

Let $z_1^*,z_2^*\in X^*$, $\|z_1^*\|=\|z_2^*\|=1$, $z_1^*(T^{i_0}x_1)=\|T^{i_0}x_1\|$ and $z_2^*(T^{i_0+1}x_1)=\|T^{i_0+1}x_1\|$. Since $\ker z_1^*\cap\ker z_2^*$ is finite codimensional and T is one-to-one, by (13) we have that

$$\dim(X/T^{-i_0}(\ker z_1^*\cap\ker z_2^*))<\infty. \tag{19}$$

Apply Corollary 5 for F, k and η replaced by $F\cap T^{-i_0}(\ker z_1^*\cap\ker z_2^*)$, i_0+2 and η_2, respectively, to obtain an infinite dimensional subspace Z_2 of Y such that for all $z\in Z_2$ and for all $i=1,2,\dots,i_0+2$,

$$T^{i-1}z\in F\cap T^{-i_0}(\ker z_1^*\cap\ker z_2^*)\quad\text{and}\quad\|T^iz\|\leq\eta_2\|T^{i-1}z\|. \tag{20}$$

Let $x_1^*\in X^*$ with $\|x_1^*\|=x_1^*(x_1)=1$ and let $x_2\in Z_2\cap\ker x_1^*$ with

$$\|T^{i_0}x_1\|=\|T^{i_0}x_2\| \tag{21}$$

and let $x=(x_1+x_2)/\|x_1+x_2\|$. We will show that x satisfies (a′) and (b′) for $k=1$.

We first show that (a′) is satisfied for $k=1$. Since $x_1,Tx_1,\dots,T^{i_0+1}x_1\in F$ (by (16)) and $x_2,Tx_2,\dots,T^{i_0+1}x_2\in F$ (by (20)) we have that $x,Tx,\dots,T^{i_0+1}x\in F$. Before showing that the norm estimate of (a′) is satisfied, we need some preliminary estimates: (22)–(30).

If $1 \le i < i_0$ (assuming that $2 \le i_0$) then, by (18), (20) and (21),

$$\|T^i x_1\| = \frac{\|T^{i_0} x_1\|}{2\dfrac{\|T^{i_0} x_1\|}{\|T^i x_1\|}} \le \frac{\|T^{i_0} x_1\|}{2\eta_2} = \frac{\|T^{i_0} x_2\|}{2\eta_2} \le \frac{\eta_2^{i_0-i}\|T^i x_2\|}{2\eta_2} \le \frac{\|T^i x_2\|}{2} \tag{22}$$

Thus, by (22), for $1 \le i < i_0$ (assuming that $2 \le i_0$) we have

$$\|T^i x\|\,\|x_1+x_2\| = \|T^i x_1 + T^i x_2\| \le \|T^i x_1\| + \|T^i x_2\| \le \tfrac{3}{2}\|T^i x_2\| \tag{23}$$

and

$$\|T^i x\|\,\|x_1+x_2\| = \|T^i x_1 + T^i x_2\| \ge \|T^i x_2\| - \|T^i x_1\| \ge \tfrac{1}{2}\|T^i x_2\|. \tag{24}$$

Also notice that, by (21),

$$\|T^{i_0} x\|\,\|x_1+x_2\| = \|T^{i_0} x_1 + T^{i_0} x_2\| \le \|T^{i_0} x_1\| + \|T^{i_0} x_2\| = 2\|T^{i_0} x_1\| \tag{25}$$

and

$$\begin{aligned} \|T^{i_0} x\|\,\|x_1+x_2\| &= \|T^{i_0} x_1 + T^{i_0} x_2\| \ge z_1^*(T^{i_0} x_1 + T^{i_0} x_2) \\ &= z_1^*(T^{i_0} x_1) = \|T^{i_0} x_1\| \end{aligned} \tag{26}$$

(by (20) for $z = x_2$ and $i = 1$). Also for $i_0 < i \le i_0 + 2$ we have that by applying (20) for $z = x_2$, $i - i_0$ times, we obtain, using (18) and (21) and the fact that $\eta_2 < 1$,

$$\begin{aligned} \|T^i x_2\| &\le \eta_2^{i-i_0}\|T^{i_0} x_2\| \le \eta_2 \|T^{i_0} x_1\| \\ &= \eta_2 \frac{2\|T^{i_0} x_1\|}{\|T^i x_1\|}\frac{1}{2}\|T^i x_1\| \le \frac{1}{2} f(\eta)\|T^i x_1\| \le \frac{1}{2}\|T^i x_1\|. \end{aligned} \tag{27}$$

Thus for $i_0 < i \le i_0 + 2$ we have

$$\|T^i x\|\,\|x_1+x_2\| = \|T^i x_1 + T^i x_2\| \le \|T^i x_1\| + \|T^i x_2\| \le \tfrac{3}{2}\|T^i x_1\|. \tag{28}$$

Also for $i_0 < i \le i_0 + 2$ we have by (27),

$$\|T^i x\|\,\|x_1+x_2\| = \|T^i x_1 + T^i x_2\| \ge \|T^i x_1\| - \|T^i x_2\| \ge \tfrac{1}{2}\|T^i x_1\|. \tag{29}$$

Later in the course of this proof we will also need that, using (27) and the fact that $f(\eta) < 1$,

$$\begin{aligned} \|T^{i_0+1} x\|\,\|x_1+x_2\| &= \|T^{i_0+1} x_1 + T^{i_0+1} x_2\| \\ &\ge \|T^{i_0+1} x_1\| - \|T^{i_0+1} x_2\| \\ &\ge \frac{2}{f(\eta)}\|T^{i_0+1} x_2\| - \|T^{i_0+1} x_2\| \\ &= \frac{2-f(\eta)}{f(\eta)}\|T^{i_0+1} x_2\| \\ &\ge \frac{1}{f(\eta)}\|T^{i_0+1} x_2\|. \end{aligned} \tag{30}$$

Finally we will show that for $1 \le i \le i_0+2$ we have $\|T^i x\| \le \eta \|T^{i-1}x\|$. Indeed if $i=1$ then, using (16), (18), (20) and the facts that $\|x_1\|=1=x_1^*(x_1+x_2)$ and $\|x_1^*\|=1$,

$$\|T^i x\| = \frac{\|Tx_1+Tx_2\|}{\|x_1+x_2\|} \le \frac{\|Tx_1\|+\|Tx_2\|}{\|x_1+x_2\|} \le \frac{\frac{1}{4}\eta\|x_1\|+\eta_2\|x_2\|}{\|x_1+x_2\|}$$

$$\le \frac{\frac{1}{4}\eta\|x_1\|+\eta_2(\|x_1+x_2\|+\|x_1\|)}{\|x_1+x_2\|} = \frac{\left(\frac{1}{4}\eta+\eta_2\right)x_1^*(x_1+x_2)}{\|x_1+x_2\|}+\eta_2 \le \frac{\eta}{4}+2\eta_2 \le \eta. \tag{31}$$

If $1<i<i_0$ (assuming that $3 \le i_0$) we have that, by (18), (20), (23) and (24),

$$\frac{\|T^i x\|}{\|T^{i-1}x\|} \le \frac{\frac{3}{2}\|T^i x_2\|}{\frac{1}{2}\|T^{i-1}x_2\|} < 3\eta_2 < \eta. \tag{32}$$

If $i=i_0>1$ then, by (18), (20), (21), (24) and (25),

$$\frac{\|T^i x\|}{\|T^{i-1}x\|} \le \frac{2\|T^{i_0}x_1\|}{\frac{1}{2}\|T^{i_0-1}x_2\|} = 4\frac{\|T^{i_0}x_2\|}{\|T^{i_0-1}x_2\|} < 4\eta_2 < \eta. \tag{33}$$

If $i_0<i\le i_0+2$ then, using (16), (28) and (29),

$$\frac{\|T^i x\|}{\|T^{i-1}x\|} \le \frac{\frac{3}{2}\|T^i x_1\|}{\frac{1}{2}\|T^{i-1}x_1\|} < \eta. \tag{34}$$

Now (31)–(34) yield that for $1 \le i \le i_0+2$ we have $\|T^i x\| \le \eta\|T^{i-1}x\|$. Thus x satisfies (a′) for $k=1$. Before proving that x satisfies (b′) for $k=1$ we need some preliminary estimates: (35)–(40). By (17) there exist scalars a_0 and a_1 with $\max\{|a_0|,|a_1|\}=1$ and $\|w\|<f(\eta)$, where

$$w = a_0\frac{T^{i_0}x_1}{\|T^{i_0}x_1\|}+a_1\frac{T^{i_0+1}x_1}{\|T^{i_0+1}x_1\|}. \tag{35}$$

Therefore

$$\big||a_0|-|a_1|\big| = \left|\left\|a_0\frac{T^{i_0}x_1}{\|T^{i_0}x_1\|}\right\|-\left\|a_1\frac{T^{i_0+1}x_1}{\|T^{i_0+1}x_1\|}\right\|\right| \le \|w\| < f(\eta). \tag{36}$$

Thus $1-f(\eta) \le |a_0|, |a_1| \le 1$ and hence

$$\frac{|a_1|}{|a_0|} \le \frac{1}{|a_0|} \le \frac{1}{1-f(\eta)}. \tag{37}$$

Also by (35) we obtain that

$$T^{i_0}x_1 = \frac{\|T^{i_0}x_1\|}{a_0} w - \|T^{i_0}x_1\| \frac{a_1}{a_0} \frac{T^{i_0+1}x_1}{\|T^{i_0+1}x_1\|}$$

and thus

$$T^{i_0}x = \frac{1}{\|x_1+x_2\|}\left(\frac{\|T^{i_0}x_1\|}{a_0} w - \|T^{i_0}x_1\| \frac{a_1}{a_0} \frac{T^{i_0+1}x_1}{\|T^{i_0+1}x_1\|} + T^{i_0}x_2\right). \tag{38}$$

Let

$$\widetilde{w} = T^{i_0}x + \frac{\|T^{i_0}x_1\|}{\|x_1+x_2\|} \frac{a_1}{a_0} \frac{T^{i_0+1}x_1}{\|T^{i_0+1}x_1\|} - \frac{T^{i_0}x_2}{\|x_1+x_2\|}. \tag{39}$$

Notice that (38) and (39) imply that $\widetilde{w} = (\|T^{i_0}x_1\|/\|x_1+x_2\|a_0)w$ and hence by (15), (20), (37), the choice of z_1^* and the fact that $\|w\| < f(\eta)$,

$$\begin{aligned}\|\widetilde{w}\| &= \frac{\|T^{i_0}x_1\|}{\|x_1+x_2\|\,|a_0|}\|w\| \le \frac{\|T^{i_0}x_1\|}{\|x_1+x_2\|}\frac{f(\eta)}{1-f(\eta)} \le 2f(\eta)\frac{\|T^{i_0}x_1\|}{\|x_1+x_2\|} = 2f(\eta)\frac{z_1^*(T^{i_0}x_1)}{\|x_1+x_2\|} \\ &= 2f(\eta)\frac{z_1^*(T^{i_0}x_1+T^{i_0}x_2)}{\|x_1+x_2\|} \le 2f(\eta)\frac{\|T^{i_0}(x_1+x_2)\|}{\|x_1+x_2\|} = 2f(\eta)\|T^{i_0}x\|. \end{aligned} \tag{40}$$

Now we are ready to estimate $\mathrm{bc}\{T^{i_0}x/\|T^{i_0}x\|, T^{i_0+1}x/\|T^{i_0+1}x\|\}$. Let the scalars A_0 and A_1 be such that

$$\left\| A_0 \frac{T^{i_0}x}{\|T^{i_0}x\|} + A_1 \frac{T^{i_0+1}x}{\|T^{i_0+1}x\|}\right\| = 1.$$

We want to estimate $\max\{|A_0|, |A_1|\}$. By (30), (39), (40) and the triangle inequality we have

$$\begin{aligned} 1 &= \left\| \frac{A_0}{\|T^{i_0}x\|}\left(\widetilde{w} - \frac{\|T^{i_0}x_1\|}{\|x_1+x_2\|}\frac{a_1}{a_0}\frac{T^{i_0+1}x_1}{\|T^{i_0+1}x_1\|} + \frac{T^{i_0}x_2}{\|x_1+x_2\|}\right) + A_1 \frac{T^{i_0+1}x}{\|T^{i_0+1}x\|}\right\| \\ &= \left\| \frac{A_0\|T^{i_0}x_2\|}{\|T^{i_0}x\|\,\|x_1+x_2\|}\frac{T^{i_0}x_2}{\|T^{i_0}x_2\|} \right. \\ &\quad + \left(\frac{-A_0\|T^{i_0}x_1\|}{\|T^{i_0}x\|\,\|x_1+x_2\|}\frac{a_1}{a_0} + \frac{A_1\|T^{i_0+1}x_1\|}{\|T^{i_0+1}x\|\,\|x_1+x_2\|}\right)\frac{T^{i_0+1}x_1}{\|T^{i_0+1}x_1\|} \\ &\quad \left. + \frac{A_0}{\|T^{i_0}x\|}\widetilde{w} + \frac{A_1T^{i_0+1}x_2}{\|T^{i_0+1}x\|\,\|x_1+x_2\|}\right\| \\ &\ge \left\| \frac{A_0\|T^{i_0}x_2\|}{\|T^{i_0}x\|\,\|x_1+x_2\|}\frac{T^{i_0}x_2}{\|T^{i_0}x_2\|} \right. \\ &\quad \left. + \left(\frac{-A_0\|T^{i_0}x_1\|}{\|T^{i_0}x\|\,\|x_1+x_2\|}\frac{a_1}{a_0} + \frac{A_1\|T^{i_0+1}x_1\|}{\|T^{i_0+1}x\|\,\|x_1+x_2\|}\right)\frac{T^{i_0+1}x_1}{\|T^{i_0+1}x_1\|}\right\| \\ &\quad - 2f(\eta)|A_0| - f(\eta)|A_1|. \end{aligned} \tag{41}$$

By (20) for $i=1$ we have that $T^{i_0}x_2\in\ker z_2^*$ and since $z_2^*(T^{i_0+1}x_1)=\|T^{i_0+1}x_1\|$ it is easy to see that $\mathrm{bc}\{T^{i_0}x_2/\|T^{i_0}x_2\|,\ T^{i_0+1}x_1/\|T^{i_0+1}x_1\|\}\le 2$. Thus (41) implies that

$$\left|-\frac{A_0\|T^{i_0}x_1\|}{\|T^{i_0}x\|\,\|x_1+x_2\|}\frac{a_1}{a_0}+\frac{A_1\|T^{i_0+1}x_1\|}{\|T^{i_0+1}x\|\,\|x_1+x_2\|}\right|\le 2+4f(\eta)|A_0|+2f(\eta)|A_1| \tag{42}$$

and

$$\frac{|A_0|\,\|T^{i_0}x_2\|}{\|T^{i_0}x\|\,\|x_1+x_2\|}\le 2+4f(\eta)|A_0|+2f(\eta)|A_1|. \tag{43}$$

Notice that (43) implies that

$$|A_0|\le 4+8f(\eta)|A_0|+4f(\eta)|A_1|, \tag{44}$$

since

$$\frac{\|T^{i_0}x\|\,\|x_1+x_2\|}{\|T^{i_0}x_2\|}=\frac{\|T^{i_0}x_1+T^{i_0}x_2\|}{\|T^{i_0}x_2\|}\le\frac{\|T^{i_0}x_1\|+\|T^{i_0}x_2\|}{\|T^{i_0}x_2\|}=2,$$

by (21). Also by (42) we obtain

$$\frac{|A_1|\,\|T^{i_0+1}x_1\|}{\|T^{i_0+1}x\|\,\|x_1+x_2\|}-\frac{|A_0|\,\|T^{i_0}x_1\|}{\|T^{i_0}x\|\,\|x_1+x_2\|}\frac{|a_1|}{|a_0|}\le 2+4f(\eta)|A_0|+2f(\eta)|A_1|.$$

Thus

$$\frac{2}{3}|A_1|-\frac{1}{1-f(\eta)}|A_0|\le 2+4f(\eta)|A_0|+2f(\eta)|A_1| \tag{45}$$

by (28) for $i=i_0+1$, (37) and

$$\frac{\|T^{i_0}x_1\|}{\|T^{i_0}x\|\,\|x_1+x_2\|}=\frac{\|T^{i_0}x_1\|}{\|T^{i_0}x_1+T^{i_0}x_2\|}\le\frac{\|T^{i_0}x_1\|}{z_1^*(T^{i_0}x_1+T^{i_0}x_2)}=\frac{\|T^{i_0}x_1\|}{z_1^*(T^{i_0}x_1)}=1,$$

which hold by (20) and the choice of z_1^*. Notice that (45) implies that

$$|A_1|\le 6+\tfrac{28}{5}|A_0|, \tag{46}$$

since $f(\eta)<\frac{1}{6}$ by (15). By substituting (46) into (44) we obtain

$$|A_0|\le 4+8f(\eta)|A_0|+4f(\eta)\big(6+\tfrac{28}{5}|A_0|\big)=4+24f(\eta)+\tfrac{112}{5}f(\eta)|A_0|\le 5+\tfrac{1}{2}|A_0|,$$

since $f(\eta)<\frac{5}{224}$ by (15). Thus $|A_0|\leq 10$. Hence (46) gives that $|A_1|\leq 62$. Therefore, by (15),

$$\mathrm{bc}\left\{\frac{T^{i_0}x}{\|T^{i_0}x\|},\frac{T^{i_0+1}x}{\|T^{i_0+1}x\|}\right\}\leq 62\leq\frac{1}{f(\eta)}.$$

We now proceed to the inductive step. Assuming the inductive statement for some integer k, let F be a finite codimensional subspace of X, $f\colon(0,1)\to(0,1)$ with $f(\eta)\searrow 0$, as $\eta\searrow 0$, and $i_0\in\mathbf{N}\cup\{0\}$. By the inductive statement for i_0, f and η replaced by i_0+1, $f^{1/4}$ and $\frac{1}{4}\eta$, respectively, there exists η_1 such that for $0<\eta<\eta_1$ there exists $x_1\in X$, $\|x_1\|=1$, such that

$$T^{i-1}x_1\in F \quad\text{and}\quad \|T^ix_1\|\leq\tfrac{1}{4}\eta\|T^{i-1}x_1\| \quad\text{for } i=1,2,\dots,(i_0+1)+k+1 \tag{47}$$

and

$$\mathrm{bc}\left\{\frac{T^{i_0+1}x_1}{\|T^{i_0+1}x_1\|},\frac{T^{i_0+2}x_1}{\|T^{i_0+2}x_1\|},\dots,\frac{T^{i_0+1+k}x_1}{\|T^{i_0+1+k}x_1\|}\right\}\leq\frac{1}{f(\eta)^{1/4}}. \tag{48}$$

Let η_0 satisfy

$$\eta_0<\eta_1,\quad f(\eta_0)<\frac{1}{288^2}\quad\text{and}\quad f(\eta_0)<\left(\frac{1}{144(k+1)}\right)^2, \tag{49}$$

let $0<\eta<\eta_0$ and let $x_1\in X$, $\|x_1\|=1$, satisfy (47) and (48). If

$$\mathrm{bc}\left\{\frac{T^{i_0}x_1}{\|T^{i_0}x_1\|},\frac{T^{i_0+1}x_1}{\|T^{i_0+1}x_1\|},\dots,\frac{T^{i_0+k+1}x_1}{\|T^{i_0+k+1}x_1\|}\right\}\leq\frac{1}{f(\eta)}$$

then x_1 satisfies the inductive statement for k replaced by $k+1$. Thus we may assume that

$$\mathrm{bc}\left\{\frac{T^{i_0}x_1}{\|T^{i_0}x_1\|},\frac{T^{i_0+1}x_1}{\|T^{i_0+1}x_1\|},\dots,\frac{T^{i_0+k+1}x_1}{\|T^{i_0+k+1}x_1\|}\right\}>\frac{1}{f(\eta)}. \tag{50}$$

Let

$$0<\eta_2<\frac{\eta}{4}\wedge\min_{1\leq i\leq i_0}\frac{\|T^{i_0}x_1\|}{2\|T^ix_1\|}\wedge\min_{i_0<i\leq i_0+k+1}\frac{\|T^ix_1\|}{2\|T^{i_0}x_1\|}f(\eta). \tag{51}$$

Let $J\subset\{2,3,\dots\}$ be a finite index set and $z_1^*,(z_j^*)_{j\in J}$ be norm one functionals such that

$$z_1^*(T^{i_0}x_1)=\|T^{i_0}x_1\|, \tag{52}$$

and

(53) for $z\in\operatorname{span}\{T^{i_0+1}x_1,\dots,T^{i_0+k+1}x_1\}$ there is $j_0\in J$ with $|z^*_{j_0}(z)|\geq\frac{1}{2}\|z\|$.

Since T is one-to-one we obtain by (13) that $\dim(X/T^{-i_0}(\bigcap_{j\in\{1\}\cup J}\ker z^*_j))<\infty$. Apply Corollary 5 for F, k, η replaced by $F\cap T^{-i_0}(\bigcap_{j\in\{1\}\cup J}\ker z^*_i)$, i_0+k+2 and η_2, respectively, to obtain an infinite dimensional subspace Z of Y such that for all $z\in Z$ and for all $i=1,2,\dots,i_0+k+2$,

$$T^{i-1}z\in F\cap T^{-i_0}\Bigl(\bigcap_{j\in\{1\}\cup J}\ker z^*_j\Bigr)\quad\text{and}\quad \|T^iz\|\leq\eta_2\|T^{i-1}z\|. \tag{54}$$

Let $x^*_1\in X^*$, $\|x^*_1\|=1=x^*_1(x_1)$, let $x_2\in Z\cap\ker x^*_1$ with

$$\|T^{i_0}x_1\|=\|T^{i_0}x_2\| \tag{55}$$

and let $x=(x_1+x_2)/\|x_1+x_2\|$. We will show that x satisfies the inductive statement for k replaced by $k+1$.

We first show that x satisfies (a′) for k replaced by $k+1$. The proof is identical to the verification of (a′) for $k=1$. The formulas (27), (28), (29) and (34) are valid for $i_0<i\leq i_0+k+2$, and (30) is valid if i_0+1 is replaced by any $i\in\{i_0+1,\dots,i_0+k+1\}$, and *this will be assumed in the rest of the proof when we refer to these formulas.*

We now prove that (b′) is satisfied for k replaced by $k+1$. By (50) there exist scalars a_0, $a_1,\dots,a_{k+1}$ with $\max\{|a_0|,|a_1|,\dots,|a_{k+1}|\}=1$ and $\|w\|<f(\eta)$, where

$$w=\sum_{i=0}^{k+1}a_i\frac{T^{i_0+i}x_1}{\|T^{i_0+i}x_1\|}. \tag{56}$$

We claim that

$$|a_0|\geq\tfrac{1}{2}f(\eta)^{1/4}. \tag{57}$$

Indeed, if $|a_0|<\frac{1}{2}f(\eta)^{1/4}$ then $\max\{|a_1|,\dots,|a_{k+1}|\}=1$ and

$$\Bigl\|\sum_{i=1}^{k+1}a_i\frac{T^{i_0+1}x_1}{\|T^{i_0+i}x_1\|}\Bigr\|=\Bigl\|w-a_0\frac{T^{i_0}x_1}{\|T^{i_0}x_1\|}\Bigr\|\leq\|w\|+|a_0|<f(\eta)+\tfrac{1}{2}f(\eta)^{1/4}<f(\eta)^{1/4},$$

since $f(\eta)<\frac{1}{4}$ by (49), which contradicts (48). Thus (57) is proved. By (56) we obtain

$$T^{i_0}x_1=\frac{\|T^{i_0}x_1\|}{a_0}w-\sum_{i=1}^{k+1}\frac{a_i}{a_0}\|T^{i_0}x_1\|\frac{T^{i_0+i}x_1}{\|T^{i_0+i}x_1\|}$$

and thus

$$T^{i_0}x=\frac{1}{\|x_1+x_2\|}\left(\frac{\|T^{i_0}x_1\|}{a_0}w-\sum_{i=1}^{k+1}\frac{a_i}{a_0}\|T^{i_0}x_1\|\frac{T^{i_0+i}x_1}{\|T^{i_0+i}x_1\|}+T^{i_0}x_2\right). \tag{58}$$

Let

$$\widetilde{w}=T^{i_0}x+\sum_{i=1}^{k+1}\frac{a_i}{a_0}\frac{\|T^{i_0}x_1\|}{\|x_1+x_2\|}\frac{T^{i_0+i}x_1}{\|T^{i_0+i}x_1\|}-\frac{T^{i_0}x_2}{\|x_1+x_2\|}. \tag{59}$$

Notice that (58) and (59) imply that $\widetilde{w}=(\|T^{i_0}x_1\|/(\|x_1+x_2\|a_0))w$ and hence, using (52), (54), (57) and the facts that $\|w\|\le f(\eta)$ and $\|z_1^*\|=1$,

$$\begin{aligned}\|\widetilde{w}\| &= \frac{\|T^{i_0}x_1\|}{\|x_1+x_2\|\,|a_0|}\|w\| < \frac{2f(\eta)^{3/4}\|T^{i_0}x_1\|}{\|x_1+x_2\|} = \frac{2f(\eta)^{3/4}z_1^*(T^{i_0}x_1)}{\|x_1+x_2\|}\\ &= \frac{2f(\eta)^{3/4}z_1^*(T^{i_0}x_1+T^{i_0}x_2)}{\|x_1+x_2\|} \le \frac{2f(\eta)^{3/4}\|T^{i_0}(x_1+x_2)\|}{\|x_1+x_2\|} = 2f(\eta)^{3/4}\|T^{i_0}x\|. \end{aligned} \tag{60}$$

Now we are ready to estimate

$$\mathrm{bc}\left\{\frac{T^{i_0}x_1}{\|T^{i_0}x_1\|},\dots,\frac{T^{i_0+k+1}x_1}{\|T^{i_0+k+1}x_1\|}\right\}.$$

Let the scalars A_0, $A_1,\dots,A_{k+1}$ be such that

$$\left\|\sum_{i=0}^{k+1}A_i\frac{T^{i_0+i}x}{\|T^{i_0+i}x\|}\right\|=1.$$

We want to estimate the $\max\{|A_0|,|A_1|,\dots,|A_{k+1}|\}$. By (30), (59), (60) and recalling the paragraph before (56) we have

$$\begin{aligned}1 &= \left\|\frac{A_0}{\|T^{i_0}x\|}\left(\widetilde{w}-\sum_{i=1}^{k+1}\frac{a_i}{a_0}\frac{\|T^{i_0}x_1\|}{\|x_1+x_2\|}\frac{T^{i_0+i}x_1}{\|T^{i_0+i}x_1\|}+\frac{T^{i_0}x_2}{\|x_1+x_2\|}\right)+\sum_{i=1}^{k+1}A_i\frac{T^{i_0+i}x}{\|T^{i_0+i}x\|}\right\|\\ &= \left\|\frac{A_0\|T^{i_0}x_2\|}{\|T^{i_0}x\|\,\|x_1+x_2\|}\frac{T^{i_0}x_2}{\|T^{i_0}x_2\|}\right.\\ &\quad+\sum_{i=1}^{k+1}\left(\frac{a_i}{a_0}\frac{-A_0\|T^{i_0}x_1\|}{\|T^{i_0}x\|\,\|x_1+x_2\|}+\frac{A_i\|T^{i_0+i}x_1\|}{\|T^{i_0+i}x\|\,\|x_1+x_2\|}\right)\frac{T^{i_0+i}x_1}{\|T^{i_0+i}x_1\|}\\ &\quad\left.+\frac{A_0}{\|T^{i_0}x\|}\widetilde{w}+\sum_{i=1}^{k+1}A_i\frac{T^{i_0+i}x_2}{\|T^{i_0+i}x\|\,\|x_1+x_2\|}\right\| \end{aligned} \tag{61}$$

$$\geq \left\| \frac{A_0\|T^{i_0}x_2\|}{\|T^{i_0}x\|\,\|x_1+x_2\|}\frac{T^{i_0}x_2}{\|T^{i_0}x_2\|} + \sum_{i=1}^{k+1}\left(\frac{a_i}{a_0}\frac{-A_0\|T^{i_0}x_1\|}{\|T^{i_0}x\|\,\|x_1+x_2\|}+\frac{A_i\|T^{i_0+i}x_1\|}{\|T^{i_0+i}x\|\,\|x_1+x_2\|}\right)\frac{T^{i_0+i}x_1}{\|T^{i_0+i}x_1\|}\right\| - 2f(\eta)^{3/4}|A_0| - \sum_{i=1}^{k+1} f(\eta)|A_i|.$$

By (54) for $i=1$ and $z=x_2$ we obtain that $T^{i_0}x_2\in\bigcap_{j\in J}\ker z_j^*$ and by (53) and (48) it is easy to see that

$$\mathrm{bc}\left\{\frac{T^{i_0}x_2}{\|T^{i_0}x_2\|},\frac{T^{i_0+1}x_1}{\|T^{i_0+1}x_1\|},\dots,\frac{T^{i_0+k+1}x_1}{\|T^{i_0+k+1}x_1\|}\right\}\leq\frac{2}{f(\eta)^{1/4}}\vee 3.$$

Since $f(\eta)<\left(\frac{2}{3}\right)^4$ (by (49)), we have that $3\leq 2/f(\eta)^{1/4}$. Hence

$$\mathrm{bc}\left\{\frac{T^{i_0}x_2}{\|T^{i_0}x_2\|},\frac{T^{i_0+1}x_2}{\|T^{i_0+1}x_2\|},\dots,\frac{T^{i_0+k+1}x_1}{\|T^{i_0+k+1}x_1\|}\right\}\leq\frac{2}{f(\eta)^{1/4}}.$$

Thus (61) implies that

$$|A_0|\frac{\|T^{i_0}x_2\|}{\|T^{i_0}x\|\,\|x_1+x_2\|}\leq\frac{2}{f(\eta)^{1/4}}\left(1+2f(\eta)^{3/4}|A_0|+\sum_{j=1}^{k+1}f(\eta)|A_j|\right), \tag{62}$$

and for $i=1,\dots,k+1$,

$$\left|\frac{a_i}{a_0}\frac{-A_0\|T^{i_0}x_1\|}{\|T^{i_0}x\|\,\|x_1+x_2\|}+\frac{A_i\|T^{i_0+i}x_1\|}{\|T^{i_0+i}x\|\,\|x_1+x_2\|}\right| \leq\frac{2}{f(\eta)^{1/4}}\left(1+2f(\eta)^{3/4}|A_0|+\sum_{j=1}^{k+1}f(\eta)|A_j|\right). \tag{63}$$

Since, by (55),

$$\frac{\|T^{i_0}x\|\,\|x_1+x_2\|}{\|T^{i_0}x_2\|}=\frac{\|T^{i_0}x_1+T^{i_0}x_2\|}{\|T^{i_0}x_2\|}\leq\frac{\|T^{i_0}x_1\|+\|T^{i_0}x_2\|}{\|T^{i_0}x_2\|}=2$$

we have that (62) implies

$$|A_0|\leq\frac{4}{f(\eta)^{1/4}}+8f(\eta)^{1/2}|A_0|+4\sum_{j=1}^{k+1}f(\eta)^{3/4}|A_j|. \tag{64}$$

Notice also that (63) implies that for $i=1,\dots,k+1$,

$$|A_i|\frac{\|T^{i_0+i}x_1\|}{\|T^{i_0+i}x\|\,\|x_1+x_2\|}-|A_0|\frac{|a_i|}{|a_0|}\frac{\|T^{i_0}x_1\|}{\|T^{i_0}x\|\,\|x_1+x_2\|}\le\frac{2}{f(\eta)^{1/4}}+4f(\eta)^{1/2}|A_0|$$
$$+2\sum_{j=1}^{k+1}f(\eta)^{3/4}|A_j|.$$

Thus

$$\frac{2}{3}|A_i|-\frac{2}{f(\eta)^{1/4}}|A_0|\le\frac{2}{f(\eta)^{1/4}}+4f(\eta)^{1/2}|A_0|+2\sum_{j=1}^{k+1}f(\eta)^{3/4}|A_j| \tag{65}$$

by (28) (see the paragraph above (56)), (57) and

$$\frac{\|T^{i_0}x_1\|}{\|T^{i_0}x\|\,\|x_1+x_2\|}=\frac{\|T^{i_0}x_1\|}{\|T^{i_0}x_1+T^{i_0}x_2\|}\le\frac{\|T^{i_0}x_1\|}{|z_1^*(T^{i_0}x_1+T^{i_0}x_2)|}$$
$$=\frac{\|T^{i_0}x_1\|}{|z_1^*(T^{i_0}x_1)|}=1,$$

which follows from (52), (54) and the fact that $\|z_1^*\|=1$. For $i=1,\dots,k+1$ rewrite (65) as

$$\left(\frac{2}{3}-2f(\eta)^{3/4}\right)|A_i|\le\frac{2}{f(\eta)^{1/4}}+\left(4f(\eta)^{1/2}+\frac{2}{f(\eta)^{1/4}}\right)|A_0|+\sum_{\substack{j=1\\ j\neq i}}^{k+1}f(\eta)^{3/4}|A_j|.$$

Thus, since $f(\eta)<\left(\frac{1}{6}\right)^{4/3}\wedge\left(\frac{1}{4}\right)^{1/2}$ (by (49)), we obtain

$$\frac{1}{3}|A_i|\le\frac{2}{f(\eta)^{1/4}}+\left(1+\frac{2}{f(\eta)^{1/4}}\right)|A_0|+\sum_{\substack{j=1\\ j\neq i}}^{k+1}f(\eta)^{3/4}|A_j|.$$

Hence, since $1\le 1/f(\eta)^{1/4}$, we obtain that for $i=1,\dots,k+1$,

$$|A_i|\le\frac{6}{f(\eta)^{1/4}}+\frac{9}{f(\eta)^{1/4}}|A_0|+3\sum_{\substack{j=1\\ j\neq i}}^{k+1}f(\eta)^{3/4}|A_j|. \tag{66}$$

By substituting (64) in (66) we obtain that for $i=1,\dots,k+1$,

$$|A_i|\le\frac{6}{f(\eta)^{1/4}}+\frac{36}{f(\eta)^{1/2}}+72f(\eta)^{1/4}|A_0|+36\sum_{j=1}^{k+1}f(\eta)^{1/2}|A_j|+3\sum_{\substack{j=1\\ j\neq i}}^{k+1}f(\eta)^{3/4}|A_j|. \tag{67}$$

We claim that (64) and (67) imply that $\max\{|A_i|:0\le i\le k+1\}\le 1/f(\eta)$ which finishes the proof. Indeed, if $\max\{|A_i|:0\le i\le k+1\}=|A_0|$ then (64) implies that

$$|A_0|\le \frac{4}{f(\eta)^{1/4}}+8f(\eta)^{1/2}|A_0|+4(k+1)f(\eta)^{3/4}|A_0|\le \frac{4}{f(\eta)^{1/4}}+\frac{1}{3}|A_0|+\frac{1}{3}|A_0|$$

since

$$f(\eta)<\left(\frac{1}{24}\right)^2\wedge\left(\frac{1}{12(k+1)}\right)^{4/3}$$

by (49). Thus

$$|A_0|\le\frac{12}{f(\eta)^{1/4}}<\frac{1}{f(\eta)} \tag{68}$$

since $f(\eta)<\left(\frac{1}{12}\right)^{4/3}$ by (49). Similarly, if there exists $l\in\{1,\dots,k+1\}$ such that $\max\{|A_i|:0\le i\le k+1\}=|A_l|$ then (67) for $i=l$ implies that

$$\begin{aligned}|A_l|&\le\frac{6}{f(\eta)^{1/4}}+\frac{36}{f(\eta)^{1/2}}+72f(\eta)^{1/4}|A_l|+36(k+1)f(\eta)^{1/2}|A_l|+3kf(\eta)^{3/4}|A_l|\\&\le\frac{42}{f(\eta)^{1/2}}+\frac{1}{4}|A_l|+\frac{1}{4}|A_l|+\frac{1}{4}|A_l|\end{aligned}$$

since $1/f(\eta)^{1/4}\le 1/f(\eta)^{1/2}$ and $f(\eta)<\left(\frac{1}{288}\right)^4\wedge(1/144(k+1))^2$ by (49). Hence

$$|A_l|\le\frac{168}{f(\eta)^{1/2}}\le\frac{1}{f(\eta)}, \tag{69}$$

since $f(\eta)<\left(\frac{1}{168}\right)^2$ by (49). By (68) and (69) we have that $\max\{|A_i|:0\le i\le k+1\}\le 1/f(\eta)$ which finishes the proof. □

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