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The spectral measure of a Jacobi matrix in terms of the Fourier transform of the perturbation

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Abstract. We study the spectral properties of Jacobi matrices. By combining Killip's technique [12] with the technique of Killip and Simon [13] we obtain a result relating the properties of the elements of Jacobi matrices and the corresponding spectral measures. This theorem is a natural extension of a recent result of Laptev–Naboko–Safronov [17].

0. Introduction

Let S be the shift operator on $l^2(\mathbf{N})$ whose action on the canonical orthonormal basis $\{e_n\}_{n=0}^{\infty}$ is given by $Se_n=e_{n+1}$. Let A and B be selfadjoint diagonal operators: $Ae_n=\alpha_n e_n$, $Be_n=\beta_n e_n$, $\alpha_n>-1$, $\beta\in\mathbf{R}$. We study the spectrum of the operator

$$J=S+S^*+Q, \quad \text{where } Q=SA+AS^*+B.$$

Such an operator can be identified with the Jacobi matrix

$$J=\begin{pmatrix} \beta_0 & 1+\alpha_0 & 0 & 0 & \dots \\ 1+\alpha_0 & \beta_1 & 1+\alpha_1 & 0 & \dots \\ 0 & 1+\alpha_1 & \beta_2 & 1+\alpha_2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{0.1}$$

If the entries of this matrix are bounded then J is a bounded operator on $l^2(\mathbf{N})$. To J we associate the measure μ given by

$$m_\mu(z):=(e_0,(J-z)^{-1}e_0)=\int_{\mathbf{R}}\frac{d\mu(t)}{t-z}, \quad z\in\mathbf{C}. \tag{0.2}$$

([1]) The author thanks Sergei Naboko for useful discussions and Barry Simon for pointing out the conjecture.

The spectral significance of this function is represented by the relation

$$\mu(\delta)=(E_J(\delta)e_0,e_0), \tag{0.3}$$

where E_J denotes the spectral measure of J and $\delta\subset\mathbf{R}$ is a Borel set.

Conversely, to each measure μ with a compact support containing infinitely many points there is a standard procedure of constructing a Jacobi matrix via the corresponding orthogonal polynomials (see [13] for historical references and bibliography). Therefore similar problems were intensively studied by specialists in this field [21], [1].

A Jacobi matrix can be understood as a discrete approximation of a second order differential operator, whose properties play an important role in quantum mechanics. For example, the spectral measure μ of a Schrödinger operator is an object which can be measured by a physicist in the corresponding experiment. So a question, similar to the one we ask in the case of Jacobi matrices, would be translated by a physicist in the following way: what can we tell about the interaction in the system, if we know everything about the "distribution" of the spectrum?

Since there is a one-to-one correspondence between Jacobi matrices and probability measures, it is natural to ask how the properties of the entries of Jacobi matrices are related to the properties of probability measures. We are interested in a class of matrices J "close" to the "free" matrix J_0 for which $\alpha_n=0$ and $\beta_n=0$, $n=0,1,\dots$.

Obviously,

$$\mu(\mathbf{R})=\|e_0\|^2=1.$$

Denote by μ' the derivative of the absolutely continuous part of the measure μ. Most of our questions will be related to convergence of the integral of $\log(1/\mu'(2\cos\theta))$ with respect to the weight

$$\varrho(e^{i\theta})\sin^2(\theta);\quad \varrho(k)=W(k)^2,\quad W(k)=\sum_{n=-n_0}^{n_0} w_n k^n, \tag{0.4}$$

where $W(e^{i\theta})$ is real-valued and n_0 is finite. In order to state our main result we denote by $\mathfrak{S}_p$ the standard Schatten classes of compact operators:

$$\mathfrak{S}_p=\{T:\|T\|_{\mathfrak{S}_p}^p=\operatorname{tr}(T^*T)^{p/2}<\infty\}.$$

Theorem 0.1. *Let J be a Jacobi matrix, μ the corresponding measure* (0.3) *and the weight ϱ be given by* (0.4) *with $W(1)W(-1)\neq 0$. Assume that the operator $Q=J-J_0$ satisfies*

$$Q\in\mathfrak{S}_3. \tag{0.5}$$

Let further $\hat{A}(\theta)$ *and* $\widehat{B}(\theta)$ *be the sums of the Fourier series* $\sum_{n=-\infty}^{\infty} e^{in\theta}\alpha_n$ *and* $\sum_{n=-\infty}^{\infty} e^{in\theta}\beta_n$*, respectively. Then*

$$\int_{-\pi}^{\pi} |2e^{i\theta}\hat{A}(2\theta)+\widehat{B}(2\theta)|^2 \varrho(e^{i\theta})\,d\theta<\infty \tag{0.6}$$

if and only if μ *satisfies the following three properties*:

(1) $\operatorname{supp}\mu=[-2,2]\cup\{E_j^+\}_{j=1}^{N_+}\cup\{E_j^-\}_{j=1}^{N_-}$, *where* $\pm E_j^\pm>2$ *and* $0\le N_\pm\le\infty$;

(2) (Quasi-Szegő condition)

$$\int_{-\pi}^{\pi}\log\left[\frac{1}{\mu'(2\cos\theta)}\right]\varrho(e^{i\theta})\sin^2\theta\,d\theta<\infty;$$

(3) (Lieb–Thirring bound)

$$\sum_{j=1}^{N_+}|E_j^+-2|^{3/2}+\sum_{j=1}^{N_-}|E_j^-+2|^{3/2}<\infty.$$

Remark 1. One can generalize this result to the case $W(1)=W(-1)=0$, where instead of the Lieb–Thirring bound with the power $\frac{3}{2}$ we would have the one with the power $\frac{5}{2}$. In this case condition (3) can simply be omitted.

Remark 2. The question whether the condition (0.5) can be substituted by a weaker assumption is still open. However one can try to work with the class $\mathfrak{S}_4$ using two trace formulas for J_0+Q and J_0-Q simultaneously.

Although this theorem is a natural generalization of a recent result of Killip–Simon, it has some disadvantages. Namely, for a given measure μ we are not able to check in advance whether the condition (0.5) is fulfilled for the corresponding Jacobi matrix J. However, in the cases $\varrho(e^{i\theta})=1$ [13] and $\varrho(e^{i\theta})=\cos^2\theta$ [17] one is able to avoid this obstacle and obtain stronger results, where one does not have the *a priori* condition (0.5).

It should be mentioned that the proof of (0.6) for our class of Jacobi matrices is based on a technique developed in [13], namely, on results related to the notion of entropy. In spite the fact that [12] was written earlier than [13], the ideas of [12] combined with [13] lead to the local logarithmic integrability of the derivative of the spectral measure and it might seem that in terms of the "forward spectral problem" our result is weaker than the results of [12]. However, this is not the case, since Theorem 0.1 allows us in many situations not only to find the location of exponential zeros of μ', but also to say something about the order of these zeros.

Let us remark that some new information about the measure can be obtained by considering both "forward" and "inverse" directions of the theorem simultaneously. However, if one splits the two statements apart, then one gets two statements whose usefulness without the other part of the theorem would be limited. It is also interesting to combine Theorem 0.1 with the results of [13]. In particular, for a given set of finitely many points in $[-2,2]$ it is easy to find a Jacobi matrix whose spectral measure μ' vanishes at each of these points.

The main idea of this paper is to modify the known trace formulae. For J's with $Q=J-J_0$ of finite rank, the standard trace formulae are due to Case [2], [3]. Recently, Killip–Simon [13] found how to exploit these sum rules as a spectral tool. In particular, Killip and Simon have shown the importance of extension of sum rules to a larger class of Q's. In particular they gave a complete characterization of spectral measures μ corresponding to Q's from the Hilbert–Schmidt class. In its turn the impressive paper [13] was motivated by work on Schrödinger operators by Deift–Killip [6] and Denisov [7], [8]. In the paper [6] the authors prove that if $V\in L^2(\mathbf{R})$, then the absolutely continuous spectrum of $-d^2/dx^2+V$ is "essentially supported" by $(0,+\infty)$, which roughly speaking means that the derivative of the spectral measure is positive almost everywhere on $(0,+\infty)$. Another consequence of trace formulae has been observed by Denisov [7], who noticed that the singular component of the spectral measure of the Schrödinger operator with a square integrable potential can be more or less arbitrary. Finally, the author would like to mention the paper [26], the results of which are used in this work.

The subject we study here has a long history of results, which would be difficult to describe in such a short paper. However we would like to mention that it was A. Kiselev [14] who first started the investigation of the absolutely continuous spectrum of Schrödinger operators with slowly decaying potentials. We also supply the article with many relevant references, which can introduce the reader to the theory of orthogonal polynomials.

1. Preliminaries

For each $T\in\mathfrak{S}_1$, one can define a complex-valued function $\det(1+T)$, so that

$$|\det(1+T)|\le \exp\|T\|_{\mathfrak{S}_1}.$$

For $T\in\mathfrak{S}_3$ one defines

$$\det{}_3(1+T)=\det((1+T)e^{-T+T^2/2}). \tag{1.1}$$

Then

$$|\det{}_3(1+T)|\le \exp c\|T\|^3_{\mathfrak{S}_3},\quad c>0. \tag{1.2}$$

Lemma 1.1. *Let A and B be of finite rank. There is a constant C such that one of the conditions*

$$|\operatorname{Im} z| > 1 \quad \textit{or} \quad |\operatorname{Re} z| > 3 + \|Q\| \tag{1.3}$$

imply

$$|\log \det_3(I + Q(J_0 - z)^{-1})| \le C \|Q\|_{\mathfrak{S}_3}^3. \tag{1.4}$$

Proof. Let $z = \lambda + i\varepsilon$, where λ and ε are real. One can repeat the arguments of Proposition 5.2 [12] to show that

$$\begin{aligned} \left| \frac{d}{d\varepsilon} \log \det_3(I + Q(J_0 - z)^{-1}) \right| &= |\operatorname{tr}(J_0 - z)^{-1} Q (J - z)^{-1} - \operatorname{tr} Q (J_0 - z)^{-2} \\ &\quad + \operatorname{tr} Q (J_0 - z)^{-1} Q (J_0 - z)^{-2}| \\ &= |\operatorname{tr}((J_0 - z)^{-1} Q (J_0 - z)^{-1} Q (J - z)^{-1} (J_0 - z)^{-1} Q)| \\ &\le \|(J_0 - z)^{-1}\|^3 \|(J - z)^{-1}\| \, \|Q\|_{\mathfrak{S}_3}^3. \end{aligned} \tag{1.5}$$

On the other hand,

$$\lim_{\varepsilon \to \infty} \det_3(I + Q(J_0 - z)^{-1}) = 1.$$

Therefore the estimate (1.2) follows from (1.5) by the fundamental theorem of calculus. □

2. Trace formulae

Let A and B be of finite rank. By H_0 we denote the operator on $l^2(\mathbf{Z})$ defined on the standard basis $\{e_n\}_{-\infty}^{\infty}$ by

$$H_0 e_n = e_{n+1} + e_{n-1}, \quad n \in \mathbf{Z}.$$

It is convenient to replace m_μ by

$$M_\mu(k) = -m_\mu(z(k)) = -m_\mu(k + k^{-1}) = \int_{\mathbf{R}} \frac{k \, d\mu(t)}{1 - tk + k^2}, \quad |k| < 1. \tag{2.1}$$

It is known (see [22]) that the limit

$$M(e^{i\theta}) = \lim_{r \uparrow 1} M_\mu(re^{i\theta}),$$

exists almost everywhere on the unit circle and

$$\frac{\operatorname{Im} M(e^{i\theta})}{\pi} = \mu'(2\cos\theta), \quad \theta \in [0,\pi].$$

We borrow from [13] one important relation which expresses the spectral measure in terms of the perturbation determinant

$$|L(e^{i\theta})|^2 \operatorname{Im} M(e^{i\theta}) = \sin(\theta) \det(1+A)^2, \tag{2.2}$$

where

$$L(k) = \det(1+Q(J_0-(k+1/k)I)^{-1}).$$

Therefore for $z=2\cos\theta$,

$$\begin{aligned} \frac{1}{2}\log\left[\frac{\sin\theta}{\operatorname{Im} M(e^{i\theta})}\right] &= -\log|\det(I+A)| + \operatorname{Re}\operatorname{tr} Q(J_0-z)^{-1} \\ &\quad -\tfrac{1}{2}\operatorname{Re}\operatorname{tr}(Q(J_0-z)^{-1})^2 + \log|\det{}_3(I+Q(J_0-z)^{-1})| \\ &=: g(\theta) + \log|\det{}_3(I+Q(J_0-z)^{-1})|. \end{aligned} \tag{2.3}$$

The function $d_3(k)=\det_3(I+Q(J_0-z)^{-1})$ vanishes when $k+1/k$ is an eigenvalue of J. Let $\{\varkappa_n\}$ be the zeros of $d_3(k)$ lying in the domain $|k|<1$. We introduce the Blaschke product

$$G = \prod_n \frac{\varkappa_n - k}{1-\varkappa_n k}\frac{\varkappa_n}{|\varkappa_n|}, \quad (\bar{\varkappa}_n = \varkappa_n).$$

Clearly $|G|=1$ when $k=e^{i\theta}$. Choose $a>0$ so small that every point $z=k+1/k$ with $|k|=a$ satisfies the condition (1.3). Obviously $a<\min_n|\varkappa_n|$. Thus, by using the Cauchy theorem we find

$$\begin{aligned} \frac{2}{\pi}\int_{-\pi}^{\pi} \log|d_3(k)|\varrho(e^{i\theta})\sin^2\theta\,d\theta &= \operatorname{Re}\frac{i}{2\pi}\int_{|k|=a}\log\left(\frac{d_3(k)}{G}\right)\varrho(k)\frac{(k^2-1)^2}{k^3}\,dk \\ &= \operatorname{Re}\frac{i}{2\pi}\int_{|k|=a}\log(d_3(k))w(k)\,\frac{dk}{k} - \sum_n f(\varkappa_n), \end{aligned} \tag{2.4}$$

where

$$w(k) = \varrho(k)\frac{(k^2-1)^2}{k^2}$$

and

$$f(\varkappa) = \operatorname{Re}\frac{i}{2\pi}\int_{|k|=a}\log\left(\frac{\varkappa-k}{1-\varkappa k}\frac{\varkappa}{|\varkappa|}\right)w(k)\,\frac{dk}{k}.$$

The first identity in (2.4) holds due to the fact that $d_3(k)/G$ is an analytic function which does not have zeros in $\mathbf{C}\setminus\{0\}$. The latter integral can be computed by the residue calculus, in particular one can show that $f(\varkappa)$ is a finite linear combination of powers $\varkappa^{\pm n}$ and $\log|\varkappa|$. However, instead of calculating it explicitly we prove that

$$f(t)=\tfrac{4}{3}\varrho(\mp 1)|t\pm 1|^3+o(|t\pm 1|^3), \quad \text{as } t\to\mp 1,\ |t|<1. \tag{2.5}$$

Indeed, since $\overline{w(k)}=w(1/\bar{k})$ we observe that

$$\begin{aligned} 2f(\varkappa)&=\operatorname{Re}\frac{i}{2\pi}\int_{|k|=a}\log\left(\frac{\varkappa-k}{1-\varkappa k}\frac{\varkappa}{|\varkappa|}\right)w(k)\,\frac{dk}{k}\\ &\quad-\operatorname{Re}\frac{i}{2\pi}\int_{|k|=1/a}\log\left(\frac{\varkappa-k}{1-\varkappa k}\frac{\varkappa}{|\varkappa|}\right)w(k)\,\frac{dk}{k}. \end{aligned} \tag{2.6}$$

Now assume that $\varkappa=1-s$, where $s\to 0$, and let

$$F(k)=\int_1^k w(z)\,\frac{dz}{z}.$$

Then integration by parts leads us to the expression

$$\begin{aligned} 2f(\varkappa)&=\operatorname{Re}\frac{1}{2\pi i}\int_{|k|=1/a}\left(\frac{1}{\varkappa-k}-\frac{\varkappa}{1-\varkappa k}\right)F(k)\,dk\\ &\quad-\operatorname{Re}\frac{1}{2\pi i}\int_{|k|=a}\left(\frac{1}{\varkappa-k}-\frac{\varkappa}{1-\varkappa k}\right)F(k)\,dk. \end{aligned}$$

Thus, by the residue calculus we obtain

$$f(\varkappa)=\operatorname{Re}\tfrac{1}{2}(F(\varkappa)-F(1/\varkappa))$$

which proves (2.5) for the case $t\to 1$. The other case is similar.

By the inequality (1.4) the first term in (2.4) is bounded by a constant $C=C(a,\varrho,\|Q\|)$ (depending on a, ϱ and $\|Q\|$) times $\|Q\|^3_{\mathfrak{S}_3}$:

$$\left|\int_{|k|=a}\log(d_3(k))w(k)\,\frac{dk}{k}\right|\le C\|Q\|^3_{\mathfrak{S}_3}, \quad C=C(a,\varrho,\|Q\|). \tag{2.7}$$

Now, since the kernel of the resolvent $R(z)=(H_0-zI)^{-1}$ is of the form

$$G_2(n,m;z)=\frac{k}{k^2-1}k^{|n-m|}, \quad z=k+k^{-1},\ 0<|k|<1, \tag{2.8}$$

we obtain the following result.

Lemma 2.1. *For $k=e^{i\theta}$,*

$$\operatorname{Re}\operatorname{tr} R(z)QR(z)Q=-\frac{1}{4\sin^2\theta}|2e^{i\theta}\hat{A}(2\theta)+\widehat{B}(2\theta)|^2+\operatorname{tr} A^2.$$

Proof. By using (2.8) we compute

$$\begin{aligned}\operatorname{Re}\operatorname{tr} R(z)BR(z)B &= -\frac{1}{4\sin^2\theta}\sum_{m,n=0}^{\infty}\cos(2|m-n|\theta)\beta_n\beta_m\\ &= -\frac{1}{4\sin^2\theta}\operatorname{Re}\sum_{m,n=0}^{\infty}e^{2i(m-n)\theta}\beta_n\beta_m\\ &= -\frac{1}{4\sin^2\theta}|\widehat{B}(2\theta)|^2,\end{aligned}$$

where $z=2\cos\theta$. Similarly,

$$\begin{aligned}\operatorname{Re}\operatorname{tr} R(z)SAR(z)SA &= \operatorname{Re}\operatorname{tr} R(z)AS^*R(z)AS^*\\ &= -\frac{1}{4\sin^2\theta}\sum_{m,n=0}^{\infty}\cos((|m-n|+|n+2-m|)\theta)\alpha_{n+1}\alpha_m\\ &= -\frac{1}{4\sin^2\theta}\biggl(\sum_{n\le m-2}\cos(2(m-n-1)\theta)\alpha_{n+1}\alpha_m\\ &\quad+\sum_{n\ge m}\cos(2(m-n-1)\theta)\alpha_{n+1}\alpha_m+\sum_{n=0}^{\infty}\cos(2\theta)\alpha_n^2\biggr)\\ &= -\frac{1}{4\sin^2\theta}(|\hat{A}(2\theta)|^2-2\sin^2(\theta)\operatorname{tr} A^2),\end{aligned}$$

and, since S and S^* commute with H_0,

$$\operatorname{Re}\operatorname{tr} R(z)SAR(z)AS^*=\operatorname{Re}\operatorname{tr} R(z)AS^*R(z)SA=-\frac{|\hat{A}(2\theta)|^2}{4\sin^2(\theta)}.$$

Finally

$$\operatorname{Re}\operatorname{tr} R(z)SAR(z)B=\operatorname{Re}\operatorname{tr} R(z)AS^*R(z)B=-\frac{1}{4\sin^2\theta}\operatorname{Re}(e^{i\theta}\hat{A}(2\theta)\overline{\widehat{B}(2\theta)})$$

which completes the proof. □

We will need the following relation, which is valid for $k=e^{i\theta}$,

$$\operatorname{Re}\operatorname{tr} R(z)Q=\operatorname{Re}\sum_{n=0}^{\infty}\frac{\beta_n+2\alpha_n e^{i\theta}}{2i\sin\theta}=\operatorname{tr} A. \tag{2.9}$$

The kernel for the resolvent $(J_0-zI)^{-1}$ of J_0 is of the form

$$G(n,m;z)=\frac{k}{1-k^2}(k^{n+m+2}-k^{|n-m|})=G_1(n,m;z)+G_2(n,m;z).$$

The first term here does not play any role because its contribution involves only finite number of entries of J. Indeed we can prove the following result.

Proposition 2.1. *Let $\phi(\theta)$ be a boundary value of an analytic function in the unit disc except for a pole of order j at the point zero. Then the function*

$$\begin{aligned}\Phi(Q)&=\operatorname{tr}\int_{-\pi}^{\pi}(QR(z)-Q(J_0-z)^{-1})\phi(\theta)\,d\theta\\ &\quad-\frac{1}{2}\operatorname{tr}\int_{-\pi}^{\pi}((QR(z))^2-(Q(J_0-z)^{-1})^2)\phi(\theta)\,d\theta\end{aligned}\tag{2.10}$$

is a second order polynomial only of a finite number of elements α_n and β_n.

Proof. Let Γ_k be the operator on $l^2(\mathbf{N})$ with the kernel $G_1(n,m;z)$, $z=k+k^{-1}$ and P be the orthogonal projection from $l^2(\mathbf{Z})$ onto $l^2(\mathbf{N})$. Then $\Gamma_k=(J_0-z)^{-1}-PR(z)P$. Therefore the integrand in (2.10) can be rewritten in terms of $\operatorname{tr}Q\Gamma_k$, $\operatorname{tr}(Q\Gamma_k)^2$ and $\operatorname{tr}Q\Gamma_kQR(z)$. The functions $(k^2-1)\operatorname{tr}Q\Gamma_k$, $(k^2-1)^2\operatorname{tr}(Q\Gamma_k)^2$ and $(k^2-1)^2\operatorname{tr}Q\Gamma_kQR(z)$ are analytic in the disc and the coefficients of the Taylor series about zero depend on finite number of entries α_n and β_n. For example

$$(k^2-1)^2\operatorname{tr}B\Gamma_kBR(z)=k^2\sum_{n=0}^{\infty}k^{2n}\biggl(\beta_n^2+2\beta_n\sum_{m=0}^{n-1}\beta_m\biggr).\quad\square$$

From (2.3), (2.4), (2.7) and (2.9) we obtain

$$\begin{aligned}\frac{1}{\pi}\int_{-\pi}^{\pi}\log\biggl[\frac{\sin\theta}{\operatorname{Im}M(e^{i\theta})}\biggr]\varrho(e^{i\theta})\sin^2\theta\,d\theta&=\frac{2}{\pi}\int_{-\pi}^{\pi}g(\theta)\varrho(e^{i\theta})\sin^2\theta\,d\theta\\ &\quad-\sum_n f(\varkappa_n)+\Psi(Q),\end{aligned}$$

where $\Psi(Q)$ is finite even if $Q\in\mathfrak{S}_3$ and can be estimated by a function of $\|Q\|_{\mathfrak{S}_3}$. From Lemma 2.1 we see that

$$g(\theta)=-\log|\det_3(1+A)|+\frac{1}{8\sin^2\theta}|2e^{i\theta}\hat{A}(2\theta)+\widehat{B}(2\theta)|^2+\operatorname{Re}\Phi_0(\theta),\tag{2.11}$$

where Φ_0 is the integrand in (2.10) with $\phi=1$. Thus we obtain the trace formula

$$\begin{aligned}\sum_n f(\varkappa_n)&+\frac{1}{\pi}\int_{-\pi}^{\pi}\log\biggl[\frac{\sin\theta}{\operatorname{Im}M(e^{i\theta})}\biggr]\varrho(e^{i\theta})\sin^2\theta\,d\theta\\ &=\frac{1}{4\pi}\int_{-\pi}^{\pi}|2e^{i\theta}\hat{A}(2\theta)+\widehat{B}(2\theta)|^2\varrho(e^{i\theta})\,d\theta+\Lambda(Q),\end{aligned}\tag{2.12}$$

where the absolute value of $\Lambda(Q)$ is bounded by a certain function of $\|Q\|_{\mathfrak{S}_3}$.

3. Perturbations of infinite rank

Given a perturbation of class $\mathfrak{S}_3$ there is a beautiful technique which enables one to prove that if one of the sides in (2.12) is finite, then so is the other side. This technique is due to Killip and Simon [13] and uses the following facts:

$$-\int_{-\pi}^{\pi} \log\left[\frac{\sin\theta}{\operatorname{Im} M(e^{i\theta})}\right] \varrho(e^{i\theta}) \sin^2\theta \, d\theta \le C, \quad C = C(\varrho),$$

and

$$\int_{-\pi}^{\pi} |2e^{i\theta}\hat{A}(2\theta) + \widehat{B}(2\theta)|^2 \varrho(e^{i\theta}) \, d\theta = 2\pi \sum_{n=0}^{\infty} \left| \sum_{j=0}^{\infty} 2w_{n-2j-1}\alpha_j + w_{n-2j}\beta_j \right|^2. \tag{3.1}$$

However, since we are forced to deal with the sums appearing in the right-hand side of (3.1) a better reference is the one to the paper [17] where the technique is more adjusted to the special case of the trace formulae (2.12). Below we denote by D the unit disc in $\mathbf{C}$.

Let $J^{(N)}$ and $Q^{(N)}$ be operators whose realizations in the standard basis $\{e_n\}_{n=0}^{\infty}$ are given by

$$J^{(N)} = \begin{pmatrix} \beta_{N+1} & 1+\alpha_{N+1} & 0 & \dots \\ 1+\alpha_{N+1} & \beta_{N+2} & 1+\alpha_{N+2} & \dots \\ 0 & 1+\alpha_{N+2} & \beta_{N+3} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{3.2}$$

$$Q^{(N)} = \begin{pmatrix} \beta_{N+1} & \alpha_{N+1} & 0 & \dots \\ \alpha_{N+1} & \beta_{N+2} & \alpha_{N+2} & \dots \\ 0 & \alpha_{N+2} & \beta_{N+3} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{3.3}$$

and let

$$\Lambda_\varrho(J) := \frac{1}{4\pi} \int_{-\pi}^{\pi} |2e^{i\theta}\hat{A}(2\theta) + \widehat{B}(2\theta)|^2 \varrho(e^{i\theta}) \, d\theta.$$

We introduce

$$P_{N,\varrho}(Q) := \Lambda_\varrho(J) - \Lambda_\varrho(J^{(N)}).$$

The "tails" in the sums $\Lambda_\varrho(J)$ and $\Lambda_\varrho(J^{(N)})$ cancel each other, so that the elements of the matrices B and A do not enter in the difference $P_{N,\varrho}(Q)$ starting from a certain index. Thus $P_{N,\varrho}(Q)$ is a continuous function of a finite number of elements of the matrices B and A and can be extended to arbitrary matrices B and A. Below $P_{N,\varrho}(Q)$ is extended for any B and A. Let

$$\Phi_\varrho(\mu) = \frac{1}{\pi}\int_{-\pi}^{\pi} \log\left|\frac{\sin\theta}{\operatorname{Im} M_\mu}\right| \varrho(e^{i\theta})\sin^2\theta\,d\theta + \sum_n f(\varkappa_n). \tag{3.4}$$

It is important for us to find conditions when

$$\Phi_\varrho(\mu) - \Phi_\varrho(J^{(N)}) = P_{N,\varrho}(Q) + \Lambda(Q) - \Lambda(Q^{(N)}). \tag{3.5}$$

The identity (3.5) is valid if Q is of finite rank. The case when the function M_μ is meromorphic in the neighborhood of the unit disc follows from Proposition 4.3 and Theorem 4.4 in [13] and the fact that

$$\varrho(k) = \left(\sum_{n=-n_0}^{n_0} w_n k^n\right)^2$$

is a finite linear combination of powers $k^{\pm m}$.

Notice that $y \mapsto -\log y$ is convex. Assume that

$$\int_{-\pi}^{\pi} \varrho(e^{i\theta})\sin^2\theta\,d\theta = 1.$$

Employing Jensen's inequality we find that

$$\begin{aligned}
\int_{-\pi}^{\pi} \log\left[\frac{\sin\theta}{\operatorname{Im} M_\mu(e^{i\theta})}\right] \varrho(e^{i\theta})\sin^2\theta\,d\theta &= -2\int_0^{\pi} \log\left[\frac{\operatorname{Im} M_\mu}{\sin\theta}\right]\sin^2\theta\varrho\,d\theta \\
&\ge -\log\left[2\int_0^{\pi}(\operatorname{Im} M_\mu)\sin\theta\,d\theta\right] \\
&\quad -2\int_0^{\pi}\log[\varrho(e^{i\theta})]\sin^2(\theta)\varrho\,d\theta \\
&= -\log[\pi\mu_{\mathrm{ac}}(-2,2)] \\
&\quad -2\int_0^{\pi}\log[\varrho(e^{i\theta})]\sin^2(\theta)\varrho\,d\theta \\
&\ge -\log\pi - 2\int_0^{\pi}\log[\varrho(e^{i\theta})]\sin^2(\theta)\varrho\,d\theta \\
&=: C(\varrho),
\end{aligned} \tag{3.6}$$

where we use that $\mu_{\mathrm{ac}}(-2,2)\le 1$. Formulae (3.5) and (3.6) imply that

$$P_{N,\varrho}(Q) \le \Phi_\varrho(\mu) - C(\varrho) + F(\|Q\|_{\mathfrak{S}_3}, \varrho),$$

where the quantity $F(\|Q\|_{\mathfrak{S}_3}, \varrho)$ depends only on $\|Q\|_{\mathfrak{S}_3}$ and ϱ. The latter inequality was obtained for M_μ meromorphic in the neighborhood of the unit disc. However, this inequality can also be extended to arbitrary measures μ satisfying conditions (1)–(3) of Theorem 0.1.

Indeed, assume that condition (2) of Theorem 0.1 is fulfilled. According to Remark 1 following after Theorem 2.1 in [26],

$$\lim_{r\uparrow 1} \int \log[\operatorname{Im} M(re^{i\theta})]\varrho(e^{i\theta}) \sin^2\theta \, d\theta = \int \log[\operatorname{Im} M(e^{i\theta})]\varrho(e^{i\theta}) \sin^2\theta \, d\theta. \tag{3.7}$$

Now, given any J and M-function $M(z)$ associated to it, there is a natural approximating family of M-functions meromorphic in a neighborhood of the closure of the unit disc $\overline{D}$. The next result is proved in [13], Lemma 8.3.

Lemma 3.1. *Let M_μ be the M-function of a probability measure μ obeying condition* (1) *of Theorem* 0.1. *Define*

$$M^{(r)}(z) = r^{-1} M_\mu(rz) \quad \text{for } 0<r<1. \tag{3.8}$$

Then, there is a family of probability measures $\mu^{(r)}$ such that $M^{(r)} = M_{\mu^{(r)}}$.

The poles of $M_{\mu^{(r)}}$ are given by

$$\varkappa_j(\mu^{(r)}) = \frac{\varkappa_j}{r},$$

where we consider only those j for which $|\varkappa_j|<r$. Thus if condition (3) of Theorem 0.1 is satisfied then $\sum f(\varkappa_j(\mu^{(r)}))$ is a continuous function of r whose limit is equal to $\sum f(\varkappa_j)$, as $r\uparrow 1$. Moreover, the convergence $M_{\mu^{(r)}}(z)\to M_\mu(z)$ is uniform on compact subsets of the upper half of D, which means that the coefficients of the Jacobi matrices must converge. Thus for any N,

$$\begin{aligned} P_{N,\varrho}(Q) &= \lim_{r\uparrow 1} P_{N,\varrho}(Q(r)) \le \lim_{r\uparrow 1} \Phi_\varrho(\mu^{(r)}) - C(\varrho) + F(\|Q\|_{\mathfrak{S}_3}, \varrho) \\ &= \Phi_\varrho(\mu) - C(\varrho) + F(\|Q\|_{\mathfrak{S}_3}, \varrho). \end{aligned}$$

Therefore $\sup_N P_{N,\varrho}(Q) < \infty$ which guarantees (0.6).

Conversely, suppose that the conditions (0.6) and (0.5) are fulfilled. We would like to establish that

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \log\left[\frac{\sin\theta}{\operatorname{Im} M_\mu(e^{i\theta})}\right] \sin^2(\theta) \varrho \, d\theta < \infty. \tag{3.9}$$

Definition. Let ν and μ be finite Borel measures on a compact Hausdorff space X. The entropy $S(\nu|\mu)$ of ν relative to μ is defined by

(3.10)

$$S(\nu\,|\,\mu)=\begin{cases} -\displaystyle\int_X \log\left(\frac{d\nu}{d\mu}\right)d\nu, & \text{if } \nu \text{ is absolutely continuous with respect to } \mu,\\ -\infty, & \text{otherwise.}\end{cases}$$

The following result is proved in the paper of Simon and Killip, [13], Corollary 5.3.

Lemma 3.2. *The entropy $S(\nu|\mu)$ is weakly upper semi-continuous in μ, that is, if $\mu_n \xrightarrow{w} \mu$, then*

$$S(\nu\,|\,\mu)\geq \limsup_{n\to\infty} S(\nu\,|\,\mu_n).$$

Let us use the fact that the formulae (2.12) are valid at least for finite rank operators A and B. Suppose now that A and B are arbitrary compact selfadjoint operators such that (0.5) and (0.6) hold. It is then clear that the right-hand side of (2.12) is finite. Now let the sequences of operators A_n and B_n converge to A and B in $\mathfrak{S}_3$ so that

$$\int_{-\pi}^{\pi} |2e^{i\theta}\hat{A}_n(2\theta)+\widehat{B}_n(2\theta)|^2\varrho(e^{i\theta})\,d\theta\to\int_{-\pi}^{\pi} |2e^{i\theta}\hat{A}(2\theta)+\widehat{B}(2\theta)|^2\varrho(e^{i\theta})\,d\theta,$$

as $n\to\infty$. Let $Q_n=SA_n+A_nS^*+B_n$, $J_n=S+S^*+Q_n$ and $\mu_n(\delta)=(E_{J_n}(\delta)e_0,e_0)$, where δ is an arbitrary Borel set. Since $(J_n-z)^{-1}$ converges to $(J-z)^{-1}$ uniformly on compact subsets of the upper half-plane we obtain that μ_n is weakly convergent to μ,

$$\mu_n \xrightarrow{w} \mu,\quad \text{as } n\to\infty.$$

Applying Lemma 3.2 we obtain that if $d\nu=\varrho(e^{i\theta})\sin^2\theta\,d\theta$ and μ is the spectral measure of J, then

$$S(\nu\,|\,\mu)>-\infty.$$

This is exactly what is needed for (3.9).

In order to complete the proof we only have to show that (0.6) and (0.5) imply condition (3) of Theorem 0.1. This can be done by a simple trick (see [17]) whose essence is that one first has to consider a finite sum $\sum_{j=1}^{p} f(\varkappa_j)$, prove that this sum can be approximated by the corresponding sum for A_n and B_n and then let $p\to\infty$.

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