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Geodesic inversion and Sobolev spaces on Heisenberg type groups

Francesca Astengo and Bianca Di Blasio

Abstract. Let σ be the geodesic inversion on a Heisenberg type group N with homogeneous dimension Q, and denote by S the jacobian of σ. We prove that, for $-\frac{1}{2}Q<\alpha<\frac{1}{2}Q$, the operators $T_\alpha: f\mapsto S^{1/2-\alpha/Q}(f\circ\sigma)$ are bounded on certain homogeneous Sobolev spaces $\mathcal{H}^\alpha(N)$ if and only if N is an Iwasawa N-group.

1. Introduction

The class of Heisenberg type groups was introduced by Kaplan in [8] as a class of two-step nilpotent Lie groups whose standard sublaplacians admit fundamental solutions analogous to that known for the Heisenberg group. It includes all Iwasawa N-groups associated to real rank one simple Lie groups. The formalism of Heisenberg type groups provides a unified way for studying many problems on real rank one simple Lie groups that can be reduced to problems on the associated Iwasawa N-group [3], [4], [5].

In [3] it was proved that the Iwasawa N-groups are characterised among all Heisenberg type groups by a Lie-algebraic condition, the so called J^2-condition; moreover it was proved that the geodesic inversion σ on N is conformal if and only if the J^2-condition holds.

In this paper we study some properties of the action of the inversion σ on functions. We consider the operators T_α defined on $C_c^\infty(N)$ by the formula

$$T_\alpha f=S^{1/2-\alpha/Q}\,(f\circ\sigma),\quad f\in C_c^\infty(N),\ -\tfrac{1}{2}Q<\alpha<\tfrac{1}{2}Q,$$

where S denotes the jacobian of the map σ. Clearly the operator T_α extends to an

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isometry on $L^p(N)$, for

$$\frac{1}{p}=\frac{1}{2}-\frac{\alpha}{Q}.$$

We prove that the J^2-condition holds if and only if the operators T_α are bounded on suitable homogeneous Sobolev spaces $\mathcal{H}^\alpha(N)$, defined in terms of the standard sublaplacian on N. An important application of our result is an elementary proof of the uniform boundedness of certain representations of real rank one simple Lie groups on $\mathcal{H}^\alpha(N)$ (see [10], [11], [12] for the first papers on this subject). The approach used in this paper appeared in [2], where M. Cowling attacked this problem for the real, complex and the quaternionic cases before the introduction of Heisenberg type groups. Additional results concerning uniformly bounded representations can be found in the paper by Cowling and the authors [1].

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2. Heisenberg type groups

Let $\mathfrak{n}$ be a two-step real nilpotent Lie algebra, with an inner product $\langle\cdot,\cdot\rangle$. Write $\mathfrak{n}$ as an orthogonal sum $\mathfrak{n}=\mathfrak{v}\oplus\mathfrak{z}$, where $\mathfrak{z}$ is the center of $\mathfrak{n}$. For each Z in $\mathfrak{z}$, define the map $J_Z\colon\mathfrak{v}\to\mathfrak{v}$ by the formula

$$\langle J_Z X, Y\rangle=\langle [X,Y],Z\rangle,\qquad X,Y\in\mathfrak{v}.$$

Following Kaplan [8], we say that the Lie algebra $\mathfrak{n}$ is *H-type* if

$$J_Z^2=-|Z|^2 I_{\mathfrak{v}},\qquad Z\in\mathfrak{z},\tag{1}$$

where $I_{\mathfrak{v}}$ is the identity on $\mathfrak{v}$. A connected and simply connected Lie group N whose Lie algebra is an H-type algebra is said to be an *H-type group.* The Iwasawa N-groups associated to all real rank one simple groups are H-type. Note that from property (1), it follows that $\mathfrak{z}=[\mathfrak{v},\mathfrak{v}]$, and moreover the dimension of $\mathfrak{v}$ is even. We denote by Q the number $\mathrm{d}_{\mathfrak{v}}+2\mathrm{d}_{\mathfrak{z}}$.

In Section 5 we shall need the following properties of the map J. These properties are proved in [3, Section 1]:

$$\begin{aligned} J_Z J_{Z'}+J_{Z'}J_Z&=-2\langle Z,Z'\rangle, && Z,Z'\in\mathfrak{z},\\ \langle J_Z X, J_{Z'}X'\rangle+\langle J_{Z'}X,J_Z X'\rangle&=2\langle Z,Z'\rangle\langle X,X'\rangle, && Z,Z'\in\mathfrak{z},\ X,X'\in\mathfrak{v},\\ J_{[X,X']}X&=|X|^2\,P_{J_{\mathfrak{z}}X}X', && X,X'\in\mathfrak{v},\\ [X,J_Z X]&=|X|^2 Z, && X\in\mathfrak{v},\ Z\in\mathfrak{z}, \end{aligned}\tag{2}$$

where $P_{J_{\mathfrak{z}}X}X'$ is the projection of the vector X' on the space $J_{\mathfrak{z}}X=\{J_Z X : Z\in\mathfrak{z}\}$.

Since N is a nilpotent Lie group, the exponential mapping is surjective. Let X be in $\mathfrak{v}$ and Z be in $\mathfrak{z}$; we denote by (X,Z) the element $\exp(X+Z)$ of the group N and by $\log(X,Z)$ the element $X+Z$ of the Lie algebra $\mathfrak{n}$.

By the Baker–Campbell–Hausdorff formula, the group law is given by

$$(X,Z)\,(X',Z')=\big(X+X',Z+Z'+\tfrac{1}{2}[X,X']\big),\quad X,X'\in\mathfrak{v},\ Z,Z'\in\mathfrak{z}.$$

The group N is unimodular and a Haar measure dn on N is dX, dZ, where dX and dZ are the Lebesgue measures on the real vector spaces $\mathfrak{v}$ and $\mathfrak{z}$ respectively.

The Iwasawa N-groups are characterized, among all H-type groups, by an algebraic condition, called the J^2-condition.

Definition. ([3]) We say that $\mathfrak{n}$ satisfies the *J^2-condition* if, for any X in $\mathfrak{v}$ and Z,Z' in $\mathfrak{z}$, such that $\langle Z,Z'\rangle=0$, there exists Z'' in $\mathfrak{z}$ such that

$$J_Z J_{Z'} X = J_{Z''} X.$$

In Section 5 we shall see that this condition is strictly linked with the geometric properties of the inversion σ on N.

When N is abelian, σ is the classical inversion $\sigma\colon x\mapsto -|x|^{-2}x$ on $\mathbf{R}^{d_{\mathfrak{z}}}\simeq N$. In the general case it is the limit on the boundary N of the geodesic inversion on the Damek–Ricci space associated to N (see [4] and [5] for further details). The map σ has been studied in [3]; it is given by

$$\sigma(X,Z)=(-\mathcal{B}(X,Z)^{-1}\bar{\mathcal{A}}(X,Z)X,-\mathcal{B}(X,Z)^{-1}Z),\quad (X,Z)\in N\setminus\{0\},$$

where

$$\mathcal{A}(X,Z)=\tfrac{1}{4}|X|^2+J_Z,\quad \bar{\mathcal{A}}(X,Z)=\tfrac{1}{4}|X|^2-J_Z\quad\text{and}\quad \mathcal{B}(X,Z)=\tfrac{1}{16}|X|^4+|Z|^2.$$

In [3] it is proved that σ is conformal if and only if the J^2-condition holds, i.e., if and only if N is an Iwasawa N-group.

Let t be a positive real number. We define the *homogeneous dilation* δ_t on N by

$$\delta_t(X,Z)=(tX,t^2Z),\quad (X,Z)\in N.$$

It is easy to check that δ_t is a group automorphism and that the number $Q=\mathrm{d}_{\mathfrak{v}}+2\mathrm{d}_{\mathfrak{z}}$ is the homogeneous dimension of N. A homogeneous gauge on N is the function $\mathcal{B}^{1/4}$. Let Σ be the unit sphere with respect to this gauge, i.e.,

$$\Sigma=\{n\in N:\mathcal{B}(n)=1\};$$

there exists a unique C^∞ measure $d\eta$ on Σ such that the following polar coordinate integral formula holds:

$$\int_N f(n)\,dn = \int_\Sigma \int_0^\infty f(\delta_t \eta)\, t^{Q-1}\,dt\,d\eta, \quad f\in C_c^\infty(N).$$

We fix orthonormal bases $\{E_j\}_{j=1}^{\mathrm{d}_{\mathfrak{v}}}$ and $\{U_k\}_{k=1}^{\mathrm{d}_{\mathfrak{z}}}$ of $\mathfrak{v}$ and $\mathfrak{z}$, respectively. For X in $\mathfrak{v}$ and Z in $\mathfrak{z}$, we write $X=\sum_{j=1}^{\mathrm{d}_{\mathfrak{v}}} x_j E_j$ and $Z=\sum_{k=1}^{\mathrm{d}_{\mathfrak{z}}} z_k U_k$. Given a vector V in $\mathfrak{n}$, we denote by $\widetilde{V}$ the left-invariant vector field associated to it, hence we write

$$\widetilde{V} f(n) = \frac{d}{dt}\bigg|_{t=0} f(n \exp tV).$$

We shall refer to vectors in $\mathfrak{v}$ as *horizontal tangent vectors.*

It is easy to check that, for a smooth function f on N,

$$\begin{aligned} \widetilde{E}_j f(X,Z) &= \partial_{x_j} f(X,Z) + \frac{1}{2}\sum_{k=1}^{\mathrm{d}_{\mathfrak{z}}} \langle J_{U_k} X, E_j\rangle \partial_{z_k} f(X,Z), \\ \widetilde{U}_k f(X,Z) &= \partial_{z_k} f(X,Z), \end{aligned} \tag{3}$$

where $j=1,\dots,\mathrm{d}_{\mathfrak{v}}$ and $k=1,\dots,\mathrm{d}_{\mathfrak{z}}$.

From now on we shall write

$$J_k = J_{U_k}, \quad k=1,\dots,\mathrm{d}_{\mathfrak{z}}.$$

Finally, we use the "variable constant convention", according to which constants are denoted by C, and these are not necessarily equal at different occurrences. All "constants" are positive.

3. Fractional powers of the sublaplacian

In this section, we recall some properties of homogeneous distributions which we will use in the next sections. For further details, the reader can refer to [2] and [6].

Let d be in $\mathbf{C}$. A function f on N is said to be *homogeneous* of degree d if

$$f\circ\delta_t = t^d f, \quad t\in\mathbf{R}^+.$$

Since $\mathcal{B}\circ\delta_t = t^4\mathcal{B}$, any homogeneous function f of degree d satisfies

$$f(n) = \Omega(n)\mathcal{B}(n)^{d/4}, \quad n\in N\setminus\{0\},$$

where Ω is homogeneous of degree 0.

We say that a distribution K on N is a *kernel of type* α if it coincides with a homogeneous function f_K of degree $\alpha-Q$ on $N\setminus\{0\}$. In what follows, we shall use the same notation K for the distribution K and the associated function f_K.

Let Δ be the *sublaplacian*, defined by

$$\Delta=-\sum_{j=1}^{d_{\mathfrak{v}}}\widetilde{E}_j^2,$$

where $\widetilde{E}_j$ are the left-invariant vector fields given by formula (3). The operator Δ is a densely defined, essentially self-adjoint, positive operator on $L^2(N)$. Hence it has a spectral resolution given by

$$\Delta=\int_0^\infty \lambda\, dA_\lambda.$$

Moreover 0 is not an eigenvalue of Δ, as proved by Folland [6, Proposition 3.9]; therefore if α is in $\mathbf{C}$, we may define the operators Δ^α by the formula

$$\Delta^\alpha=\int_0^\infty \lambda^\alpha\, dA_\lambda.$$

Proposition 1. ([6, Theorem 3.15, Propositions 3.17 and 3.18]) *The operators* Δ^α *have the following properties*:

(i) *the operator* Δ^α *is closed on* $L^2(N)$ *for every* α *in* $\mathbf{C}$;

(ii) *if* f *is in* $\operatorname{Dom}(\Delta^\alpha)\cap\operatorname{Dom}(\Delta^{\alpha+\beta})$, *then* $\Delta^\alpha f$ *is in* $\operatorname{Dom}(\Delta^\beta)$ *and* $\Delta^\beta\Delta^\alpha f=\Delta^{\alpha+\beta}f$; *in particular,* $\Delta^{-\alpha}=(\Delta^\alpha)^{-1}$;

(iii) *if* $0<\operatorname{Re}(\alpha)<Q$, *there exists a kernel* $\mathcal{R}_\alpha$ *of type* α *such that if* f *is in* $\operatorname{Dom}(\Delta^{-\alpha/2})$, *then* $\Delta^{-\alpha/2}f=f*\mathcal{R}_\alpha$.

4. Lorentz and Sobolev spaces

Let f be a measurable function on the group N. The *nonincreasing rearrangement* of f is the function f^* on $\mathbf{R}^+$ defined by

$$f^*(t)=\inf\{s\in\mathbf{R}^+:|\{n\in N:|f(n)|>s\}|\le t\},\quad t\in\mathbf{R}^+,$$

where $|E|$ denotes the Haar measure of a measurable subset E of N. The function f^* is nonincreasing, nonnegative, equimeasurable with f and right-continuous. For any given measurable function f on N, we define

$$\|f\|_{L^{p,q}}=\left(\frac{q}{p}\int_0^\infty (s^{1/p}f^*(s))^q\,\frac{ds}{s}\right)^{1/q},\quad 1<p<\infty,\ 1\le q<\infty,$$

and

$$\|f\|_{L^{p,\infty}}=\sup\{s^{1/p}f^*(s):s\in\mathbf{R}^+\},\quad 1<p<\infty.$$

Definition. Let $1<p<\infty$ and $1\leq q\leq\infty$. The *Lorentz space* $L^{p,q}(N)$ consists of those measurable functions f on N such that $\|f\|_{L^{p,q}}$ is finite.

It is easy to check that $L^{p,p}(N)$ coincides with the usual Lebesgue space $L^p(N)$, with equality of norms. Moreover, if $q_1<q_2$, then $L^{p,q_1}(N)$ is contained in $L^{p,q_2}(N)$ and, if $1<p,q<\infty$, the dual space of $L^{p,q}(N)$ is $L^{p',q'}(N)$, where

$$\frac{1}{p}+\frac{1}{p'}=\frac{1}{q}+\frac{1}{q'}=1.$$

A good reference for Lorentz spaces is [7]. We recall a few facts from that paper.

Lemma 2. ([7, p. 273]) *Let p and r be in $(1,\infty)$. Then there exists a constant $C(p,r)$, depending only on p and r, such that for every f in $L^{p,2}(N)$ and g in $L^{r,\infty}(N)$*

$$\|f*g\|_{L^{q,2}}\leq C(p,r)\|g\|_{L^{r,\infty}}\|f\|_{L^{p,2}},$$

where

$$\frac{1}{p}+\frac{1}{r}=\frac{1}{q}+1$$

and $1<p<q<\infty$.

Lemma 3. ([7, p. 271]) *Let p and r be in $(1,\infty)$. Then there exists a constant $C(p,r)$, depending only on p and r, such that for every f in $L^{p,2}(N)$ and m in $L^{r,\infty}(N)$*

$$\|mf\|_{L^{q,2}}\leq C(p,r)\|m\|_{L^{r,\infty}}\|f\|_{L^{p,2}},$$

where

$$\frac{1}{q}=\frac{1}{p}+\frac{1}{r}$$

and $1<q<p<\infty$.

Definition. For real α, we define the *homogeneous Sobolev space* $\mathcal{H}^\alpha(N)$ to be the completion of the space of smooth functions with compact support on N with respect to the norm

$$\|f\|_{\mathcal{H}^\alpha}=\|\Delta^{\alpha/2}f\|_{L^2},\quad f\in C_c^\infty(N).$$

The spaces $\mathcal{H}^\alpha(N)$ and $\mathcal{H}^{-\alpha}(N)$ are dual with respect to the pairing

$$(f,g)=\int_N f(n)g(n)\,dn,\quad f\in\mathcal{H}^\alpha(N),\ g\in\mathcal{H}^{-\alpha}(N).$$

We have the following Sobolev immersion properties.

Proposition 4. *If* $-\frac{1}{2}Q<\alpha\leq 0$ *and*

$$\frac{1}{p}=\frac{1}{2}-\frac{\alpha}{Q},$$

then $L^{p,2}(N)$ *is contained in* $\mathcal{H}^\alpha(N)$*, and moreover*

$$\|f\|_{\mathcal{H}^\alpha}\leq C(\alpha)\|f\|_{L^{p,2}}, \quad f\in\mathcal{H}^\alpha(N).$$

Proof. If $\alpha=0$, the proposition is clearly true. If α is in $\left(-\frac{1}{2}Q,0\right)$, by item (iii) of Proposition 1, we have for f in $C_c^\infty(N)$,

$$\Delta^{\alpha/2}f=f*\mathcal{R}_{-\alpha},$$

where $\mathcal{R}_{-\alpha}=\Omega\mathcal{B}^{-(\alpha+Q)/4}$ and Ω is homogeneous of degree 0 and smooth away from the identity. Since $|\mathcal{R}_{-\alpha}(n)|\leq C\mathcal{B}^{-(\alpha+Q)/4}(n)$, we obtain

$$(\mathcal{R}_{-\alpha})^*(t)\leq C(\mathcal{B}^{-(\alpha+Q)/4})^*(t), \quad t\in(0,+\infty).$$

Now we compute $(\mathcal{B}^{-(\alpha+Q)/4})^*$. For $t>0$ we have

$$\begin{aligned}(\mathcal{B}^{-(\alpha+Q)/4})^*(t)&=\inf\{s\in\mathbf{R}^+:\left|\{n\in N:|\mathcal{B}^{-(\alpha+Q)/4}(n)|>s\}\right|\leq t\}\\&=\inf\left\{s\in\mathbf{R}^+:\int_0^{s^{-1/(Q+\alpha)}}u^{Q-1}\,du\leq\frac{t}{|\Sigma|}\right\}=Ct^{-(1+\alpha/Q)}.\end{aligned}$$

It follows that $\mathcal{R}_{-\alpha}$ is in $L^{r,\infty}(N)$ when $1/r=1+\alpha/Q$. If

$$\frac{1}{p}=\frac{1}{2}-\frac{\alpha}{Q},$$

then $1<p<2$ and

$$\frac{1}{p}+\frac{1}{r}=\frac{1}{2}+1.$$

Therefore, by Lemma 2, we obtain

$$\|\Delta^{\alpha/2}f\|_{L^2}=\|f*\mathcal{R}_{-\alpha}\|_{L^2}\leq C\|\mathcal{R}_{-\alpha}\|_{L^{r,\infty}}\|f\|_{L^{p,2}},$$

as required. □

By duality, we obtain the following result.

Corollary 5. *If* $0<\alpha<\frac{1}{2}Q$ *and*

$$\frac{1}{p}=\frac{1}{2}-\frac{\alpha}{Q},$$

then $\mathcal{H}^\alpha(N)$ *is contained in* $L^{p,2}(N)$, *and moreover*

$$\|f\|_{L^{p,2}}\le C(\alpha)\|f\|_{\mathcal{H}^\alpha},\quad f\in\mathcal{H}^\alpha(N).$$

We need the following characterization of Sobolev spaces, proved by Folland in the nonhomogeneous case for $\alpha>0$; his proof can be adapted without substantial changes to the case of homogeneous Sobolev spaces of any order.

If f is in $\mathcal{H}^\alpha(N)$, we shall write $\widetilde{E}_j f$ for the distributional derivatives of f.

Theorem 6. ([6, Theorem 4.10]) *Let* α *be real. Then* f *is in* $\mathcal{H}^{\alpha+1}(N)$ *if and only if for every* $j=1,\dots,\mathrm{d}_{\mathfrak{v}}$ *we have that* $\widetilde{E}_j f$ *is in* $\mathcal{H}^\alpha(N)$; *moreover the norms* $\|f\|_{\mathcal{H}^{\alpha+1}}$ *and* $\sum_{j=1}^{\mathrm{d}_{\mathfrak{v}}}\|\widetilde{E}_j f\|_{\mathcal{H}^\alpha}$ *are equivalent.*

The following multiplier theorem was proved in [2] for the real, complex and the quaternionic Iwasawa N-groups.

Theorem 7. *Let* $\mathcal{M}^d(N)$ *be the space of functions in* $C^\infty(N\setminus\{0\})$, *which are homogeneous of degree* d, *where* d *is in* $\mathbf{C}$ *and* $\mathrm{Re}(d)\le 0$. *If* $-\frac{1}{2}Q<\alpha\le\beta<\frac{1}{2}Q$, m *is in* $\mathcal{M}^d(N)$ *and* $\alpha-\beta=\mathrm{Re}(d)$, *then pointwise multiplication by* m *defines a bounded operator from* $\mathcal{H}^\beta(N)$ *to* $\mathcal{H}^\alpha(N)$.

Proof. For m in $\mathcal{M}^d(N)$, denote by $\Lambda(m)$ the operator defined by $\Lambda(m)f=mf$ for every measurable function f on N. We divide the proof into four steps.

(i) If $\alpha\le 0$ and $\beta\ge 0$, then $\Lambda(m)$ is bounded from $\mathcal{H}^\beta(N)$ to $\mathcal{H}^\alpha(N)$, for m in $\mathcal{M}^d(N)$ with $\mathrm{Re}(d)=\alpha-\beta$.

(ii) If $-1\le\alpha=\beta\le 1$, then $\Lambda(m)$ is bounded on $\mathcal{H}^\alpha(N)$, for m in $\mathcal{M}^d(N)$ with $\mathrm{Re}(d)=0$.

(iii) If $0<\alpha=\beta<\frac{1}{2}Q$, then $\Lambda(m)$ is bounded on $\mathcal{H}^\alpha(N)$, for m in $\mathcal{M}^d(N)$ with $\mathrm{Re}(d)=0$

(iv) By duality and complex interpolation $\Lambda(m)$ is bounded from $\mathcal{H}^\beta(N)$ to $\mathcal{H}^\alpha(N)$, for $-\frac{1}{2}Q<\alpha\le\beta<\frac{1}{2}Q$, $\mathrm{Re}(d)=\alpha-\beta$ and m in $\mathcal{M}^d(N)$.

Let us proceed with the proof.

(i) The case where $\alpha=\beta=0$ is trivial because if $\mathrm{Re}(d)=0$ any function in $\mathcal{M}^d(N)$ is bounded. Let $-\frac{1}{2}Q<\alpha\le 0\le\beta<\frac{1}{2}Q$, but $\alpha\ne\beta$. If m is in $\mathcal{M}^d(N)$, with $\mathrm{Re}(d)=\alpha-\beta$, then

$$|m(n)|\le C\mathcal{B}^{\mathrm{Re}(d)/4}(n),\quad n\in N\setminus\{0\}.$$

Proceeding as in the proof of Proposition 4, one can check that m is in $L^{r,\infty}(N)$ for $1/r=-\mathrm{Re}(d)/Q$. Moreover, for every f in $C_c^\infty(N)$, by Lemma 3, Proposition 4 and Corollary 5, we have

$$\|mf\|_{\mathcal{H}^\alpha} \leq C\|mf\|_{L^{q,2}} \leq C\|m\|_{L^{r,\infty}}\|f\|_{L^{p,2}} \leq C\|m\|_{L^{r,\infty}}\|f\|_{\mathcal{H}^\beta},$$

where

$$\frac{1}{q}=\frac{1}{2}-\frac{\alpha}{Q} \quad \text{and} \quad \frac{1}{p}=\frac{1}{q}-\frac{1}{r}=\frac{1}{2}-\frac{\beta}{Q}.$$

We conclude that $\Lambda(m)$ is bounded from $\mathcal{H}^\beta(N)$ to $\mathcal{H}^\alpha(N)$ for m in $\mathcal{M}^d(N)$ with $\mathrm{Re}(d)=\alpha-\beta$.

(ii) Let m be in $\mathcal{M}^d(N)$ with $\mathrm{Re}(d)=0$ and let f be in $\mathcal{H}^1(N)$. We shall prove that mf is in $\mathcal{H}^1(N)$ by using Theorem 6, i.e., we shall verify that $\widetilde{E}_j(mf)$ is in $\mathcal{H}^0(N)=L^2(N)$ for every $j=1,\dots,\mathrm{d}_{\mathfrak{v}}$. We have

$$\widetilde{E}_j(mf)=\widetilde{E}_j(m)f+m\widetilde{E}_j(f).$$

Since f is in $\mathcal{H}^1(N)$, by Theorem 6 the function $\widetilde{E}_j(f)$ is in $\mathcal{H}^0(N)$. Moreover the function m is in $\mathcal{M}^0(N)$, so $\widetilde{E}_j(m)$ is in $\mathcal{M}^{-1}(N)$. Therefore by step (i) we obtain

$$\|\widetilde{E}_j(mf)\|_{L^2} \leq \|\widetilde{E}_j m\|_{L^{Q,\infty}}\|f\|_{\mathcal{H}^1}+\|m\|_{L^\infty}\|\widetilde{E}_j f\|_{L^2} \leq C\|f\|_{\mathcal{H}^1}.$$

Thus, by Theorem 6, $\Lambda(m)$ is bounded on $\mathcal{H}^1(N)$. By real interpolation and duality, $\Lambda(m)$ is bounded on $\mathcal{H}^\alpha(N)$ for α in $[-1,1]$. Indeed, we have seen that if $\mathrm{Re}(d)=0$, then $\Lambda(m)$ is bounded on $\mathcal{H}^0(N)$ and on $\mathcal{H}^1(N)$. Hence $\Lambda(m)$ is bounded on $\mathcal{H}^\alpha(N)$ for every α in $[0,1]$ by real interpolation. By duality, $\Lambda(m)$ is bounded on $\mathcal{H}^\alpha(N)$ also for every α in $[-1,0]$.

(iii) Let $0<\alpha<\frac{1}{2}Q$ and let m be in $\mathcal{M}^d(N)$ with $\mathrm{Re}(d)=0$. For any positive integer s, denote by D^s a left-invariant differential operator of the form $D^s=\prod_{k=1}^s \widetilde{E}_{j_k}$, where $j_1,\dots,j_s$ are in $\{1,\dots,\mathrm{d}_{\mathfrak{v}}\}$. Let k be the positive integer such that $\alpha-k$ is in $[-1,0)$. By Theorem 6 it suffices to prove that $D^k(mf)$ is in $\mathcal{H}^{\alpha-k}(N)$, for every f in $\mathcal{H}^\alpha(N)$ and for every differential operator D^k of order k. By Leibniz' rule we have

$$D^k(mf)=\sum_{s=0}^k \binom{s}{k}(D^s m)(D^{k-s}f),$$

where D^0 is the identity operator. Note that, for $s=0,1,\dots,k$, the function $D^s m$ is in $\mathcal{M}^{d-s}(N)$ and the function $D^{k-s}f$ is in $\mathcal{H}^{\alpha-k+s}(N)$, by Theorem 6. Therefore, by steps (i) and (ii), the function $D^k(mf)$ is in $\mathcal{H}^{\alpha-k}(N)$ and $\Lambda(m)$ is bounded on $\mathcal{H}^\alpha(N)$.

(iv) Consider the case where $0<\alpha<\beta<\frac{1}{2}Q$ and let m be in $\mathcal{M}^d(N)$ with $\mathrm{Re}(d)=\alpha-\beta$. Consider the analytic family of operators $z\mapsto\Lambda(m^{-z\beta/d})$. Note that $m^{-z\beta/d}$ is in $\mathcal{M}^{-z\beta}(N)$. For $\mathrm{Re}(z)=0$, the operator $\Lambda(m^{-z\beta/d})$ is bounded on $\mathcal{H}^\beta(N)$, by step (iii). For $\mathrm{Re}(z)=1$, by step (i), the operator $\Lambda(m^{-z\beta/d})$ is bounded from $\mathcal{H}^\beta(N)$ to $\mathcal{H}^0(N)$. Therefore, by Stein's complex interpolation theorem [15], the operator $\Lambda(m^{-z\beta/d})$ is bounded from $\mathcal{H}^\beta(N)$ to $\mathcal{H}^{(1-\mathrm{Re}(z))\beta}(N)$ for $0\le\mathrm{Re}(z)\le1$. For $z=-d/\beta$ we have $(1-\mathrm{Re}(z))\beta=\alpha$. Thus the operator $\Lambda(m)$ is bounded from $\mathcal{H}^\beta(N)$ to $\mathcal{H}^\alpha(N)$. Duality or complex interpolation can be used for the case where $-\frac{1}{2}Q<\alpha<\beta<0$. This concludes the proof of the theorem. □

5. Inversion and Sobolev spaces

Denote by S the jacobian of the inversion σ, i.e.,

$$\int_N f(n)\,dn=\int_N f(\sigma(n))S(n)\,dn, \quad f\in C_c^\infty(N).$$

In [3] it is proved that, if the J^2-condition holds, then $S=\mathcal{B}^{-Q/2}$. The expression of the density S in the general case is not known; however it is easy to prove that S is homogeneous of degree $-2Q$.

For real α, define a linear operator T_α by

$$T_\alpha f(n)=S(n)^{1/2-\alpha/Q}f(\sigma(n)), \quad n\in N,$$

for every measurable function f on N.

It is easy to check that the operator T_0 is bounded on $L^2(N)$. In this section we prove that the operator T_α is bounded on $\mathcal{H}^\alpha(N)$, for every $-\frac{1}{2}Q<\alpha<\frac{1}{2}Q$, if and only if N is an Iwasawa N-group (see Theorem 9). The proof uses the fact, proved in [3], that the inversion σ on N is conformal if and only if the J^2-condition holds.

We recall that a map $\mu: N\to N$ is conformal if $d\mu$ maps horizontal tangent vectors to horizontal tangent vectors, and restricted to the space of such vectors is a multiple of an isometry at each point. Actually, in the proof of Theorem 9, we use the fact that $d\sigma$ maps horizontal tangent vectors to horizontal tangent vectors if and only if the J^2-condition holds.

Lemma 8. *The map $d\sigma$ maps horizontal tangent vectors to horizontal tangent vectors if and only if the J^2-condition holds.*

Proof. Note that $d\sigma$ maps horizontal tangent vectors to horizontal tangent vectors if and only if for every n in N and every V in $\mathfrak{v}$ there exists a vector V_n' in

$\mathfrak{v}$ such that

$$\widetilde{V}(f\circ\sigma)(n)=[\widetilde{V}_n'f](\sigma(n)),\quad f\in C_c^\infty(N).$$

It is enough to verify that the latter condition holds for the vectors of the basis $\{E_j\}_{j=1}^{\mathrm{d}_\mathfrak{v}}$ of $\mathfrak{v}$.

We have

$$(4)\qquad \begin{aligned}\widetilde{E}_j(f\circ\sigma)(n)&=\sum_{i=1}^{\mathrm{d}_\mathfrak{v}}(\widetilde{E}_j\sigma_{E_i})(n)\partial_{x_i}f(\sigma(n))+\sum_{k=1}^{\mathrm{d}_\mathfrak{z}}(\widetilde{E}_j\sigma_{U_k})(n)\partial_{z_k}f(\sigma(n))\\&=\sum_{i=1}^{\mathrm{d}_\mathfrak{v}}(\widetilde{E}_j\sigma_{E_i})(n)(\widetilde{E}_if)(\sigma(n))+\sum_{k=1}^{\mathrm{d}_\mathfrak{z}}G_{j,k}(n)\partial_{z_k}f(\sigma(n)),\end{aligned}$$

where, for V in $\mathfrak{n}$, we denote by σ_V the function on N given by $\sigma_V(n)=\langle\log\sigma(n),V\rangle$, and for $n=(X,Z)$,

$$G_{j,k}(n)=(\widetilde{E}_j\sigma_{U_k})(n)+\frac{1}{2}\sum_{i=1}^{\mathrm{d}_\mathfrak{v}}\langle\mathcal{B}(X,Z)^{-1}J_k\bar{\mathcal{A}}(X,Z)X,E_i\rangle(\widetilde{E}_j\sigma_{E_i})(n).$$

It follows that the theorem is proved if we show that

$$G_{j,k}\equiv0,\quad j=1,\dots,\mathrm{d}_\mathfrak{v},\ k=1,\dots,\mathrm{d}_\mathfrak{z},$$

if and only if the J^2-condition holds. We claim that

$$(5)\qquad G_{j,k}(X,Z)=\tfrac{1}{4}|X|^2\mathcal{B}(X,Z)^{-2}\langle J_kJ_ZX+X\langle Z,U_k\rangle-P_{J_\mathfrak{z}X}(J_kJ_ZX),E_j\rangle$$

for every (X,Z) in N, and for every $j=1,\dots,\mathrm{d}_\mathfrak{v}$ and $k=1,\dots,\mathrm{d}_\mathfrak{z}$.

It is easy to see that if the J^2-condition holds then this expression vanishes; on the other hand from this equality it follows that if $G_{j,k}\equiv0$, then

$$P_{J_\mathfrak{z}X}(J_WJ_ZX)-\langle Z,W\rangle X=-J_WJ_ZX,\quad X\in\mathfrak{v},\ W,Z\in\mathfrak{z}.$$

In particular for $\langle W,Z\rangle=0$, the last equality implies that J_WJ_ZX belongs to $J_\mathfrak{z}X$, i.e., the J^2-condition holds.

We now prove the claim, i.e., equality (5). Let $\bar{\mathcal{A}}_V(X,Z)=\langle\bar{\mathcal{A}}(X,Z)X,V\rangle$. Using formula (3) (see also [3, pp. 27–28]) we have

$$\widetilde{E}_j\mathcal{B}(X,Z)=\langle\mathcal{A}(X,Z)X,E_j\rangle$$

and

$$\widetilde{E}_j\bar{\mathcal{A}}_V(X,Z)=\tfrac{1}{2}\langle X,E_j\rangle\langle X,V\rangle+\langle\bar{\mathcal{A}}(X,Z)E_j,V\rangle-\tfrac{1}{2}\langle J_{[X,E_j]}X,V\rangle.$$

From now on we shall write

$$\mathcal{B}=\mathcal{B}(X,Z),\quad \mathcal{A}X=\mathcal{A}(X,Z)X \quad\text{and}\quad \bar{\mathcal{A}}X=\bar{\mathcal{A}}(X,Z)X.$$

Since $\sigma_{U_k}(X,Z)=-\mathcal{B}^{-1}z_k$ and $\sigma_{E_i}(X,Z)=-\mathcal{B}^{-1}\langle\bar{\mathcal{A}}X,E_i\rangle$, by (3) we obtain

$$\begin{aligned}(\widetilde{E}_j\sigma_{U_k})(X,Z)&=\mathcal{B}^{-2}(\widetilde{E}_j\mathcal{B})z_k-\tfrac12\mathcal{B}^{-1}\langle J_kX,E_j\rangle\\&=\mathcal{B}^{-2}\langle\mathcal{A}X,E_j\rangle z_k+\tfrac12\mathcal{B}^{-1}\langle X,J_kE_j\rangle\\&=\mathcal{B}^{-2}\left[\tfrac14|X|^2x_jz_k+\langle J_ZX,E_j\rangle z_k+\tfrac12\mathcal{B}\langle X,J_kE_j\rangle\right]\end{aligned}$$

and

$$\begin{aligned}(\widetilde{E}_j\sigma_{E_i})(X,Z)&=\mathcal{B}^{-2}(\widetilde{E}_j\mathcal{B})\langle\bar{\mathcal{A}}X,E_i\rangle-\mathcal{B}^{-1}(\widetilde{E}_j\bar{\mathcal{A}}_{E_i})(X,Z)\\&=\mathcal{B}^{-2}\langle\mathcal{A}X,E_j\rangle\langle\bar{\mathcal{A}}X,E_i\rangle-\tfrac12\mathcal{B}^{-1}\langle X,E_j\rangle\langle X,E_i\rangle-\mathcal{B}^{-1}\langle\bar{\mathcal{A}}E_j,E_i\rangle\\&\quad+\tfrac12\mathcal{B}^{-1}\langle J_{[X,E_j]}X,E_i\rangle.\end{aligned}\tag{6}$$

Therefore, since $\langle J_k\bar{\mathcal{A}}X,\bar{\mathcal{A}}X\rangle=0$, we get

$$\begin{aligned}G_{j,k}(n)&=(\widetilde{E}_j\sigma_{U_k})(n)\\&\quad+\mathcal{B}^{-2}\left[-\tfrac14\langle J_k\bar{\mathcal{A}}X,X\rangle x_j-\tfrac12\langle J_k\bar{\mathcal{A}}X,\bar{\mathcal{A}}E_j\rangle+\tfrac14\langle J_k\bar{\mathcal{A}}X,J_{[X,E_j]}X\rangle\right].\end{aligned}$$

Using properties (2), we shall treat the terms in the square brackets separately.

For the first term, we have

$$-\tfrac14\langle J_k\bar{\mathcal{A}}X,X\rangle x_j=-\tfrac1{16}|X|^2\langle J_kX,X\rangle x_j-\tfrac14\langle J_ZX,J_kX\rangle x_j=-\tfrac14|X|^2z_kx_j.$$

For the second term, we have

$$\begin{aligned}-\tfrac12\langle J_k\bar{\mathcal{A}}X,\bar{\mathcal{A}}E_j\rangle&=-\tfrac18|X|^2\langle J_k\bar{\mathcal{A}}X,E_j\rangle+\tfrac12\langle J_k\bar{\mathcal{A}}X,J_ZE_j\rangle\\&=\tfrac1{32}|X|^4\langle X,J_kE_j\rangle-\tfrac18|X|^2\langle J_ZX,J_kE_j\rangle\\&\quad+\tfrac18|X|^2\langle J_kX,J_ZE_j\rangle-\tfrac12\langle J_kJ_ZX,J_ZE_j\rangle\\&=\tfrac1{32}|X|^4\langle X,J_kE_j\rangle-\tfrac14|X|^2\langle J_ZX,J_kE_j\rangle+\tfrac14|X|^2z_kx_j\\&\quad+\tfrac12\langle J_ZJ_kX,J_ZE_j\rangle-\langle J_ZX,E_j\rangle z_k\\&=\left(\tfrac1{32}|X|^4-\tfrac12|Z|^2\right)\langle X,J_kE_j\rangle-\tfrac14|X|^2\langle J_ZX,J_kE_j\rangle\\&\quad+\tfrac14|X|^2z_kx_j-\langle J_ZX,E_j\rangle z_k.\end{aligned}$$

For the third term, we have

$$\begin{aligned}\tfrac14\langle J_k\bar{\mathcal{A}}X,J_{[X,E_j]}X\rangle&=\tfrac1{16}|X|^2\langle J_kX,J_{[X,E_j]}X\rangle-\tfrac14\langle J_kJ_ZX,J_{[X,E_j]}X\rangle\\&=\tfrac1{16}|X|^4\langle[X,E_j],U_k\rangle-\tfrac14\langle J_kJ_ZX,J_{[X,E_j]}X\rangle\\&=-\tfrac1{16}|X|^4\langle X,J_kE_j\rangle-\tfrac14\langle J_kJ_ZX,J_{[X,E_j]}X\rangle.\end{aligned}$$

Now we compute

$$\begin{aligned}\mathcal{B}^2 G_{j,k}(X,Z) &= \tfrac14|X|^2 x_j z_k + \langle J_Z X, E_j\rangle z_k + \tfrac12 \mathcal{B}\langle X, J_k E_j\rangle - \tfrac14 |X|^2 z_k x_j \\ &\quad + \left(\tfrac{1}{32}|X|^4 - \tfrac12|Z|^2\right)\langle X, J_k E_j\rangle - \tfrac14|X|^2\langle J_Z X, J_k E_j\rangle \\ &\quad + \tfrac14 |X|^2 z_k x_j - \langle J_Z X, E_j\rangle z_k - \tfrac{1}{16}|X|^4\langle X, J_k E_j\rangle \\ &\quad - \tfrac14\langle J_k J_Z X, J_{[X,E_j]}X\rangle \\ &= \tfrac14|X|^2\langle J_k J_Z X + \langle U_k, Z\rangle X, E_j\rangle - \tfrac14\langle J_k J_Z X, J_{[X,E_j]}X\rangle \\ &= \tfrac14|X|^2\langle J_k J_Z X + \langle U_k, Z\rangle X, E_j\rangle - \tfrac14|X|^2\langle J_k J_Z X, P_{J_{\mathfrak{z}}X} E_j\rangle \\ &= \tfrac14|X|^2\langle J_k J_Z X + X\langle Z, U_k\rangle - P_{J_{\mathfrak{z}}X}(J_k J_Z X), E_j\rangle.\end{aligned}$$

Equality (5) follows and the proof is complete. □

We shall use this lemma in the proof of the following theorem.

Theorem 9. *The operator T_α is bounded on $\mathcal{H}^\alpha(N)$ for every α in $\left(-\frac12 Q, \frac12 Q\right)$ if and only if the J^2-condition holds.*

Proof. The operator T_0 is clearly bounded on $\mathcal{H}^0(N)=L^2(N)$. Suppose that the J^2-condition holds. By Theorem 6, to show that $T_\alpha f$ is in $\mathcal{H}^\alpha(N)$, it suffices to prove that $\widetilde{E}_j(T_\alpha f)$ is in $\mathcal{H}^{\alpha-1}(N)$ for every $j=1, \dots, \mathrm{d}_{\mathfrak{v}}$.

First consider $\alpha=1$ and let f be in $\mathcal{H}^1(N)$. By formula (4) in the proof of Lemma 8, we have

$$\begin{aligned}\widetilde{E}_j(T_1 f) &= (\widetilde{E}_j S^{1/2-1/Q})(f\circ\sigma) + S^{1/2-1/Q}\widetilde{E}_j(f\circ\sigma) \\ &= (\widetilde{E}_j S^{1/2-1/Q})(f\circ\sigma) + S^{1/2-1/Q}\sum_{i=1}^{\mathrm{d}_{\mathfrak{v}}}(\widetilde{E}_j\sigma_{E_i})(\widetilde{E}_i f)\circ\sigma.\end{aligned}$$

Remember that S is homogeneous of degree $-2Q$, so that $S^{1/2}(\widetilde{E}_j S^{1/2-1/Q})\circ\sigma$ is homogeneous of degree -1, and by Theorem 7 we conclude that

$$\|(\widetilde{E}_j S^{1/2-1/Q})(f\circ\sigma)\|_{L^2} = \|S^{1/2}((\widetilde{E}_j S^{1/2-1/Q})\circ\sigma) f\|_{L^2} \le C\|f\|_{\mathcal{H}^1}.$$

Analogously, since $S^{1/Q}((\widetilde{E}_j\sigma_{E_i})\circ\sigma)$ is homogeneous of degree 0, we obtain

$$\begin{aligned}\|S^{1/2-1/Q}(\widetilde{E}_j\sigma_{E_i})((\widetilde{E}_i f)\circ\sigma)\|_{L^2} &= \|S^{1/Q}((\widetilde{E}_j\sigma_{E_i})\circ\sigma)\widetilde{E}_i f\|_{L^2} \\ &\le C\|\widetilde{E}_i f\|_{L^2} \le C\|f\|_{\mathcal{H}^1}.\end{aligned}$$

Therefore T_1 is bounded on $\mathcal{H}^1(N)$.

The case where $0<\alpha<1$ follows by complex interpolation arguing as in Theorem 7.

We write $[0,\frac{1}{2}Q)=\bigcup_{h=1}^{Q/2}[h-1,h)$ and proceed by induction on h. We have just proved that T_α is bounded on $\mathcal{H}^\alpha(N)$, when α is in $[0,1)$.

Now suppose that T_α is bounded on $\mathcal{H}^\alpha(N)$ when α is in $[h-1,h)$ and let α be in $[h,h+1)$. We have

$$\begin{aligned}\widetilde{E}_j(T_\alpha f)&=S^{1/2-\alpha/Q-1}(\widetilde{E}_jS)(f\circ\sigma)+S^{1/2-\alpha/Q}\sum_{i=1}^{p}(\widetilde{E}_j\sigma_{E_i})(\widetilde{E}_if)\circ\sigma\\&=T_{\alpha-1}[S^{1+1/Q}((\widetilde{E}_jS)\circ\sigma)f]+T_{\alpha-1}\left[S^{1/Q}\left(\sum_{i=1}^{\mathrm{d}_\mathfrak{v}}(\widetilde{E}_j\sigma_{E_i})\circ\sigma\right)\widetilde{E}_if\right].\end{aligned}$$

Note that the functions $S^{1+1/Q}((\widetilde{E}_jS)\circ\sigma)$ and $S^{1/Q}\left(\sum_{i=1}^{\mathrm{d}_\mathfrak{v}}(\widetilde{E}_j\sigma_{E_i})\circ\sigma\right)$ are homogeneous of degrees -1 and 0, respectively. Therefore, from the induction hypothesis and from Theorems 6 and 7, it follows that, for every $j=1,\dots,\mathrm{d}_\mathfrak{v}$,

$$\begin{aligned}\|\widetilde{E}_j(T_\alpha f)\|_{\mathcal{H}^{\alpha-1}}&\le C\|S^{1+1/Q}((\widetilde{E}_jS)\circ\sigma)f\|_{\mathcal{H}^{\alpha-1}}\\&\quad+C\left\|S^{1/Q}\left(\sum_{i=1}^{\mathrm{d}_\mathfrak{v}}(\widetilde{E}_j\sigma_{E_i})\circ\sigma\right)\widetilde{E}_if\right\|_{\mathcal{H}^{\alpha-1}}\\&\le C\|f\|_{\mathcal{H}^\alpha}+\sum_{i=1}^{\mathrm{d}_\mathfrak{v}}C_i\|\widetilde{E}_if\|_{\mathcal{H}^{\alpha-1}}\le C\|f\|_{\mathcal{H}^\alpha}.\end{aligned}$$

The case where $-\frac{1}{2}Q<\alpha<0$ follows by duality, since $T_\alpha=T^*_{-\alpha}$.

Conversely, we show that the operator T_α is unbounded on $\mathcal{H}^1(N)$ when N is not an Iwasawa group.

Let φ be in $C_c^\infty(N)$ such that $\varphi\equiv1$ in a neighbourhood $\mathcal{U}$ of the identity 0. Suppose that the J^2-condition does not hold and let $\tilde{n}$ be in $N\backslash\mathcal{U}$ such that $G_{j,k}(\tilde{n})\neq0$ for some j and k, where $G_{j,k}$ was defined in Lemma 8. Let $-4<\beta<-2$ and define

$$f(n)=\varphi(\tilde{n}^{-1}n)\mathcal{B}(\tilde{n}^{-1}n)^{-(\beta+Q)/8},\quad n\in N.$$

Note that the function f is in $\mathcal{H}^1(N)$, because it is compactly supported and it behaves like $\mathcal{B}(\tilde{n}^{-1}\cdot)^{-(\beta+Q)/8}$ in a neighbourhood of $\tilde{n}$ and $\beta<-2$.

We now show that T_1f is not in $\mathcal{H}^1(N)$. From formula (4) it follows that

$$\begin{aligned}\widetilde{E}_j(T_1f)(n)&=(\widetilde{E}_jS^{1/2-1/Q})(n)f(\sigma(n))+S(n)^{1/2-1/Q}\sum_{i=1}^{\mathrm{d}_\mathfrak{v}}(\widetilde{E}_j\sigma_{E_i})(n)(\widetilde{E}_if)(\sigma(n))\\&\quad+S(n)^{1/2-1/Q}\sum_{k=1}^{\mathrm{d}_\mathfrak{z}}G_{j,k}(n)\partial_{z_k}f(\sigma(n)).\end{aligned}$$

As proved before, all the functions in the previous sum are in $L^2(N)$ except for $S^{1/2-1/Q}G_{j,k}(\partial_{z_k}f)\circ\sigma$. Indeed, since $\widetilde{U}_k=\partial_{z_k}$ is left-invariant and homogeneous of degree -2, the function $[S^{1/Q}\,G_{j,k}\circ\sigma(\partial_{z_k}f)]^2$ behaves like $\mathcal{B}(\tilde{n}^{-1}\cdot)^{-(\beta+Q+4)/4}$ in a neighbourhood of $\tilde{n}$, hence it is not locally integrable, since $\beta>-4$. It follows that $\widetilde{E}_j(T_1f)$ is not in $L^2(N)$ so that, by Theorem 6, T_1f is not in $\mathcal{H}^1(N)$. □

6. Uniformly bounded representations

In this section, we assume that the J^2-condition holds, i.e., that N is the Iwasawa nilpotent subgroup of a connected, real-rank-one simple Lie group.

The group G of conformal transformations of $N\cup\{\infty\}$ is generated by translations, rotations, dilations and the inversion σ and it is isomorphic to the aforementioned simple group [9], [13], [14]. For g in G denote by $n\mapsto g^{-1}\cdot n$ the action of g on $N\cup\{\infty\}$ and by $\mathcal{J}_g$ its jacobian, i.e.,

$$\int_N f(n)\,dn=\int_N f(g^{-1}\cdot n)\mathcal{J}_g(n)\,dn,\quad f\in C_c^\infty(N).$$

Let α be a real number and define a representation π_α of G on $C_c^\infty(N)$ by the formula

$$\pi_\alpha(g)f(n)=\mathcal{J}_g(n)^{1/2-\alpha/Q}f(g^{-1}\cdot n),\quad g\in G,\ n\in N,\ f\in C_c^\infty(N).$$

Corollary 10. *Let $-\frac{1}{2}Q<\alpha<\frac{1}{2}Q$, then π_α is uniformly bounded on $\mathcal{H}^\alpha(N)$.*

Proof. Any element of G may be written as a product of translations, rotations, dilations and inversions, and there is a bound on the number of factors required in the product.

Let $-\frac{1}{2}Q<\alpha<\frac{1}{2}Q$. When g is a (left) translation or a rotation then $\pi_\alpha(g)$ acts unitarily on $\mathcal{H}^\alpha(N)$, since

$$\mathcal{J}_g\equiv 1\quad\text{and}\quad \Delta^{\alpha/2}\circ\pi_\alpha(g)=\pi_\alpha(g)\circ\Delta^{\alpha/2}.$$

When g is the inversion σ, then

$$\pi_\alpha(g)=T_\alpha$$

which is bounded on $\mathcal{H}^\alpha(N)$ by Theorem 9.

Let g be a dilation δ_t, for t real. We have

$$\pi_\alpha(g)f=t^{Q/2-\alpha}f\circ\delta_t$$

and

$$\Delta^{\alpha/2}(f\circ\delta_t)=t^{\alpha}(\Delta^{\alpha/2}f)\circ\delta_t.$$

Therefore

$$\begin{aligned}\|\pi_\alpha(g)\,f\|_{\mathcal{H}^\alpha}&=\|\Delta^{\alpha/2}(\pi_\alpha(g)f)\|_{L^2}=\|t^{Q/2-\alpha}\Delta^{\alpha/2}(f\circ\delta_t)\|_{L^2}\\&=\|t^{Q/2}(\Delta^{\alpha/2}f)\circ\delta_t\|_{L^2}=\|f\|_{\mathcal{H}^\alpha}.\quad\square\end{aligned}$$

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Francesca Astengo
Dipartimento di Matematica
Università di Genova
IT-16146 Genova
Italia
email: astengo@dima.unige.it

Bianca Di Blasio
Dipartimento di Matematica
Università di Roma "Tor Vergata"
IT-00133 Roma
Italia
email: diblasio@mat.uniroma2.it