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On a class of strongly hyperbolic systems

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Abstract. In this paper we prove a well-posedness result for the Cauchy problem. We study a class of first order hyperbolic differential [2] operators of rank zero on an involutive submanifold of $T^*\mathbf{R}^{n+1}\backslash\{0\}$ and prove that under suitable assumptions on the symmetrizability of the lifting of the principal symbol to a natural blow up of the "singular part" of the characteristic set, the operator is strongly hyperbolic.

1. Introduction

In this paper we are concerned with strong hyperbolicity for a class of first order systems with involutive characteristics. It is well known that the well-posedness of the Cauchy problem in the C^∞ category usually requires some conditions both on the principal symbol and on the lower order terms of the operator (see e.g. [5], [6] and [10]). However this is not always the case and we say that the Cauchy problem is *strongly hyperbolic* if it is well-posed for any lower order term.

It is thus natural to ask when a given system is strongly hyperbolic. Actually most of the existent literature is concerned with necessary conditions for strong hyperbolicity. We would like to refer to some of the most important contributions in this subject related to the present paper. In [8] Nishitani proved that, assuming that the coefficients of the system are real-analytic, strong hyperbolicity implies that the principal symbol at a multiple characteristic point has Jordan blocks of size at most 2 in its canonical form. In particular a strongly hyperbolic system of size m has rank $\leq \left[\frac{1}{2}m\right]$ at a characteristic point of multiplicity m. It is also easy to exhibit examples of even order m strongly hyperbolic systems with characteristic roots of multiplicity m whose principal symbol has rank $\left[\frac{1}{2}m\right]$ at those points. Actually these examples exhibit a propagation cone transverse to the characteristic manifold and this fact mimics what happens in the scalar case for effectively hyperbolic operators. On the other hand, when the rank exceeds $\left[\frac{1}{2}m\right]$ at a characteristic of multiplicity m, we in general expect a non-strongly hyperbolic operator, i.e. Levi conditions appear, as was pointed out, e.g., in [3].

It seems thus natural to ask about the strong hyperbolicity for operators with non-transverse (or even tangent) propagation cone.

When the multiple characteristic set (or manifold: we clarify this in the following) is not involutive we still expect that, even in the case of zero rank systems, there is no strong hyperbolicity. In the last section of the present paper we present an example of a non-strongly 3×3 hyperbolic system with rank zero on its non-involutive characteristic manifold consisting of points of multiplicity 3. The commutation relations of the symplectic coordinates prevent well-posedness independently of the lower order terms.

It is therefore natural to study a case in which the propagation cone at a multiple point is tangent to the characteristic manifold and the characteristic manifold is involutive.

In the present paper we deal with a first order system

$$L(x,D)=D_0+\sum_{j=1}^{n} A_j(x)D_j+B(x)=L_1(x,D)+L_0(x), \tag{1.1}$$

where $A_j(x)$ and $B(x)$ are C^∞, $m\times m$ matrices and we consider the Cauchy problem

$$\begin{cases} L(x,D)u=f, & x_0>0,\\ u|_{x_0=0}=g. \end{cases} \tag{1.2}$$

Denote by $h(x,\xi)=\det L_1(x,\xi)$ the determinant of the principal symbol L_1 and assume that h is hyperbolic with respect to ξ_0. Here $x=(x_0,x_1,\dots,x_n)=(x_0,x')\in \mathbf{R}^{n+1}$ and x_0 is the time variable.

We suppose that the characteristic set of h is a stratified involutive manifold; this means that each layer of the stratification is an involutive manifold Σ_j having the property that if $\varrho\in\Sigma_j$, then $T^\sigma_\varrho\Sigma_j\subset T_\varrho\Sigma_j$ and ϱ is a characteristic of h of multiplicity j. Here $T^\sigma_\varrho\Sigma_j$ denotes the dual of $T_\varrho\Sigma_j$ with respect to the symplectic form σ.

A typical situation in which this happens is when the localization of h at a point $\varrho\in\Sigma_m$, where m is the maximum multiplicity, is strictly hyperbolic on the normal space $N_\varrho\Sigma_m=T_\varrho\Sigma_m^\perp$ (here $T_\varrho\Sigma_m^\perp$ denotes the euclidean orthogonal of $T_\varrho\Sigma_m$); then there are only two layers corresponding to multiplicity m and multiplicity 1.

According to what has been said before, we consider a zero rank situation: this means that, if $\varrho\in\Sigma_j$, ϱ is a characteristic of multiplicity j. Thus the matrix $L_1(x,\xi)$, and hence the whole symbol L, can be decomposed in block diagonal form

$$\begin{bmatrix} L_{11} & 0\\ 0 & L_{22}\end{bmatrix},$$

where $L_{22}(\varrho)$ is elliptic and $(L_1)_{11}(\varrho)=0$.

Even in this geometrical framework we know that the assumptions do not guarantee that the operator L is a strongly hyperbolic operator. For instance the symbol

$$L_1(x,\xi)=\begin{bmatrix}\xi_0 & \xi_1 & 0\\ 0 & \xi_0 & 0\\ 0 & 0 & \xi_0-\xi_1\end{bmatrix}$$

satisfies the conditions we assumed up to now, but is not strongly hyperbolic. Its triple characteristic manifold is given by $\Sigma_3=\{(x,\xi)\,|\,\xi_0=\xi_1=0\}$. If $\xi_1\neq 0$ then we are at a double characteristic point and the canonical form of the principal symbol exhibits a 2×2 Jordan block. Hence $\Sigma=\Sigma_3\cup\Sigma_2\cup\Sigma_1$, where $\Sigma_1=\{(x,\xi)\,|\,\xi_0=\xi_1\neq 0\}$ and $\Sigma_2=\{(x,\xi)\,|\,\xi_0=0 \text{ and } \xi_1\neq 0\}$; of course Σ_j is involutive for every $j=1,2,3$.

In order to rule out the above cases we must add an extra assumption on the symmetrizability of the localization of L_1 on the normal space $N_\varrho\Sigma_j$ if $\Sigma_{j'}$, $1<j'<j$, is present. The above example does not satisfy this assumption. Actually this assumption turns out to be also necessary, according to Theorem 5.2 in [9].

The technique employed in this paper is inspired of the Melrose–Uhlmann paper [7]. We do not construct parametrices though, so that in order to construct a symmetrizer for our operator we need a precise way to define the composition within a class of singular pseudodifferential operators. Section 3 is devoted to this task. Once this is accomplished, the strong hyperbolicity is a consequence of the usual energy estimates; the latter can be deduced easily because of the uniform microlocal symmetrizability assumption (see (2.3) below for a precise statement) in Section 4. Section 5 is devoted to proving that a specific example of system with non-involutive characteristic manifold is not strongly hyperbolic. In particular the determinant of this system is a third order hyperbolic symbol with triple characteristics and its well-posedness is well known due to [1]. We conclude that in general we cannot expect strong hyperbolicity for systems with non-involutive characteristic manifold.

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2. Statement of the result

We consider the following Cauchy problem

$$\text{(2.1)}\qquad \begin{cases} L(x,D)u(x)=f(x) & \text{in } \{x\in U\,|\,x_0>0\},\\ u|_{x_0=0}=g & \text{in } \{x\in U\,|\,x_0=0\},\end{cases}$$

where U is an open subset of $\mathbf{R}^{n+1}$, $0\in U$, and f and g are given distributions. Here $x\in\mathbf{R}^{n+1}$, $x=(x_0,x_1,\dots,x_n)=(x_0,x')$, and L is a pseudodifferential first order matrix operator of size m of the form

$$L(x,D)=D_0+A(x,D')+B(x,D')=L_1(x,D)+L_0(x,D'), \tag{2.2}$$

where $A(x,\xi')$ (resp. $B(x,\xi')$) is an $m\times m$ matrix whose entries are homogeneous symbols of order 1 (resp., polyhomogeneous symbols of order 0) with respect to ξ'.

The operator L satisfies the assumption:

(H1) L is hyperbolic with respect to the direction $(0,e_0)=(x=0,\xi=(1,\dots,0))$ i.e., writing $h(x,\xi)=\det L_1(x,\xi)=\det(\xi_0+A(x,\xi'))$, the polynomial $\xi_0\mapsto h(x,\xi_0,\xi')$ has only real roots in the time covariable ξ_0.

Let us denote by $\Sigma=\{(x,\xi)\in T^*U\setminus\{0\}\,|\,h(x,\xi)=0\}$ the characteristic set of L. We assume that, for $1<l\le m$,

$(\mathrm{H2})_l$ The multiplicity of the points of Σ is at most m. Moreover, denoting by

$$\Sigma_l=\{(x,\xi)\in\Sigma\,|\,d^jh(x,\xi)=0,\ \ j=0,1,\dots,l-1,\ \text{ and } d^lh(x,\xi)\neq 0\},$$

the set of points of multiplicity l, we suppose that Σ_l is a non-radial involutive C^∞ submanifold of T^*U of codimension k_l+1, $k_l\ge 2$.

We further assume that:

$$\operatorname{Ker} L_1(x,\xi)\cap\operatorname{Im} L_1(x,\xi)=\{0\} \text{ for } (x,\xi)\in\Sigma_l\cap T^*U\setminus\{0\} \quad\text{and}\quad \dim\operatorname{Ker} L_1|_{\Sigma_l}=l. \tag{2.3}$$

Define now the smooth submanifold Σ_l' as the flowout of Σ_l along the Hamilton vector field of $-x_0$, i.e.

$$\Sigma_l'=\exp(tH_{-x_0})\Sigma_l,\quad |t|\le\delta, \tag{2.4}$$

where δ is a suitable positive number such that $\Sigma_l'\subset T^*U\setminus\{0\}$. Let $\bar z\in\Sigma_l$ and consider a (conic) neighborhood of $\bar z$, $V_{\bar z}\subset T^*U\setminus\{0\}$. We now want to blow up $T^*U\setminus\{0\}$ along the submanifold Σ_l'. As a set this blow up $(T^*U\setminus\{0\})^{\wedge}_{\Sigma_l'}$ is defined to be the union of $(T^*U\setminus\{0\})\setminus\Sigma_l'$ and $\mathrm{PN}\Sigma_l'$, the projective normal bundle of Σ_l' (see e.g. [4]).

The above mentioned neighborhood $V_{\bar z}$ is then lifted to an open subset $\widehat{V}_{\bar z}\subset(T^*U\setminus\{0\})^{\wedge}_{\Sigma_l'}$ and $A(x,\xi')$ can be extended to a smooth function—for the C^∞ structure of $(T^*U\setminus\{0\})^{\wedge}_{\Sigma_l'}$—defined in $\widehat{V}_{\bar z}$.

The final assumption is:

$(\mathrm{H3})_l$ A is uniformly symmetrizable in $\widehat{V}_{\bar z}$.

We can then state our first result.

Theorem 2.1. *Under the assumptions* (H1), $(\mathrm{H2})_l$ *and* $(\mathrm{H3})_l$ *the Cauchy problem* (2.1) *is microlocally strongly well-posed in a neighborhood of a point in* Σ_l.

A local version of the above theorem is the following result.

Theorem 2.2. *Assume that* Σ *is stratified as* $\bigcup_{j-1}^{m}\Sigma_j$, *where* Σ_j *are* C^∞ *manifolds of multiplicity* j. *Denote by* $\pi: T^*U\backslash\{0\}\to U$ *the natural projection and consider* $\pi^{-1}(U)\cap\Sigma$. *Assume that* (H1), $(\mathrm{H2})_j$ *and* $(\mathrm{H3})_j$ *are satisfied for* $j\geq 2$ *on* $\pi^{-1}(U)\cap\Sigma_j$. *Then* L *is strongly hyperbolic.*

When the localization h_z of $\det L_1$ is strictly hyperbolic we can prove the the following theorem.

Theorem 2.3. *Assume that* (H1) *and* $(\mathrm{H2})_m$ *hold and that* $T_z\Sigma_m=\Lambda(h_z)$, $z\in\Sigma_m$. *Then the Cauchy problem* (2.1) *is microlocally strongly well-posed in a neighborhood of a point in* Σ_l.

3. Some preparations

In this section we assume that $(\mathrm{H2})_m$ and $(\mathrm{H3})_m$ hold, i.e. we make our assumptions in the case of maximal multiplicity. The other cases, modulo a block reduction, are quite similar.

By assumption (H2) there are smooth functions $\varphi_j(x,\xi)$, $j=0,1,\dots,k$, homogeneous of degree 1 with respect to ξ such that

$$\Sigma_m=\{(x,\xi)\mid\varphi_j(x,\xi)=0,\ j=0,1,\dots,k\}.$$

We argue in a neighborhood of a fixed point $\bar{z}\in\Sigma_m$. It is easy to see that for at least one $j\in\{0,\dots,k\}$ we have $(\partial\varphi_j/\partial\xi_0)(\bar{z})\neq 0$. Let us say that $(\partial\varphi_0/\partial\xi_0)(\bar{z})\neq 0$. Hence we may assume that $\varphi_0(x,\xi)=\xi_0-\widetilde{\varphi}_0(x,\xi')$ and $\varphi_j=\varphi_j(x,\xi')$ for $j=1,\dots,k$, $\widetilde{\varphi}_0$ being homogeneous of degree 1 with respect to ξ'. By means of a canonical transformation leaving the hyperplane $\{x\,|\,x_0=0\}$ unchanged we may reduce $\varphi_0(x,\xi)$ to ξ_0 so that $\Sigma_m=\{(x,\xi)\,|\,\xi_0=0$ and $\varphi_j(x,\xi')=0,\ j=1,\dots,k\}$. This choice of coordinates yields that

$$\Sigma_m'=\{(x,\xi)\mid\varphi_j(x,\xi')=0,\ j=1,\dots,k\}.$$

By assumption (H2), Σ_m is an involutive submanifold of $T^*\mathbf{R}^{n+1}\backslash\{0\}$; this implies that

$$\{\xi_0,\varphi_j\}|_{\Sigma_m}=0\quad\text{and}\quad\{\varphi_i,\varphi_j\}|_{\Sigma_m}=0,$$

for every $i,\ j\in\{1,\dots,k\}$. Let us first consider the first set of brackets.

Lemma 3.1. *Due to our assumptions, the functions* φ_j, $j=1,\dots,k$, *can be chosen to be independent of the variable* x_0.

Proof. From the above mentioned bracket vanishing property we obtain that there are smooth functions $\alpha_{jl}(x,\xi')$, $j,l=1,\dots,k$, such that

$$\frac{\partial\varphi_j}{\partial\xi_0}(x,\xi')=\sum_{l=1}^{k}\alpha_{jl}(x,\xi')\varphi_l(x,\xi'),\quad j=1,\dots,k. \tag{3.1}$$

Let $U(x,\xi')$ be the $k\times k$ smooth matrix obtained by solving the Cauchy problem

$$\begin{cases}\dfrac{\partial U}{\partial x_0}(x,\xi')=-U(x,\xi')A(x,\xi'),\\ U|_{x_0=0}=I_k,\end{cases} \tag{3.2}$$

where A denotes the $k\times k$ matrix whose entries are the α_{jl} above. Set $\widetilde{\varphi}(x,\xi')=U(x,\xi')\varphi(x,\xi')$, where $\widetilde{\varphi}$ and φ denote two vectors with k components. Since $\partial_{x_0}\widetilde{\varphi}=(\partial_{x_0}U)\varphi+U(\partial_{x_0}\varphi)=(-UA+UA)\varphi\equiv 0$ we conclude that $\widetilde{\varphi}$ are actually functions of (x',ξ') only. Furthermore, U is a non-singular matrix, at least for small values of the variable x_0, i.e. $\varphi=0$ if and only if $\widetilde{\varphi}=0$. $\square$

As a consequence of Lemma 3.1 we may assume that the submanifold

$$\Sigma_m=\{(x,\xi)\in T^*\mathbf{R}^{n+1}\setminus\{0\}\mid \xi_0=0 \text{ and } \varphi_j(x',\xi')=0,\ j=1,\dots,k\},$$

where the functions φ_j have independent differentials at the points of Σ_m. It is then obvious that the manifold

$$\Sigma'_m=\{(x,\xi)\in T^*\mathbf{R}^{n+1}\setminus\{0\}\mid \varphi_j(x',\xi')=0,\ j=1,\dots,k\}$$

is cylindrical with respect to (x_0,ξ_0) and of course involutive. Hence there are homogeneous symplectic coordinates, which we still denote by (x,ξ), such that

$$\Sigma_m=\{(x,\xi)\in T^*\mathbf{R}^{n+1}\setminus\{0\}\mid \xi_0=\xi_1=\dots=\xi_k=0\}.$$

We point out that this homogeneous canonical transformation can be realized by a Fourier integral operator of order zero, elliptic at $\bar{z}\in\Sigma_m$, leaving the hyperplane $\{(x,\xi)x_0=0\}$ unchanged, due to Lemma 3.1. This will lead to a microlocal energy estimate for the system (2.1). Using these new coordinates (2.2) becomes

$$L(x,D)=D_0+\sum_{j=1}^{k}A_j(x,D')D_j+B(x,D')=L_1(x,D)+L_0(x,D'), \tag{3.3}$$

where the $A_j(x,\xi')$ are $m\times m$ matrices of order zero, homogeneous of degree zero with respect to ξ', depending smoothly on the parameter x_0. We notice that, in order to obtain the form (3.3) for L, the rank zero assumption has been used.

Next we consider the symbol $\sum_{j=1}^k A_j(x,\xi')\xi_j$ and want to regularize it by using a blow up technique.

Let us consider $(T^*U\backslash\{0\})^{\wedge}_{\Sigma'_m}$, the blow up of $T^*U\backslash\{0\}$ along the submanifold $\Sigma'_m=\{(x,\xi)\in T^*U\backslash\{0\}\,|\,\xi_1=\ldots=\xi_k=0\}$. It will be useful to adopt the following notation: for $j\in\{1,\ldots,n\}$, denote by $x_{(j)}$ the j-tuple $x_{(j)}=(x_1,\ldots,x_j)$ and by $x^{(j)}$ the $(n-j+1)$-tuple $x^{(j)}=(x_j,\ldots,x_n)$, so that $x=(x_0,x_{(j)},x^{(j+1)})$. Let us now take a look at $(T^*U\backslash\{0\})^{\wedge}_{\Sigma'_m}$. By definition it is the union of $(T^*U\backslash\{0\})\backslash\Sigma'_m$ and of the projective normal bundle of Σ'_m. Using our special coordinates the latter can be identified with the quotient of

$$\begin{aligned}\{((x,\xi_0,\xi_{(k)}=0,\xi^{(k+1)}),\delta\xi_{(k)},0)\,|\,\xi^{(k+1)}\neq 0 \text{ and } (\delta\xi)_{(k)}\neq 0\}\\ \equiv\{(x,\xi_0,\xi^{(k+1)},\delta\xi_{(k)})\,|\,\xi^{(k+1)}\neq 0 \text{ and } \delta\xi_{(k)}\neq 0\}\end{aligned}$$

by the equivalence relation obtained identifying two normal vectors when they are multiple one of the other via a non-zero real factor. This means that the above normal bundle can be described as

$$\{(x,\xi_0,\xi^{(k+1)},\delta\xi_{(k)})\,|\,\xi^{(k+1)}\neq 0 \text{ and } |\delta\xi_{(k)}|=1\}$$

and each couple of antipodal points on the unit $(k-1)$-dimensional sphere is identified with one point.

It can be easily seen (see e.g. [4] for a proof of this fact) that $(T^*U\backslash\{0\})^{\wedge}_{\Sigma'_m}$ can be identified with the quotient of

$$\mathbf{R}\times\mathbf{S}^{k-1}\times\widehat{U}_{(x,\xi_0,\xi^{(k+1)})},$$

where

$$\widehat{U}_{(x,\xi_0,\xi^{(k+1)})}=\{(x,\xi_0,\xi^{(k+1)})\,|\,\text{there exists a } \xi_{(k)} \text{ such that } (x,\xi)\in U\},$$

via the identification of $(x,\varrho,\omega,\xi_0,\xi^{(k+1)})$ with

$$i(x,\varrho,\omega,\xi_0,\xi^{(k+1)})=(x,-\varrho,-\omega,\xi_0,\xi^{(k+1)}).$$

Using these coordinates the symbol $L_1(x,\xi)$ in (3.3) becomes

$$L_1(x,\varrho,\omega,\xi_0,\xi^{(k+1)})=\xi_0+\varrho\sum_{j=1}^k A_j(x,\varrho,\omega,\xi^{(k+1)})\omega_j \tag{3.4}$$

on $(\mathbf{R}_\varrho\times\mathbf{S}^{k-1}_\omega\times\widehat{U})/i$. Now assumption (H3) means that there is a smooth matrix $S(x,\varrho,\omega,\xi^{(k+1)})$ defined in $(\mathbf{R}_\varrho\times\mathbf{S}^{k-1}_\omega\times\widehat{U})/i$ such that

(i) $S=S^*$;

(ii) S is positive definite;

(iii) $SA=A^*S$,

where $A=A(x,\varrho,\omega,\xi^{(k+1)})=\varrho\sum_{j=1}^k A_j(x,\varrho\omega,\xi^{(k+1)})\omega_j$. Let us now introduce a class of symbols naturally related to the blow up geometry. Without loss of generality we may assume that $\bar{z}=(0,e_n)$.

Definition 3.1. Let Ω be an open subset of $\mathbf{R}^{n+1}$ containing the origin and $l,m\in\mathbf{R}$. We say that

$$a(x,\varrho,\omega,\xi^{(k+1)})\in S^l(\mathbf{R}^+;S^m(\Omega\times\mathbf{S}^{k-1}_\omega\times\mathbf{R}^{k+1}_{\xi^{(k+1)}}))=S^l(\mathbf{R}^+;S^m)$$

if for every $K\Subset\Omega$ there exists a positive constant $C_{K,\beta,\alpha,s,Q}$ such that

$$|\partial_x^\beta\partial_\varrho^s\partial^\alpha_{\xi^{(k+1)}}Q(\omega,\partial_\omega)a(x,\varrho,\omega,\xi^{(k+1)})|\le C_{K,\beta,\alpha,s,Q}(1+\varrho)^{l-s}(1+|\xi^{(k+1)}|)^{m-|\alpha|},$$

where Q is a differential operator on $\mathbf{S}^{k-1}$.

From (iii) above it is easily seen that, possibly dividing S by $(\varrho^2+|\xi^{(k+1)}|^2)^{1/2}$, we can choose S as a function positively homogeneous of degree 0 with respect to ϱ and $\xi^{(k+1)}$.

This immediately implies that $S(x,\varrho,\omega,\xi^{(k+1)})\in S^0(\mathbf{R}^+;S^0(\Omega\times\mathbf{S}^{k-1}\times\mathbf{R}^{k+1}))$.

Remark. Assumption (H3) forces us to place ourselves in a neighborhood of $\bar{z}=(0,e_n)$ of the form $|\xi_{(k)}|\le\varepsilon|\xi^{(k+1)}|$, for a positive number ε. In the blow up coordinates this means that $\varrho\le\varepsilon|\xi^{(k+1)}|$. Therefore choosing $\sigma\in C_0^\infty(\mathbf{R})$, $\operatorname{supp}\sigma\subset[-1,1]$, $\sigma\equiv1$ on $\left[-\frac12,\frac12\right]$, we will use a cut-off function of the form

$$\sigma_1(\varrho,\xi^{(k+1)})=\sigma\left(\frac{\varrho}{\varepsilon|\xi^{(k+1)}|}\right),$$

and thus $\sigma_1\in S^0(\mathbf{R}^+;S^0)$. In particular the symbols of Definition 3.1 can be quantized in the usual way, provided they are supported (or cut off) in a region of the above form.

We now turn to the precise computation of the composition of two symbols in the above defined class.

Lemma 3.2. *Let* $a\in S^l(\mathbf{R}^+;S^m)$, $b\in S^{l'}(\mathbf{R}^+;S^{m'})$. *Then* $\operatorname{op}(a)\circ\operatorname{op}(b)=C_1+C_2+C_3+C_4$, *where* $\sigma(C_1)\in S^{l+l'}(\mathbf{R}^+;S^{m+m'})$, $\sigma(C_2)\in S^{-\infty}(\mathbf{R}^+;S^{m+m'})$, $\sigma(C_3)\in S^{l+l'}(\mathbf{R}^+;S^{-\infty})$, $\sigma(C_4)\in S^{-\infty}(\mathbf{R}^+;S^{-\infty})$.

Proof. A straightforward computation shows that

$$\partial_{\xi_{(k)}}=\omega\partial_\varrho+\frac{1}{\varrho}(I-\omega\otimes\omega)\partial_\omega, \tag{3.5}$$

where I denotes the $k\times k$ identity matrix, so that

$$\partial_{\xi_{(k)}}^{\alpha_{(k)}}=\sum_{s=0}^{|\alpha_{(k)}|}Q_{|\alpha_{(k)}|-s}(\omega,\partial_\omega)\varrho^{-|\alpha_{(k)}|+s}\partial_\varrho^s, \tag{3.6}$$

where $Q_{|\alpha_{(k)}|-s}(\omega,\partial_\omega)$ denotes a differential operator on $\mathbf{S}^{k-1}$ of order $|\alpha_{(k)}-s|$. Formula (3.6) implies that singularities at $\varrho=0$ appear whenever we take the composition of two symbols. Thanks to (3.6) the latter can be expressed as

$$a\#b=\sum_{\alpha\geq0}\frac{1}{\alpha!}\sum_{s=0}^{|\alpha_{(k)}|}Q_{|\alpha_{(k)}|-s}(\omega,\partial_\omega)\varrho^{-|\alpha_{(k)}|+s}\partial_\varrho^s\partial_{\xi^{(k+1)}}^{\alpha^{(k+1)}}a\cdot D_x^\alpha b=\sum_{i,j\geq0}\gamma_{i,j}, \tag{3.7}$$

where $\gamma_{i,j}\in S^{l+l'-i}(\mathbf{R}^+;S^{m+m'-j})$ for $i,j\geq0$. In order to make sense of the second sum in (3.7), at first we apply the usual asymptotic summation technique to $\sum_{i,j\geq0}\gamma_{i,j}$ for fixed values of the index i (see e.g. [4]). This is possible since the cut-off functions used to perform the sum $\sum_{i,j\geq0}\gamma_{i,j}$ are functions of $\xi^{(k+1)}$ alone and the product of a usual symbol in the variable $(x^{(k+1)},\xi^{(k+1)})$ by a symbol in the classes $S^l(\mathbf{R}^+;S^m)$ yields a symbol in the classes $S^l(\mathbf{R}^+;S^m)$. Moreover, the index i being fixed, there is no uniformity problem with respect to i. Therefore we obtain

$$\sum_{i,j\geq0}\gamma_{i,j}=g_i+r_i,$$

where $g_i\in S^{l+l'-i}(\mathbf{R}^+;S^{m+m'})$ and $r_i\in S^{l+l'-i}(\mathbf{R}^+;S^{-\infty})$. As a second step we take

$$\sum_{i\geq0}g_i+\sum_{i\geq0}r_i. \tag{3.8}$$

We do this by means of the same Borel summation technique with respect to the variable ϱ (the fact that $\varrho\in\mathbf{R}^+$ has no influence on the procedure). Again, analogously to the preceding argument only the variable ϱ matters. From the first summand in (3.8) we obtain a symbol $g+r$, where $g\in S^{l+l'}(\mathbf{R}^+;S^{m+m'})$ and $r\in S^{-\infty}(\mathbf{R}^+;S^{m+m'})$. On the other hand the same argument for the second summand in (3.8) yields a symbol $\tilde r_1+\tilde r_2$, with $\tilde r_1\in S^{l+l'}(\mathbf{R}^+;S^{-\infty})$ and $\tilde r_2\in S^{-\infty}(\mathbf{R}^+;S^{-\infty})$. This completes the proof of the lemma. □

Lemma 3.3. *Let $a\in S^l(\mathbf{R}^+;S^m)$; then*

$$a^*(x,D)=\gamma_1(x,D)+\gamma_2(x,D)+\gamma_3(x,D)+\gamma_4(x,D),$$

where $\gamma_1\in S^l(\mathbf{R}^+;S^m)$ with symbol coinciding with $\bar{a}$,

$$\gamma_2\in S^{l-1}(\mathbf{R}^+;S^m),\quad \gamma_3\in S^l(\mathbf{R}^+;S^{m-1})$$

and

$$\gamma_4(x,D)u=\int e^{i\langle x^{(k+1)}-y^{(k+1)},\xi^{(k+1)}\rangle}\gamma_4(x,y_{(k)},\xi^{(k+1)})u(y)\,dy\,d\xi^{(k+1)}, \tag{3.9}$$

where $\gamma_4(x,y_{(k)},\xi^{(k+1)})\in S^m(\mathbf{R}^n_x\times\mathbf{R}^k_{y_{(k)}}\times\mathbf{R}^{n-k}_{\xi^{(k+1)}})$.

Proof. It is analogous to that of the preceding lemma. □

Remark. We point out that $\operatorname{op}(\tilde{r}_1)$ is a smoothing operator. Moreover $\sigma_1\tilde{r}_1\in S^{-\infty}(\mathbf{R}^+;S^{-\infty})$, where σ_1 is the cut-off function introduced in the remark after Definition 3.1, so that the operator corresponding to $\sigma_1\tilde{r}_1$ is also a smoothing operator.

We want to take a look at $C_2=\operatorname{op}(c_2)$, $c_2\in S^{-\infty}(\mathbf{R}^+;S^m)$ for some $m\in\mathbf{R}$.

Lemma 3.4. *Let $a\in S^{-\infty}(\mathbf{R}^+;S^m)$. Then $\operatorname{op}(a)$ can be written as a smoothing operator in the $x_{(k)}$ variables with values in $OPS^m(\mathbf{R}^{n-k})$.*

Proof. From the expression

$$\begin{aligned}\operatorname{op}(a)u=\int & e^{i\varrho\langle x_{(k)}-y_{(k)},\omega\rangle+i\langle x^{(k+1)}-y^{(k+1)},\xi^{(k+1)}\rangle}\\ &\times a(x,\varrho,\omega,\xi^{(k+1)})u(y_{(k)},y^{(k+1)})\varrho^{k-1}\,d\varrho\,d\omega\,d\xi^{(k+1)}\,dy,\end{aligned}$$

we obtain that the distribution kernel of $\operatorname{op}(a)$ is

$$\begin{aligned}k_a(x,y)&=\int e^{i\varrho\langle x_{(k)}-y_{(k)},\omega\rangle+i\langle x^{(k+1)}-y^{(k+1)},\xi^{(k+1)}\rangle}a(x,\varrho,\omega,\xi^{(k+1)})\varrho^{k-1}\,d\varrho\,d\omega\,d\xi^{(k+1)}\\ &=\int e^{i\langle x^{(k+1)}-y^{(k+1)},\xi^{(k+1)}\rangle}a_L(x,y_{(k)},\xi^{(k+1)})\,d\xi^{(k+1)},\end{aligned}$$

since the (ϱ,ω)-integral is absolutely convergent. Thus the partial differential operator corresponding to the above kernel is

$$\begin{aligned}\operatorname{op}(a_L)u=\int & e^{i\langle x^{(k+1)}-y^{(k+1)},\xi^{(k+1)}\rangle}a_L(x,y_{(k)},\xi^{(k+1)})\\ &\times u(y_{(k)},y^{(k+1)})\,dy_{(k)}\,dy^{(k+1)}\,d\xi^{(k+1)}.\end{aligned}$$

It is straightforward to conclude that

$$a_L \in C^\infty(\mathbf{R}^k_{y_{(k)}}; S^m(\mathbf{R}^n_x \times \mathbf{R}^{n-k}_{\xi^{(k+1)}})).$$

This proves the lemma. □

It will be useful to refer to the L^2 continuity for our class of partial differential operators.

Lemma 3.5. *Let $a \in S^0(\mathbf{R}^+; S^0)$, then $a(x, D)$ is bounded in L^2.*

Proof. The proof proceeds along the lines of Theorem 18.1.11 in [4]. We start assuming that $a \in S^{-k-1}(\mathbf{R}^+; S^{-(n-k)-1})$; the standard calculus guarantees that, denoting by $K(x, y)$ the distribution kernel of a, $(x-y)^\alpha K(x, y)$ is the kernel of a symbol in

$$S^{-k-1-|\alpha_{(k)}|}(\mathbf{R}^+; S^{-(n-k)-1-|\alpha^{(k+1)}|})$$

for every $\alpha \geq 0$. As a consequence, by use of the Schur lemma, we can conclude that a is L^2 continuous if $a \in S^{-k-1}(\mathbf{R}^+; S^{-(n-k)-1})$. Let now $a \in S^l(\mathbf{R}^+; S^m)$, where $l, m \leq -1$. For $u \in \mathcal{S}$ we have

$$\|a(x, D)u\|^2 = \langle a^*(x, D)a(x, D)u, u\rangle,$$

where $a^*a(x, D) \in OPS^{2l}(\mathbf{R}^+; S^{2m})$. By the Cauchy–Schwarz inequality we are then reduced to the proof of the L^2 continuity for an operator having the symbol in the class $S^{2l}(\mathbf{R}^+; S^{2m})$. We get L^2 continuity if $l \leq -\frac{1}{2}(k+1)$ and $m \leq -\frac{1}{2}(n-k+1)$. Arguing by induction, in a finite number of steps, we conclude that $a \in S^l(\mathbf{R}^+; S^m)$ is L^2 continuous if $l, m \leq -1$. Let now $l \leq 1$. We want to show that if $a \in S^{-l}(\mathbf{R}^+; S^0)$, then $a(x, D)$ is L^2 continuous. This again uses an induction argument. As above we compute

$$\|a(x, D)u\|^2 = \langle a^*(x, D)a(x, D)u, u\rangle = \langle b_1(x, D)u, u\rangle,$$

where $b_1 \in S^{-2l}(\mathbf{R}^+; S^0)$. Now $\langle b_1(x, D)u, u\rangle \leq \|u\|\,\|b_1(x, D)u\|$, so that we are left with the proof of L^2 continuity for

$$b_1 \in S^{-2l}(\mathbf{R}^+; S^0).$$

Iterating this argument we obtain

$$\|a(x, D)u\|^2 \leq \|u\|\,\|b_1(x, D)u\| \leq \|u\|^{3/2}\|b_1(x, D)u\| \leq \ldots$$
$$\leq \|u\|^{1+1/2+\ldots+1/2^r}\|b_{r+1}(x, D)u\|^{1/2^r},$$

where $b_j\in S^{-2^j l}(\mathbf{R}^+;S^0)$. After a finite number of steps we reach the point where for instance $2^r l>2k+2$, so that

$$b_{r+1}(x,D)u(x)=\int e^{i\langle x^{(k+1)}-y^{(k+1)},\xi^{(k+1)}\rangle}\tilde{b}(x,y_{(k)},\xi^{(k+1)})u(y)\,dy\,d\xi^{(k+1)},$$

whose L^2 continuity is a consequence of the classical result, $\tilde{b}$ being a sufficiently smooth kernel in the variable $y_{(k)}$ and a symbol of order 0 in $(x^{(k+1)},\xi^{(k+1)})$. Thus

$$\|a(x,D)u\|^2\le C\|u\|^{1+1/2+\ldots+1/2^r+1/2^r}=C\|u\|^2.$$

Let us now consider $a\in S^0(\mathbf{R}^+;S^0)$. Let $M>2\sup|a|^2$ and set $c(x,\varrho,\omega,\xi^{(k+1)})=(M-|a(x,\varrho,\omega,\xi^{(k+1)})|^2)^{1/2}$. It is easy to see that c is a well-defined symbol in $S^0(\mathbf{R}^+;S^0)$. Now we have that

$$\begin{aligned}0&\le\langle c^*(x,D)c(x,D)u,u\rangle\\&=\langle(M-a^*(x,D)a(x,D))u,u\rangle+\langle\gamma_1(x,D)u,u\rangle+\langle\gamma_2(x,D)u,u\rangle,\end{aligned}$$

where

$$\gamma_1(x,\varrho,\omega,\xi^{(k+1)})\in S^{-1}(\mathbf{R}^+;S^0)\quad\text{and}\quad\gamma_2(x,\varrho,\omega,\xi^{(k+1)})\in S^0(\mathbf{R}^+;S^{-1}).$$

It is obvious that the L^2 continuity of γ_j yields the statement. That γ_1 is L^2 continuous follows from what has been said before. In order to get the L^2 continuity of γ_2 we must recall the microlocal nature of our symbols, namely γ_2 is a symbol having its support in a region where $\varrho\le\varepsilon|\xi^{(k+1)}|$. Since, in such a region, $(1+\varrho)/(1+|\xi^{(k+1)}|)\le 1$, we obtain that $\gamma_2\in S^0(\mathbf{R}^+;S^{-1})\subset S^{-1}(\mathbf{R}^+;S^0)$. This ends the proof of the lemma. ☐

4. The energy estimate

The proof of Theorem 2.1 relies on the deduction of an energy estimate for the operator L as in (3.4). Actually the deduction of such an estimate goes through a number of steps, the most important of which is an energy estimate in a conic neighborhood of the multiple characteristic manifold. In this section we accomplish such a task.

Let $S(x,D)$ be the operator obtained from the symbol

$$S(x,\varrho,\omega,\xi^{(k+1)})\in S^0(\mathbf{R}^+;S^0)$$

satisfying conditions (i)–(iii) following equation (3.4). Writing

$$L(x,D)=D_0+A(x,D')+B(x,D'), \tag{4.1}$$

where

$$A(x,\varrho,\omega,\xi^{(k+1)})=\varrho\sum_{j=1}^{k}A_j(x,\varrho,\omega,\xi^{(k+1)})\omega_j\in S^1(\mathbf{R}^+;S^0), \tag{4.2}$$

$$B(x,\varrho,\omega,\xi^{(k+1)})\in S^0(\mathbf{R}^+;S^0). \tag{4.3}$$

Let us compute, with $u=\mathrm{op}(\sigma_1)v$, $v\in C_0^\infty(\Omega)$,

$$2i\,\mathrm{Im}\langle(D_0+A(x,D')+B(x,D'))u,S(x,D')u\rangle=I_1+I_2+I_3.$$

We have

$$I_1=2i\,\mathrm{Im}\langle D_0u,S(x,D')u\rangle=D_0\langle u,Su\rangle-\langle u,D_0(S^*-S)u\rangle-\langle u,[S^*,D_0]u\rangle.$$

Let us consider the second term in the right-hand side above:

$$\langle u,D_0(S^*-S)u\rangle=\langle u,(S^*-S)D_0u\rangle+\langle u,[D_0,S^*-S]u\rangle.$$

The first summand can be written as

$$\langle(S-S^*)u,D_0u\rangle=\langle(S-S^*)u,Lu\rangle-\langle(S-S^*)u,Au+Bu\rangle \tag{4.4}$$

By Lemma 3.3 and conditions (i)–(iii) on $S(x,\varrho,\omega,\xi^{(k+1)})$ we obtain that

$$S-S^*=\gamma_2(x,D')+\gamma_3(x,D')+\gamma_4(x,D'),$$

where $\gamma_2(x,\varrho,\omega,\xi^{(k+1)})\in S^{-1}(\mathbf{R}^+;S^0)$, $\gamma_3(x,\varrho,\omega,\xi^{(k+1)})\in S^0(\mathbf{R}^+;S^{-1})$ and γ_4 has the form (3.9) where

$$\gamma_4(x,y_{(k)},\xi^{(k+1)})\in S^{-1}(\mathbf{R}^{n+1}_{(x_0,x')}\times\mathbf{R}^k_{y_{(k)}}\times\mathbf{R}^{n-k}_{\xi^{(k+1)}}).$$

The first term in (4.4) is estimated by

$$|\langle(S-S^*)u,Lu\rangle|\le\frac{1}{\delta}\|Lu\|^2+\delta\|u\|^2,\quad \delta>0.$$

Consider now

$$|\langle (S-S^*)u, Au\rangle| \le |\langle \gamma_2 u, Au\rangle| + |\langle \gamma_3 u, Au\rangle| + |\langle \gamma_4 u, Au\rangle|.$$

On the support of the cut-off function σ_1 the second term is equivalent to the first. By Lemmas 3.2, 3.3 and 3.5, $\gamma_2^* A$ is L^2 continuous. Moreover, since on the support of the cut-off function σ_1, $S^1(\mathbf{R}^+; S^0) \subset S^0(\mathbf{R}^+; S^1)$ we can also conclude that $|\langle \gamma_4 u, Au\rangle| \le C\|u\|^2$, so that

$$|\langle (S-S^*)u, Au\rangle| \le C\|u\|^2. \tag{4.5}$$

A similar, but easier argument, yields that

$$|\langle (S-S^*)u, Bu\rangle| \le C\|u\|^2. \tag{4.6}$$

Next let us consider $\langle u, [D_0, S^*-S]u\rangle$. The commutator inside the scalar product has the same form as S^*-S and hence is L^2 continuous. We argue for $\langle u, [S, D_0]u\rangle$ in an analogous way. Thus

$$I_1 = D_0\langle u, Su\rangle + R_1(u), \tag{4.7}$$

where $|R_1(u)| \le C_1\|u\|^2 + C_2\|Lu\|^2$. Now

$$\begin{aligned} I_2 &= 2i \operatorname{Im}\langle A(x, D')u, S(x, D')u\rangle \\ &= \langle (S^*-S)(x, D')A(x, D')u, u\rangle + \langle (SA - A^*S)(x, D')u, u\rangle. \end{aligned}$$

The inequality (4.5) allows us to estimate the first term. By condition (iii),

$$\sigma(SA - A^*S) = \alpha_1 + \alpha_2,$$

$\alpha_1 \in S^0(\mathbf{R}^+; S^0)$, and $\alpha_2 \in S^0(\mathbf{R}^{n+1}_{(x_0,x')} \times \mathbf{R}^k_{y(k)} \times \mathbf{R}^{n-k}_{\xi(k+1)})$ is quantized according to (3.9), where we used the fact that there is the cut-off function σ_1 in front of everything. Hence, by Lemma 3.5,

$$|I_2| \le C\|u\|^2. \tag{4.8}$$

The same argument as above gives us that

$$|I_3| \le C\|u\|^2. \tag{4.9}$$

Summing up, all the previous estimates give

$$2i \operatorname{Im}\langle Lu, Su\rangle = D_0\langle u, Su\rangle + R(u), \tag{4.10}$$

where

$$|R(u)| \leq C_1 \|u\|^2 + C_2 \|Lu\|^2. \tag{4.11}$$

Let τ be a positive large parameter; we have

$$\begin{aligned}-\int_0^{+\infty} e^{2\tau x_0} \partial_0 \langle u, Su\rangle \, dx_0 &= \langle u(0), S(0)u(0)\rangle + 2\tau \int_0^{+\infty} e^{2\tau x_0} \langle u, Su\rangle \, dx_0 \\ &\geq C\left(\|u(0)\|^2 + 2\tau \int_0^{+\infty} e^{2\tau x_0} \|u(x_0)\|^2 \, dx_0 \right),\end{aligned}$$

due to the properties of S. On the other hand

$$\begin{aligned}-\int_0^{+\infty} e^{2\tau x_0} \partial_0 \langle u, Su\rangle \, dx_0 &= -\int_0^{+\infty} e^{2\tau x_0} (2 \operatorname{Im} \langle Lu, Su\rangle + iR(u)) \, dx_0 \\ &\leq C\left(\int_0^{+\infty} e^{2\tau x_0} \|Lu\|^2 \, dx_0 + \int_0^{+\infty} e^{2\tau x_0} \|u\|^2 \, dx_0 \right).\end{aligned}$$

Taking τ large enough gives the microlocal energy estimates

$$\|u(0)\|^2 + \int_0^{+\infty} e^{2\tau x_0} \|u(x_0)\|^2 \, dx_0 \leq C \int_0^{+\infty} e^{2\tau x_0} \|Lu(x_0)\|^2 \, dx_0. \tag{4.12}$$

This ends the proof of Theorem 2.1.

We now prove Theorem 2.2.

Let us consider a partition of unity of $T^*K \setminus \{0\}$ where $K \Subset U$. We assume that the support of each element of the partition of unity is chosen in such a way that (a) either it intersects only one layer corresponding to a certain multiplicity l or (b) it contains points of a certain multiplicity l and points of lower multiplicity $l' \leq l$.

It is then clear that we may apply the above argument in a hierarchical way, starting from the set of maximum multiplicity.

As a preliminary remark we point out that near a point of multiplicity j, $2 \leq j \leq m$, the principal symbol can be block diagonalized uniformly in a neighborhood of the point, by assumption $(\mathrm{H2})_j$. This allows us to block diagonalize globally our system and thus reduce us microlocally to a $j \times j$ system, for which we already proved an estimate of the form (4.12):

$$\tau |\!|\!| \chi_\mu u |\!|\!|^2 \leq C |\!|\!| L\chi_\mu u |\!|\!|^2,$$

where $|\!|\!| u |\!|\!|^2 = \int_0^{+\infty} e^{2\tau x_0} \|u(x_0, \cdot)\|^2 \, dx_0$. Then

$$\tau |\!|\!| u |\!|\!|^2 \leq \tau \sum_\mu |\!|\!| \chi_\mu u |\!|\!|^2 \leq C(|\!|\!| Lu |\!|\!|^2 + \sum_\mu |\!|\!| [L, \chi_{mu}] u |\!|\!|^2 \leq C'(|\!|\!| Lu |\!|\!|^2 + |\!|\!| u |\!|\!|^2),$$

since the commutator $[L, \chi_{mu}]$ is L^2 continuous. Choosing τ sufficiently large proves the local energy estimate and thus strong hyperbolicity follows.

Let us now finally prove Theorem 2.2.

In this case our assumptions imply that $\Sigma = \Sigma_1 \cup \Sigma_m$, i.e. $h_z(\delta z)$ is a strictly hyperbolic polynomial of degree m in $T_z\Sigma^\perp$. Hence $(\mathrm{H3})_m$ holds since the lifting of A to a neighborhood in the blow up has distinct eigenvalues.

5. A counterexample

In this section we consider the following differential operator

$$L(x,D) = D_0 I_3 + A D_1 + B x_1 D_n,$$

where no zero order term is present,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}.$$

Here I_3 denotes the 3×3 identity matrix. It is easily seen that this system is hyperbolic, i.e. $\det L(x,\xi) = \xi_0(\xi_0^2 - \xi_1^2 - x_1^2\xi_n^2)$. Moreover the submanifold Σ of triple points is not involutive near $(0, e_n)$ and it can be checked that A and B are not simultaneously symmetrizable.

The study of the well-posedness of scalar operators with this principal symbol has been carried out in [1].

We want to prove the following result.

Proposition 5.1. *The Cauchy problem for L is not well-posed. In particular L is not strongly hyperbolic.*

Proof. We will choose a suitable asymptotic null solution of $LV \sim 0$ of the form $V = MU$, where $M = ({}^{co}L(x,D) + iGD_n)U$ with ${}^{co}L(x,\xi)$ denoting the cofactor matrix of L and G being a suitable constant matrix to be precised later.

Computing ${}^{co}L(x,\xi)$ we get

$${}^{co}L(x,\xi) = \begin{bmatrix} \xi_0(\xi_0-\xi_1) - x_1^2\xi_n^2 & -x_1\xi_n(\xi_0-\xi_1) + x_1^2\xi_n^2 & x_1^2\xi_n^2 - x_1\xi_n\xi_0 \\ -x_1\xi_n(\xi_0-\xi_1) - x_1^2\xi_n^2 & \xi_0^2 - \xi_1^2 + x_1^2\xi_n^2 & -x_1\xi_n(\xi_0+\xi_1) + x_1^2\xi_n^2 \\ x_1\xi_n\xi_0 + x_1^2\xi_n^2 & -x_1\xi_n(\xi_0+\xi_1) - x_1^2\xi_n^2 & \xi_0(\xi_0+\xi_1) - x_1^2\xi_n^2 \end{bmatrix}.$$

Now we have

$$LV = LMU = L({}^{co}L(x,D) + iGD_n)U = \Delta U,$$

where $\Delta = \Delta_3 + \Delta_2$, $\Delta_3(x,\xi) = (\det L(x,\xi)) I_3$ and

$$\Delta_2(x,\xi) = \frac{1}{i}(\partial_{\xi_1} L(x,\xi) \partial_{x_1} \cdot {}^{\mathrm{co}} L(x,\xi) - L(x,\xi) \cdot G\xi_n).$$

Choosing

$$G = \begin{bmatrix} 0 & 1 & 0 \\ -2 & 3 & 2 \\ 0 & -1 & 0 \end{bmatrix}$$

we get that

$$\Delta_2(x,\xi) = \begin{bmatrix} 0 & -2 & -1 \\ 2 & -3 & -2 \\ 1 & 2 & 0 \end{bmatrix} \frac{\xi_0 \xi_n}{i}.$$

The matrix in Δ_2 can be diagonalized since it has 3 distinct eigenvalues, one real and two complex conjugate. Denote by λ_j these eigenvalues. After a conjugation with a non-singular matrix, H, eventually we have that $\Delta(x,D)$ can be put in diagonal form

$$\Delta(x,\xi) = \begin{bmatrix} \det L(x,\xi) - i\lambda_1 \xi_0 \xi_n & 0 & 0 \\ 0 & \det L(x,\xi) - i\lambda_2 \xi_0 \xi_n & 0 \\ 0 & 0 & \det L(x,\xi) - i\lambda_3 \xi_0 \xi_n \end{bmatrix}.$$

An asymptotic null solution violating the a priori estimates is then easily found since we are now violating the Hörmander–Ivrii–Petkov Levi condition for an operator with double characteristics.

Finally, we must make sure that the above determined asymptotic solution does not annihilate the operator M. More precisely we must ascertain that

$$({}^{\mathrm{co}} L(x,D) + iGD_n) H^{-1} \widetilde{U} \neq 0,$$

where $\widetilde{U}$ is the null asymptotic solution mentioned above and H is the constant matrix used in the diagonalization of $\Delta(x,D)$. In order to do this we point out that

$$H^{-1} \widetilde{U}(x) = e^{i\lambda^2 x_n - \lambda^2 x_1^2/2 + i\lambda\varphi(x)} \tilde{u}_\lambda(x),$$

where $\varphi(x)$ is a phase function satisfying the equation $(\partial_0 \varphi)^2 - (1 + i\lambda_3) = 0$ and $\partial_1 \varphi(0) = 0$, λ is a large parameter and the Cauchy data for the eikonal equation are chosen suitably to make $\widetilde{U}$ a rapidly decreasing function.

When we compute ${}^{\mathrm{co}}L(x,D)$ at $x=0$ we easily see that

$$ {}^{\mathrm{co}}L(0,D)=\begin{bmatrix} D_0(D_0-D_1) & 0 & 0 \\ 0 & D_0^2-D_1^2 & 0 \\ 0 & 0 & D_0(D_0+D_1) \end{bmatrix}. $$

Therefore if we consider the element $(1,1)$ of the matrix M we only have that the function

$$ D_0(D_0-D_1)e^{i\lambda^2 x_n-\lambda^2 x_1^2/2+i\lambda\varphi(x)}\tilde{u}_\lambda(x) $$

must be non-zero. The latter is equal to

$$ e^{i\lambda^2 x_n-\lambda^2 x_1^2/2+i\lambda\varphi(x)}[(D_0+\lambda\partial_0\varphi)^2-(D_0+\lambda\partial_0\varphi)(D_1+i\lambda x_1+\lambda\partial_1\varphi)]\tilde{u}_\lambda \\ \sim\lambda^2(\partial_0\varphi)^2\tilde{u}_\lambda+O(\lambda), $$

which concludes the proof, since λ can be chosen large and $\partial_0\varphi$ does not vanish at the origin. □

References

1. Bernardi, E. and Bove, A., Necessary and sufficient conditions for the well-posedness of the Cauchy problem for a class of hyperbolic operators with triple characteristics, *J. Anal. Math.* **54** (1990), 21–59.
2. Bernardi, E. and Nishitani, T., Remarks on symmetrization of 2×2 systems and the characteristic manifolds, *Osaka J. Math.* **29** (1992), 129–134.
3. Bove, A. and Nishitani, T., Necessary conditions for hyperbolic systems, *Bull. Sci. Math.* **126** (2002), 445–479.
4. Hörmander, L., *The Analysis of Linear Partial Differential Operators.* Vol. I–IV, Springer-Verlag, Berlin, 1983–85.
5. Ivrii, V. Ya. and Petkov, V. M., Necessary conditions for the correctness of the Cauchy problem for non-strictly hyperbolic equations, *Uspekhi Mat. Nauk* **29**:5 (1974), 3–70 (Russian). English transl.: *Russian Math. Surveys* **29**:5 (1974), 1–70.
6. Kajitani, K., Strongly hyperbolic systems with variable coefficients, *Publ. Res. Inst. Math. Sci.* **9** (1973/74), 597–612.
7. Melrose, R. B. and Uhlmann, G. A., Microlocal structure of involutive conical refraction, *Duke Math. J.* **46** (1979), 571–582.
8. Nishitani, T., Necessary conditions for strong hyperbolicity of first order systems, *J. Anal. Math.* **61** (1993), 181–229.

9. NISHITANI, T., On localizations of a class of strongly hyperbolic systems, *Osaka J. Math.* **32** (1995), 41–69.
10. NISHITANI, T., Hyperbolicity for systems, in *Analysis and Applications—ISAAC 2001 (Berlin)*, pp. 237-252, Kluwer, Dordrecht, 2003.

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