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Cyclicity in the Dirichlet space

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Abstract. Let $\mathcal{D}$ be the Dirichlet space, namely the space of holomorphic functions on the unit disk whose derivative is square-integrable. We give a new sufficient condition, not far from the known necessary condition, for a function $f\in\mathcal{D}$ to be *cyclic*, i.e. for $\{pf:p \text{ is a polynomial}\}$ to be dense in $\mathcal{D}$.

The proof is based on the notion of Bergman–Smirnov exceptional set introduced by Hedenmalm and Shields. Our methods yield the first known examples of such sets that are uncountable. One of the principal ingredients of the proof is a new converse to the strong-type inequality for capacity.

1. Introduction

Let X be a Banach space of functions holomorphic in the open unit disk $\mathbf{D}$, such that the shift operator $M_z\colon f(z)\mapsto zf(z)$ is a continuous map of X into itself. Given $f\in X$, we denote by $[f]_X$ the smallest closed M_z-invariant subspace of X containing f, namely

$$[f]_X=\overline{\{pf:p \text{ is a polynomial}\}}.$$

We say that f is *cyclic* for X if $[f]_X=X$.

For example, in the case $X=H^2$, where

$$H^2:=\left\{f(z)=\sum_{k\geq 0}a_kz^k:\|f\|_{H^2}^2:=\sum_{k\geq 0}|a_k|^2<\infty\right\},$$

the cyclic vectors were identified by Beurling [2]. He showed that $f\in H^2$ is cyclic if and only if it is an outer function. This was part of his classification of the shift-invariant subspaces of H^2.

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In this article we shall be interested primarily in studying cyclic vectors in the case $X=\mathcal{D}$, where $\mathcal{D}$ is the Dirichlet space, defined by

$$\mathcal{D}:=\Big\{f(z)=\sum_{k\geq 0}a_kz^k : \|f\|_{\mathcal{D}}^2:=\sum_{k\geq 0}(k+1)|a_k|^2<\infty\Big\}.$$

Equivalently, $\mathcal{D}$ is the space of holomorphic functions whose derivative is square-integrable on $\mathbf{D}$, and we have

$$\|f\|_{\mathcal{D}}^2=\|f\|_{H^2}^2+\frac{1}{\pi}\int_{\mathbf{D}}|f'(z)|^2\,dA(z),$$

where dA denotes Lebesgue area measure on $\mathbf{D}$.

By a result of Richter and Sundberg [11, Theorem 5.3], if $\mathcal{M}$ is a closed M_z-invariant subspace of $\mathcal{D}$, then $\mathcal{M}=[f]_{\mathcal{D}}\cap\theta H^2$, where f is an outer function in $\mathcal{D}$ and θ is an inner function (not necessarily in $\mathcal{D}$). Thus a complete identification of the cyclic vectors would be a significant step towards a Beurling-type classification of the shift-invariant subspaces of $\mathcal{D}$. We refer the reader to the recent book of Ross and Shapiro [13] for much more information on this theme, especially the connection with pseudocontinuation, which will play a prominent role in what follows.

The study of cyclic vectors for $\mathcal{D}$ was instigated by Brown and Shields in [4]. Among many other results, they obtained the following necessary conditions for a function to be cyclic. Given $f\in\mathcal{D}$, we write $f^*(e^{i\theta}):=\lim_{r\to 1^-}f(re^{i\theta})$. It is known that this limit exists for all $e^{i\theta}$ in the unit circle $\mathbf{T}$ outside a set of logarithmic capacity zero [1].

Theorem 1.1. ([4]) *If f is cyclic for $\mathcal{D}$, then*
(i) *f is an outer function, and*
(ii) *$Z(f^*):=\{e^{i\theta}\in\mathbf{T}:f^*(e^{i\theta})=0\}$ is a set of logarithmic capacity zero.*

Brown and Shields further conjectured that, conversely, if (i) and (ii) hold for some $f\in\mathcal{D}$, then f is cyclic for $\mathcal{D}$. Whether this is true remains an open question. A weak form of the conjecture was proved by Hedenmalm and Shields [9], and subsequently improved by Richter and Sundberg [12]. Their combined results, which will be described in more detail in Section 2, lead to the following partial converse of Theorem 1.1.

Theorem 1.2. ([9], [12]) *Let $f\in\mathcal{D}$. Then f is cyclic for $\mathcal{D}$ provided that*
(i) *f is an outer function, and*
(ii) *$\underline{Z}(f):=\{e^{i\theta}\in\mathbf{T}:\liminf_{z\to e^{i\theta}}|f(z)|=0\}$ is countable.*

Our aim is in this paper is to try to bridge the gap between these two theorems by replacing ‘countable’ by a condition much closer to ‘capacity zero’. Given $E\subset\mathbf{T}$ and $t>0$, let us write $E_t:=\{e^{i\theta}\in\mathbf{T}:d(e^{i\theta},E)\le t\}$, where d denotes the distance with respect to arclength. Also, $|E_t|$ denotes the Lebesgue measure of E_t. The following theorem is our main result.

Theorem 1.3. *Let $f\in\mathcal{D}$. Then f is cyclic for $\mathcal{D}$ provided that*
(i) *f is an outer function, and*
(ii) *$\underline{Z}(f)=C\cup E$, where C is countable and E is a perfect set satisfying*

$$\int_0 \frac{|E_t|}{(t\log(1/t)\log\log(1/t))^2}\,dt<\infty. \tag{1}$$

How close is condition (1) to ‘capacity zero’? As an example, consider what happens when E is a Cantor set constructed in the usual way from the sequence $(l_n)_{n\ge0}$, where $\sup_n l_{n+1}/l_n<\frac{1}{2}$. (Thus, we begin with a closed arc of length l_0, remove an open arc from the middle to leave two closed arcs of length l_1, remove open arcs from their middles to leave four arcs of length l_2, etc.; then E is the intersection of the resulting nested sequence of sets.) We shall see later that E satisfies (1) provided that

$$\sum_{n\ge1}\frac{2^n}{n^2\log(1/l_n)}<\infty, \tag{2}$$

whereas it is well known (see e.g. [7, Section IV, Theorem 3]) that E is of logarithmic capacity zero if and only if it satisfies the stronger condition

$$\sum_{n\ge1}\frac{\log(1/l_n)}{2^n}=\infty. \tag{3}$$

Yet these conditions are not so far apart. For example, if $l_n=e^{-2^n/n^\sigma}$, then both (2) and (3) hold if $\sigma<1$, and neither holds if $\sigma>1$. The case $\sigma=1$ shows that the two are not quite equivalent.

The proof of Theorem 1.3 has four principal ingredients: (i) the notion of Bergman–Smirnov exceptional set, as introduced by Hedenmalm and Shields, (ii) a result about spectral synthesis in the Dirichlet space, (iii) the construction of a certain holomorphic semigroup in the Dirichlet space, and (iv) the following converse to the strong-type estimate for capacity, which we believe to be of interest in its own right.

Theorem 1.4. *Let E be a proper closed subset of $\mathbf{T}$, and let $\eta\colon(0,\pi]\to\mathbf{R}^+$ be a continuous, decreasing function. Then the following are equivalent:*

(i) *there exists* $f\in\mathcal{D}$ *such that for all* $e^{i\theta}\in\mathbf{T}$ *outside a set of capacity zero*

$$|f^*(e^{i\theta})|\geq\eta(d(e^{i\theta},E));$$

(ii) *the capacity* $c(E_t)$ *satisfies*

$$\int_0 c(E_t)\,d\eta^2(t)>-\infty.$$

The rest of the paper is organized as follows. The four ingredients of the proof listed above are treated in Sections 2, 3, 4 and 5, respectively. In Section 6 we combine these ideas to obtain a criterion for Bergman–Smirnov exceptional sets sufficient to yield a weak version of Theorem 1.3, in which the log-log term is omitted from (1). In Section 7 we describe the technical refinements needed to obtain the full version, in Section 8 we complete the proof of Theorem 1.3, and finally in Section 9 we outline the calculation leading to (2). There is also an appendix, wherein we gather a few general results about measure and capacity that are used in the rest of the paper.

2. Bergman–Smirnov exceptional sets

Recall that $\mathcal{D}$ consists of those $f(z)=\sum_{k\geq0}a_kz^k$ with $\sum_{k\geq0}(k+1)|a_k|^2<\infty$. The dual of $\mathcal{D}$ can thus be naturally identified with $\mathcal{B}_e$, the Bergman space on the exterior $\mathbf{D}_e$ of the closed unit disk, defined by

$$\mathcal{B}_e:=\left\{\phi(z)=\sum_{k\geq0}\frac{b_k}{z^{k+1}}:\|\phi\|^2_{\mathcal{B}_e}:=\sum_{k\geq0}\frac{|b_k|^2}{k+1}<\infty\right\},$$

the duality being given by the pairing

$$\langle f,\phi\rangle:=\sum_{k\geq0}a_kb_k,\quad f\in\mathcal{D},\ \phi\in\mathcal{B}_e.$$

Given $S\subset\mathcal{D}$, we write $S^\perp:=\{\phi\in\mathcal{B}_e:\langle f,\phi\rangle=0 \text{ for all } f\in S\}$. By the Hahn–Banach theorem, f is cyclic for $\mathcal{D}$ if and only if $[f]^\perp_{\mathcal{D}}=\{0\}$.

In [9], Hedenmalm and Shields proved an important analytic continuation theorem for functions in $[f]^\perp_{\mathcal{D}}$. To state their result, it is convenient to introduce some notation. Let $\mathcal{N}^+$ denote the Smirnov class, namely those functions holomorphic on $\mathbf{D}$ representable as g/h, where g and h are bounded and holomorphic on $\mathbf{D}$ and h is outer. Given a closed subset E of $\mathbf{T}$, we write

$$\mathcal{H}_E(\mathcal{N}^+,\mathcal{B}_e):=\{\phi \text{ holomorphic on } \mathbf{C}\setminus E:\phi|_{\mathbf{D}}\in\mathcal{N}^+ \text{ and } \phi|_{\mathbf{D}_e}\in\mathcal{B}_e\}.$$

Finally, E is called a *Bergman–Smirnov exceptional set* if $\mathcal{H}_E(\mathcal{N}^+,\mathcal{B}_e)=\{0\}$.

Theorem 2.1. ([9, Lemma 5]) *Let $f\in\mathcal{D}$ be an outer function belonging to the disk algebra. Then $[f]^{\perp}_{\mathcal{D}}\subset\mathcal{H}_E(\mathcal{N}^+,\mathcal{B}_e)$, where E is the zero set of f. Hence, if E is a Bergman–Smirnov exceptional set, then $[f]^{\perp}_{\mathcal{D}}=\{0\}$, and so f is cyclic.*

Theorem 2.1 was subsequently improved by Richter and Sundberg. In our terminology their result can be stated as follows.

Theorem 2.2. ([12, Corollary 3.3]) *Let $f\in\mathcal{D}$ be an outer function, let $E=\{e^{i\theta}\in\mathbf{T}:\liminf_{z\to e^{i\theta}}|f(z)|=0\}$, and suppose that E is a Bergman–Smirnov exceptional set. Then f is cyclic.*

Of course, the interest of these results depends upon being able to identify which sets are Bergman–Smirnov exceptional. Hedenmalm and Shields showed that every countable closed subset of $\mathbf{T}$ is Bergman–Smirnov exceptional [9, Theorem 3]. In conjunction with Theorem 2.2 above, this is sufficient to yield Theorem 1.2.

Until now, no other examples of Bergman–Smirnov exceptional sets were known. We shall prove the following theorem, which is a source of new examples.

Theorem 2.3. *Let E be a closed subset of $\mathbf{T}$ such that*

$$\int_0 \frac{|E_t|}{(t\log(1/t)\log\log(1/t))^2}\,dt<\infty. \tag{4}$$

Then E is a Bergman–Smirnov exceptional set.

Assuming this, it is easy to deduce Theorem 1.3.

Proof of Theorem 1.3. Write $\{e^{i\theta}\in\mathbf{T}:\liminf_{z\to e^{i\theta}}|f(z)|=0\}=C\cup E$, where C is countable and E satisfies (1). By Theorem 2.3, E is a Bergman–Smirnov exceptional set. As remarked in [9, p. 104], a closed subset of $\mathbf{T}$ is Bergman–Smirnov exceptional if and only if its perfect part is. It follows that $C\cup E$ too is Bergman–Smirnov exceptional. Now apply Theorem 2.2. $\square$

It thus suffices to prove Theorem 2.3. Most of the rest of the paper is devoted to this goal.

3. Spectral synthesis in the Dirichlet space

The first step on the way is a sort of converse to Theorem 2.1. It is a statement about spectral synthesis in the Dirichlet space.

Theorem 3.1. *Let E be a closed subset of $\mathbf{T}$, let $f\in\mathcal{D}$, and suppose that*

$$|f(z)|\le C\operatorname{dist}(z,E)^2,\quad z\in\mathbf{D}, \tag{5}$$

where C is a constant. Then $\mathcal{H}_E(\mathcal{N}^+,\mathcal{B}_e)\subset[f]^{\perp}_{\mathcal{D}}$.

As an easy consequence, we deduce the following result.

Corollary 3.2. *Let E be a closed subset of* $\mathbf{T}$, *and suppose that there exists a cyclic function $f\in\mathcal{D}$ satisfying* (5). *Then E is a Bergman–Smirnov exceptional set.*

Proof. By the theorem, $\mathcal{H}_E(\mathcal{N}^+,\mathcal{B}_e)\subset[f]_{\mathcal{D}}^{\perp}$. Since f is cyclic, $[f]_{\mathcal{D}}^{\perp}=\{0\}$. □

In particular, this allows us to give a new proof of the result of Hedenmalm and Shields [9, Proposition 1] that singletons are Bergman–Smirnov exceptional.

Corollary 3.3. *If $E=\{e^{i\theta}\}$, then E is a Bergman–Smirnov exceptional set.*

Proof. It is enough to consider the case $E=\{1\}$. Let $f(z)=(z-1)^2$. Clearly f satisfies (5). If we can show that f is cyclic for $\mathcal{D}$, then the result will follow from Corollary 3.2.

Let

$$\phi(z):=\sum_{k\geq 0}\frac{b_k}{z^{k+1}}\in[z-1]_{\mathcal{D}}^{\perp}.$$

Then $b_k=b_{k+1}$ for all k. Since $\sum_{k\geq 0}|b_k|^2/(k+1)<\infty$, this forces $b_k=0$ for all k, and so $\phi=0$. Thus $[z-1]_{\mathcal{D}}^{\perp}=\{0\}$, and $[z-1]_{\mathcal{D}}=\mathcal{D}$. In particular, $1\in[z-1]_{\mathcal{D}}$, so $(z-1)\in[(z-1)^2]_{\mathcal{D}}$, and hence, finally, $[(z-1)^2]_{\mathcal{D}}=[(z-1)]_{\mathcal{D}}=\mathcal{D}$, as desired. □

In the next section, we shall consider how to extend this result to more general sets.

We now turn to the proof of Theorem 3.1, which will occupy the rest of this section. The arguments that follow were strongly influenced by ideas from [15].

Lemma 3.4. *Let E be a closed subset of* $\mathbf{T}$ *with $|E|=0$, let $\phi\in\mathcal{H}_E(\mathcal{N}^+,\mathcal{B}_e)$ and let $f\in\mathcal{D}$. Suppose that the family of functions*

$$e^{i\theta}\longmapsto f(re^{i\theta})\phi(e^{i\theta}/r),\quad \tfrac{1}{2}<r<1, \tag{6}$$

is uniformly integrable on $\mathbf{T}$. *Then $\phi\in[f]_{\mathcal{D}}^{\perp}$.*

Proof. An elementary computation shows that

$$\langle f,\phi\rangle=\lim_{r\to 1^-}\frac{1}{2\pi}\int_0^{2\pi}f(re^{i\theta})\phi(e^{i\theta}/r)e^{i\theta}\,d\theta.$$

The uniform integrability condition allows us to pass the limit inside the integral, to obtain

$$\langle f,\phi\rangle=\frac{1}{2\pi}\int_0^{2\pi}f^*(e^{i\theta})\phi(e^{i\theta})e^{i\theta}\,d\theta, \tag{7}$$

where $f^*(e^{i\theta}):=\lim_{r\to 1^-} f(re^{i\theta})$, which exists a.e. on $\mathbf{T}$. Note that $\phi(e^{i\theta})$ exists a.e. on $\mathbf{T}$ because, by assumption, $|E|=0$. The uniform integrability condition also implies that $f^*\phi\in L^1(\mathbf{T})$.

Now on the unit disk, $\phi\in\mathcal{N}^+$ and $f\in\mathcal{D}$, so $f\phi\in\mathcal{N}^+$. Also the radial limit of $f\phi$ satisfies $(f\phi)^*=f^*\phi\in L^1(\mathbf{T})$. By Smirnov's generalized maximum principle (see e.g. [8, Theorem 2.11]), it follows that $f\phi\in H^1$. Therefore the integral in (7) vanishes and $\langle f,\phi\rangle=0$.

Repeating the same argument with $f(z)$ replaced by $z^n f(z)$, $n=0,1,2,...$, we conclude that $\phi\in[f]_{\mathcal{D}}^{\perp}$, as required. □

The next result furnishes an estimate for $\phi\in\mathcal{H}_E(\mathcal{N}^+,\mathcal{B}_e)$ valid for all closed subsets E of $\mathbf{T}$. A stronger estimate will be proved in Section 7 under more restrictive conditions on E.

Lemma 3.5. *Let E be a closed subset of $\mathbf{T}$, and let $\phi\in\mathcal{H}_E(\mathcal{N}^+,\mathcal{B}_e)$. Then there exists a constant C such that*

$$|\phi(z)|\le\frac{C}{\operatorname{dist}(z,E)^2},\quad 1<|z|<2. \tag{8}$$

Proof. Since $\phi|_{\mathbf{D}_e}\in\mathcal{B}_e$, we have

$$\phi(z)=\sum_{k\ge 0}\frac{b_k}{z^{k+1}},\quad |z|>1,$$

where $\sum_{k\ge 0}|b_k|^2/(k+1)<\infty$. By Schwarz's inequality, if $|z|>1$, then

$$\Bigl|\sum_{k\ge 0}\frac{b_k}{z^{k+1}}\Bigr|^2\le\Bigl(\sum_{k\ge 0}\frac{|b_k|^2}{k+1}\Bigr)\Bigl(\sum_{k\ge 0}\frac{k+1}{|z|^{2k+2}}\Bigr)=\Bigl(\sum_{k\ge 0}\frac{|b_k|^2}{k+1}\Bigr)\Bigl(\frac{|z|}{|z|^2-1}\Bigr)^2.$$

Hence, there exists a constant C_1 such that

$$|\phi(z)|\le\frac{C_1}{|z|-1},\quad |z|>1. \tag{9}$$

Also, since $\phi|_{\mathbf{D}}\in\mathcal{N}^+$, it follows that $\phi^*(e^{i\theta}):=\lim_{r\to 1^-}\phi(re^{i\theta})$ exists a.e. on $\mathbf{T}$, that $\log|\phi^*|\in L^1(\mathbf{T})$, and that, for $|z|<1$,

$$\log|\phi(z)|\le\frac{1}{2\pi}\int_0^{2\pi}\frac{1-|z|^2}{|e^{i\theta}-z|^2}\log|\phi^*(e^{i\theta})|\,d\theta.$$

Hence, there exists a constant C_2 such that

$$\log|\phi(z)|\le\frac{C_2}{1-|z|},\quad |z|<1. \tag{10}$$

The result now follows immediately upon feeding the estimates (9) and (10) into [15, Lemmas 5.8 and 5.9]. □

It is now straightforward to deduce the main result.

Proof of Theorem 3.1. Let $\phi\in\mathcal{H}_E(\mathcal{N}^+,\mathcal{B}_e)$. By Lemma 3.5, ϕ satisfies (8). Together with (5), this implies that the family of functions (6) is uniformly integrable. Note also that the existence of an $f\in\mathcal{D}\setminus\{0\}$ satisfying (5) automatically guarantees that $|E|=0$. By Lemma 3.4, $\phi\in[f]_{\mathcal{D}}^{\perp}$. □

4. A holomorphic semigroup in the Dirichlet space

To extend Corollary 3.3 to more general closed sets E, we need to know conditions under which there exists a cyclic $f\in\mathcal{D}$ such that (5) holds. There are two obvious requirements on E. Firstly, for f to be cyclic, E must be a set of logarithmic capacity zero (this follows from Theorem 1.1). And secondly, for f to satisfy (5), E must be a so-called *Carleson set*, i.e.

$$\int_0^{2\pi}\log d(e^{i\theta},E)\,d\theta>-\infty \tag{11}$$

(see e.g. [15, p. 1269]). Each of these conditions is also sufficient. Namely, if E is of capacity zero, then there exists a cyclic $f\in\mathcal{D}$, belonging to the disk algebra, whose zero set equals E (this is a result of Brown and Cohn [3], refining earlier work of Carleson [5]). Also, if E is a Carleson set, then there exists $f\in\mathcal{D}$ satisfying (5) (see [5], [15]). However, we do not know whether, if E is both of capacity zero and a Carleson set, one can choose f having both properties simultaneously.

In this section we develop a device designed to try to circumvent this problem. Our aim is to prove the following theorem.

Theorem 4.1. *Let E be a closed subset of $\mathbf{T}$. Suppose that there exists $f\in\mathcal{D}$ such that*

$$\operatorname{Re} f(z)\ge\log\log(2/\operatorname{dist}(z,E)) \text{ and } |\operatorname{Im} f(z)|\le 1,\quad z\in\mathbf{D}. \tag{12}$$

Then E is a Bergman–Smirnov exceptional set.

Comparing this result with Corollary 3.2, we see that it has the advantage that it is no longer required that f be cyclic. Of course, the growth condition on f has changed: we shall pursue this in the next section.

The proof of the theorem depends on a lemma about a certain holomorphic semigroup in $\mathcal{D}$. In what follows, given $\alpha\in(0,\pi/2)$, we shall write

$$\Omega_\alpha:=\{\lambda\in\mathbf{C}\setminus\{0\}:|\arg\lambda|<\pi/2-\alpha\}.$$

Lemma 4.2. *Let $f\in\mathcal{D}$, and suppose that there exists $\alpha\in(0,\pi/2)$ such that*

$$|\operatorname{Im} f(z)|\le\alpha,\quad z\in\mathbf{D}.$$

For $\lambda\in\Omega_\alpha$, define

$$g_\lambda(z)=\exp(-\lambda e^{f(z)}),\quad z\in\mathbf{D}.$$

Then $g_\lambda\in\mathcal{D}$ for $\lambda\in\Omega_\alpha$, the map $\lambda\mapsto g_\lambda\colon\Omega_\alpha\to\mathcal{D}$ is holomorphic, and $\|g_\lambda-1\|_{\mathcal{D}}\to0$ as $\lambda\to0$ through positive real values.

Proof. We begin by establishing pointwise estimates for $|g_\lambda|$ and $|g'_\lambda|$, namely

$$|g_\lambda(z)|\le1 \text{ and } |g'_\lambda(z)|\le\frac{|f'(z)|}{e\cos(\arg\lambda+\alpha)},\quad z\in\mathbf{D}. \tag{13}$$

To prove these, fix $\lambda\in\Omega_\alpha$, and write and $\lambda=re^{i\theta}$ and $f=u+iv$. Then

$$|g_\lambda(z)|=\exp(-re^{u(z)}\cos(\theta+v(z))),\quad z\in\mathbf{D}.$$

By assumption, $|\theta|<\pi/2-\alpha$ and $|v(z)|\le\alpha$, so $\cos(\theta+v(z))\ge0$, and the first estimate in (13) follows. Also,

$$|g'_\lambda(z)|=|f'(z)|re^{u(z)}\exp(-re^{u(z)}\cos(\theta+v(z))),\quad (z\in\mathbf{D}).$$

Using the elementary inequality $te^{-at}\le1/ea$, for $t,a>0$, we obtain the second estimate in (13).

Since $f\in\mathcal{D}$, it follows straightaway from (13) that $g_\lambda\in\mathcal{D}$ for all $\lambda\in\Omega_\alpha$.

If $\lambda_n\to\lambda_0$ in Ω_α, then $g_{\lambda_n}(z)\to g_{\lambda_0}(z)$ and $g'_{\lambda_n}(z)\to g'_{\lambda_0}(z)$ pointwise on $\mathbf{D}$, and the estimates (13) then allow us to apply the dominated convergence theorem to prove that $\|g_{\lambda_n}-g_{\lambda_0}\|_{\mathcal{D}}\to0$. Thus $\lambda\mapsto g_\lambda$ is continuous as a map: $\Omega_\alpha\to\mathcal{D}$. The usual argument involving Cauchy's theorem and Morera's theorem then shows that this same map is in fact holomorphic.

Finally, let $(\lambda_n)_{n\ge1}$ be a sequence of positive numbers tending to 0. It is clear that $g_{\lambda_n}(z)\to1$ and $g'_{\lambda_n}(z)\to0$ pointwise on $\mathbf{D}$. Once again, using (13) with the dominated convergence theorem, we obtain that $\|g_{\lambda_n}-1\|_{\mathcal{D}}\to0$, as required. □

Remark. The family $(g_\lambda)_{\lambda\in\Omega_\alpha}$ evidently satisfies

$$g_{\lambda_1+\lambda_2}=g_{\lambda_1}g_{\lambda_2},\quad \lambda_1,\lambda_2\in\Omega_\alpha,$$

so it is in fact a holomorphic semigroup in $\mathcal{D}$. However, we shall not make use of this.

Proof of Theorem 4.1. The condition (12) on $\operatorname{Im} f$ means that we can apply the preceding lemma with $\alpha=1$. Define g_λ as in the lemma. The condition (12) on $\operatorname{Re} f$ then implies that, for λ real and positive,

$$|g_\lambda(z)| \le \exp(-\lambda e^{\operatorname{Re} f(z)} \cos 1) \le (\operatorname{dist}(z,E)/2)^{\lambda \cos 1}, \quad z \in \mathbf{D}.$$

In particular, if $\lambda \ge 2/\cos 1$, then $|g_\lambda(z)| \le C \operatorname{dist}(z,E)^2$ for some constant C, and so by Theorem 3.1 we have $\mathcal{H}_E(\mathcal{N}^+, \mathcal{B}_e) \subset [g_\lambda]_{\mathcal{D}}^{\perp}$.

Now fix $\phi \in \mathcal{H}_E(\mathcal{N}^+, \mathcal{B}_e)$, and let $k \ge 0$. By what we have just proved, we have $\langle z^k g_\lambda, \phi\rangle = 0$ for all real $\lambda \ge 2/\cos 1$. Also, the map $\lambda \mapsto \langle z^k g_\lambda, \phi\rangle$ is holomorphic on Ω_1. By the identity principle, it follows that $\langle z^k g_\lambda, \phi\rangle = 0$ for all $\lambda \in \Omega_1$. In particular, letting $\lambda \to 0$ through positive values, and recalling that $g_\lambda \to 1$ in $\mathcal{D}$, we obtain $\langle z^k, \phi\rangle = 0$. As k is arbitrary, it follows that $\phi = 0$. Thus $\mathcal{H}_E(\mathcal{N}^+, \mathcal{B}_e) = \{0\}$, and E is a Bergman–Smirnov exceptional set. □

5. A converse to the strong-type estimate for capacity

Theorem 4.1 begs the following question: for which sets $E \subset \mathbf{T}$ it is possible to construct $f \in \mathcal{D}$ satisfying (12)? Equivalently, and more conveniently, for which sets $E \subset \mathbf{T}$ does there exist $f \in \mathcal{D}$ such that

$$\operatorname{Re} f^*(e^{i\theta}) \ge \log\log(\pi/d(e^{i\theta}, E)) \text{ and } |\operatorname{Im} f^*(e^{i\theta})| \le 1 \quad \text{q.e. on } \mathbf{T}? \tag{14}$$

(Here and in what follows, q.e. denotes 'quasi-everywhere', namely everywhere outside a set of capacity zero. Recall that, if $f \in \mathcal{D}$, then f^* exists q.e. on $\mathbf{T}$.) That the two questions really are equivalent is easily established using the Poisson integral formula. The purpose of this section is to provide an answer, in the form of a converse to the strong-type estimate for capacity.

We begin with some notation. Given a (Borel) probability measure μ on $\mathbf{T}$, we define

$$I(\mu) := \iint \log\left|\frac{1}{e^{i\theta} - e^{i\phi}}\right| d\mu(\theta)\, d\mu(\phi).$$

Writing $\hat{\mu}(n) := \int e^{-in\theta}\, d\mu(\theta)$, $n \in \mathbf{Z}$, a standard calculation (see e.g. [4, p. 294]) shows that

$$I(\mu) = \sum_{n \ge 1} \frac{|\hat{\mu}(n)|^2}{n}. \tag{15}$$

Let E be a proper closed subset of $\mathbf{T}$. We define its capacity $c(E)$ by

$$\frac{1}{c(E)} := \inf\{I(\mu) : \mu \text{ is a probability measure on } E\}.$$

In so doing, we are following [7]. The usual logarithmic capacity of E is then $e^{-1/c(E)}$. In particular, E is of logarithmic capacity zero if and only if $c(E)=0$. Recall also that E_t denotes the set of $e^{i\theta}\in\mathbf{T}$ such that $d(e^{i\theta},E)\le t$, where d is arclength distance.

We can now state the main result of this section, which we believe to be of independent interest. The equivalence between (i) and (iii) was already remarked at the end of Section 1.

Theorem 5.1. *Let E be a proper closed subset of $\mathbf{T}$, and let $\eta\colon(0,\pi]\to\mathbf{R}^+$ be a continuous, decreasing function. Then the following are equivalent*:

(i) *there exists $f\in\mathcal{D}$ such that*

$$|f^*(e^{i\theta})|\ge\eta(d(e^{i\theta},E))\quad \textit{q.e. on } \mathbf{T};$$

(ii) *there exists $f\in\mathcal{D}$ such that*

$$\operatorname{Re} f^*(e^{i\theta})\ge\eta(d(e^{i\theta},E)) \textit{ and } |\operatorname{Im} f^*(e^{i\theta})|\le 1\quad \textit{q.e. on } \mathbf{T};$$

(iii) *E and η satisfy*

$$\int_0 c(E_t)\,d\eta^2(t)>-\infty. \tag{16}$$

Taking $\eta(t)=\log\log(1/t)$ for small t, we immediately derive the following corollary, which answers the question posed at the beginning of the section.

Corollary 5.2. *Let E be a proper closed subset of $\mathbf{T}$. There exists $f\in\mathcal{D}$ satisfying* (14) *if and only if*

$$\int_0 c(E_t)\frac{\log\log(1/t)}{t\log(1/t)}\,dt<\infty.$$

Proof of Theorem 5.1. The implication (ii)⇒(i) is obvious.

For the implication (i)⇒(iii), observe that, if $|f^*(e^{i\theta})|\ge\eta(d(e^{i\theta},E))$ q.e., then

$$-\int_0 c(E_t)d\eta^2(t)\le-\int_0 c(\{z:|f^*(z)|\ge\eta(t)\})\,d\eta^2(t)=\int^\infty c(\{z:|f^*(z)|\ge s\})\,ds^2.$$

The last integral is finite for all $f\in\mathcal{D}$: this is just the strong-type estimate for capacity (see e.g. [10, Theorem 3.12], [4, p. 295] or [16, §2]).

It remains to prove the implication (iii)⇒(ii); in fact this is the one we actually need. Given a probability measure μ on $\mathbf{T}$, we define the holomorphic function $f_\mu\colon\mathbf{D}\to\mathbf{C}$ by

$$f_\mu(z)=-\int\log(1-ze^{-i\theta})\,d\mu(\theta),\quad z\in\mathbf{D}.$$

The following lemma lists some basic properties of f_μ.

Lemma 5.3. (i) $f_\mu(0)=0$;
(ii) $-\log(1+|z|)\le \operatorname{Re} f_\mu(z)\le -\log(1-|z|)$, $z\in\mathbf{D}$;
(iii) $|\operatorname{Im} f_\mu(z)|\le \pi/2$, $z\in\mathbf{D}$;
(iv) $f_\mu\in\mathcal{D}$ *if and only if* $I(\mu)<\infty$, *in which case* $I(\mu)=\|f_\mu(z)/z\|_{\mathcal{D}}^2$.

Proof. The first three parts are evident. For (iv), combine (15) with the observation that

$$f_\mu(z)=\int\sum_{n\ge1}\frac{(ze^{-i\theta})^n}{n}\,d\mu(\theta)=\sum_{n\ge1}\frac{\hat\mu(n)}{n}z^n,\quad z\in\mathbf{D}.\quad\square$$

We also require an elementary lemma about Hilbert spaces, whose proof is left to the reader.

Lemma 5.4. *Let* $(h_n)_{n\ge1}$ *be vectors in a Hilbert space* H. *Suppose that*

$$(h_m-h_n,h_n)_H=0\quad\text{whenever } m\ge n.$$

Then $\sum_{n\ge1}h_n/\|h_n\|_H^2$ *converges in* H *if and only if* $\sum_{n\ge1}n/\|h_n\|_H^2<\infty$.

With these lemmas in hand, we can return to the proof of the implication (iii)⇒(ii) in Theorem 5.1. If η is a bounded function, then we can just take f to be a large positive constant. So, from now on, we assume that $\lim_{t\to0^+}\eta(t)=\infty$. Let n_0 be a positive integer with $n_0\ge\eta(\pi)$. For $n\ge n_0$, we set $\delta_n:=\eta^{-1}(n)$ and $c_n:=c(E_{\delta_n})$. The condition (16) is then equivalent to

$$\sum_{n\ge n_0}nc_n<\infty.\tag{17}$$

Increasing n_0, if necessary, we can further suppose that

$$\sum_{n\ge n_0}c_n\le\frac{2}{\pi}.\tag{18}$$

For $n\ge n_0$, let μ_n be the equilibrium measure for E_{δ_n}, and let f_{μ_n} be defined as in Lemma 5.3. Define $f\colon\mathbf{D}\to\mathbf{C}$ by

$$f(z):=\sum_{n\ge n_0}c_nf_{\mu_n}(z),\quad z\in\mathbf{D}.\tag{19}$$

Since $|f_{\mu_n}(z)|\le\pi/2-\log(1-|z|)$, independently of n, and $\sum_n c_n<\infty$, it is clear that this series converges locally uniformly on $\mathbf{D}$.

We claim that $f\in\mathcal{D}$. To see this, set $h_n(z):=f_{\mu_n}(z)/z$. By Lemma 5.3 (iv), we have $\|h_n\|_{\mathcal{D}}^2=I(\mu_n)=1/c_n$, so (19) becomes

$$f(z)=z\sum_{n\geq n_0}\frac{h_n}{\|h_n\|_{\mathcal{D}}^2}.$$

By Lemma 5.4 this converges in $\mathcal{D}$ provided that $(h_m-h_n,h_n)_{\mathcal{D}}=0$, for $m\geq n$, and $\sum_n n/\|h_n\|_{\mathcal{D}}^2<\infty$. The latter condition is just $\sum_n nc_n<\infty$, which we know to be true from (17). As for the former, we remark that, by the polarization identity,

$$(h_m,h_n)_{\mathcal{D}}=\iint\log\left|\frac{1}{e^{i\theta}-e^{i\phi}}\right|d\mu_n(\theta)\,d\mu_m(\phi)=\int U_{\mu_n}\,d\mu_m,$$

where $U_{\mu_n}(z):=\int\log|1/(e^{i\theta}-z)|\,d\mu_n(\theta)$. By Frostman's theorem, $U_{\mu_n}\equiv I(\mu_n)$ on E_{δ_n}. Hence, if $m\geq n$, then

$$(h_m,h_n)_{\mathcal{D}}=\int I(\mu_n)\,d\mu_m=I(\mu_n)=(h_n,h_n)_{\mathcal{D}},$$

as desired. Thus the claim is justified.

We next estimate $\operatorname{Im} f^*$. Using Lemma 5.3 (iii) together with (18), we have

$$|\operatorname{Im} f(z)|\leq\sum_{n\geq n_0}c_n|\operatorname{Im} f_{\mu_n}(z)|\leq\sum_{n\geq n_0}c_n\frac{\pi}{2}\leq 1,\quad z\in\mathbf{D}.$$

It follows immediately that $|\operatorname{Im} f^*|\leq 1$ q.e. on $\mathbf{T}$.

Lastly, we estimate $\operatorname{Re} f^*$. Fix $e^{i\phi}\in\mathbf{T}$ so that $f^*(e^{i\phi})$ exists and $d(e^{i\phi},E)<\delta_{n_0}$. Let N be the integer such that $\delta_{N+1}\leq d(e^{i\phi},E)<\delta_N$. We have $\operatorname{Re} f_{\mu_n}(z)=U_{\mu_n}(z)\geq -\log 2$, $z\in\mathbf{D}$, whence, using (18),

$$\operatorname{Re} f(z)=\sum_{n\geq n_0}c_nU_{\mu_n}(z)\geq\sum_{n=n_0}^{N}c_nU_{\mu_n}(z)-\frac{2}{\pi}\log 2,\quad z\in\mathbf{D}.$$

As U_{μ_n} is lower semicontinuous, it follows that

$$\operatorname{Re} f^*(e^{i\phi})\geq\sum_{n=n_0}^{N}c_nU_{\mu_n}(e^{i\phi})-\frac{2}{\pi}\log 2.$$

Now $e^{i\phi}\in E_{\delta_n}$, $n_0\leq n\leq N$, so Frostman's theorem implies that $U_{\mu_n}(e^{i\phi})=I(\mu_n)$, $n_0\leq n\leq N$. Hence

$$\begin{aligned}\operatorname{Re} f^*(e^{i\phi})&\geq\sum_{n=n_0}^{N}c_nI(\mu_n)-\frac{2}{\pi}\log 2=N+1-n_0-\frac{2}{\pi}\log 2\\&\geq\eta(d(e^{i\phi},E))-n_0-\frac{2}{\pi}\log 2.\end{aligned}$$

Thus, if we replace f by $f+C$, where C is a large enough positive constant, then we will have $\operatorname{Re} f^*(e^{i\theta})\geq\eta(d(e^{i\theta},E))$ q.e. on $\mathbf{T}$. □

6. First results on Bergman–Smirnov exceptional sets

In this short section, we piece together what we have proved so far about Bergman–Smirnov exceptional sets. Our first result furnishes a sufficient condition in terms of capacity.

Theorem 6.1. *Let E be a proper closed subset of $\mathbf{T}$ such that*

$$\int_0 c(E_t)\frac{\log\log(1/t)}{t\log(1/t)}\,dt<\infty. \tag{20}$$

Then E is a Bergman–Smirnov exceptional set.

Proof. All the work is done: just combine Theorem 4.1 and Corollary 5.2. □

It is a result of Hedenmalm and Shields [9, Lemma 2] that every Bergman–Smirnov exceptional set is of capacity zero. Thus it is perhaps not surprising that the condition (20) should also be expressed in terms of capacity (indeed, capacity zero is equivalent to $\lim_{t\to 0} c(E_t)=0$, while (20) says that $\lim_{t\to 0} c(E_t)=0$ 'fast'). However, (20) is difficult to use in practice, and it is convenient to have an easier criterion, expressed in terms of the measures $|E_t|$ rather than the capacities $c(E_t)$. The connection between the two types of condition is described in the appendix (see Proposition A.4).

Theorem 6.2. *Let E be a closed subset of $\mathbf{T}$ such that*

$$\int_0 \frac{|E_t|}{(t\log(1/t))^2}\,dt<\infty. \tag{21}$$

Then E is a Bergman–Smirnov exceptional set.

Proof. It suffices to show that (21) implies (20). This is done applying Proposition A.4 with $\eta(t)=\log\log(1/t)$ for t close to zero. □

This is not far from the main result that we are trying to prove, Theorem 2.3. The only difference is an extra factor of $(\log\log(1/t))^2$ in the denominator of (4). The next section describes the refinement needed to obtain the stronger result.

7. A refined estimate

The starting point for all our results is Lemma 3.5, which gives a growth estimate for functions $\phi\in\mathcal{H}_E(\mathcal{N}^+,\mathcal{B}_e)$. This estimate takes no account of any special properties of the set E. It is reasonable to guess that, if E satisfies a Carleson-type condition of the sort that we are assuming anyway, then one can improve upon the estimate (8), and thereby obtain a stronger eventual conclusion. This turns out indeed to be the case.

The improved estimate that we are seeking is given by the following result.

Theorem 7.1. *Let $\alpha>0$, and let E be a closed subset of $\mathbf{T}$ satisfying*

$$|E_t|=O(t(\log(1/t))^{2\alpha})\quad \text{as } t\to 0^+. \tag{22}$$

Then, given $\phi\in\mathcal{H}_E(\mathcal{N}^+,\mathcal{B}_e)$, there exists a constant C such that

$$|\phi(z)|\le \frac{C}{\operatorname{dist}(z,E)}\left(\log\frac{4}{\operatorname{dist}(z,E)}\right)^{\alpha+1},\quad 1<|z|<2.$$

We shall examine the consequences of this theorem in the next section. The rest of this section is devoted to its proof, which proceeds via a number of lemmas.

Given $w\in\mathbf{D}$ and $f\in\mathcal{D}$, we define

$$T_w f(z):=\begin{cases}\dfrac{f(z)-f(w)}{z-w}, & z\in\mathbf{D}\setminus\{w\},\\ f'(w), & z=w.\end{cases}$$

Lemma 7.2. *T_w is a bounded linear operator: $\mathcal{D}\to\mathcal{D}$ with $\|T_w\|\le 1/(1-|w|)$.*

Proof. It is clear that $T_0\colon\mathcal{D}\to\mathcal{D}$ and that $\|T_0\|=1$. For $w\neq 0$, a simple computation shows that $T_wf-T_0f=wT_wT_0f$, $f\in\mathcal{D}$. Hence $T_w=T_0(I-wT_0)^{-1}$ and $\|T_w\|\le 1/(1-|w|)$. □

In what follows, $A(\mathbf{D})$ denotes the disk algebra. Also, given $f\in A(\mathbf{D})$, we write $Z(f)$ for the zero set of f. Recall that the dual of $\mathcal{D}$ can be identified with $\mathcal{B}_e$. The next result is a formula for the analytic continuation implicit in Theorem 2.1.

Lemma 7.3. *Let $f\in\mathcal{D}\cap A(\mathbf{D})$ and let $\phi\in[f]^{\perp}_{\mathcal{D}}$. Then ϕ extends holomorphically to $\mathbf{C}\setminus Z(f)$, and*

$$\phi(w)=\frac{\langle T_wf,\phi\rangle}{f(w)},\quad w\in\mathbf{D}\setminus Z(f).$$

Proof. See [9, Lemma 5]; the formula for ϕ appears in the penultimate line of the proof. □

This leads to the following abstract estimate.

Lemma 7.4. *Let E be a closed subset of* $\mathbf{T}$ *and let* $\phi\in\mathcal{H}_E(\mathcal{N}^+,\mathcal{B}_e)$. *Then*

$$|\phi(w)|\leq \inf_{f\in I_E}\frac{\|\phi\|_{\mathcal{B}_e}\|f\|_{\mathcal{D}}}{(1-|w|)|f(w)|},\quad w\in\mathbf{D}, \tag{23}$$

where

$$I_E:=\left\{f\in\mathcal{D}\cap A(\mathbf{D}):\sup_{z\in\mathbf{D}}\frac{|f(z)|}{\operatorname{dist}(z,E)^2}<\infty\right\}.$$

Proof. Let $f\in I_E$. By Theorem 3.1 we have $\phi\in[f]_{\mathcal{D}}^{\perp}$. Hence, using the two preceding lemmas,

$$|\phi(w)|\leq\frac{|\langle T_wf,\phi\rangle|}{|f(w)|}\leq\frac{\|\phi\|_{\mathcal{B}_e}\|T_wf\|_{\mathcal{D}}}{|f(w)|}\leq\frac{\|\phi\|_{\mathcal{B}_e}\|f\|_{\mathcal{D}}}{(1-|w|)|f(w)|},\quad w\in\mathbf{D}\setminus Z(f).\quad\square$$

Of course, this result is of interest only if $I_E\neq\{0\}$. As remarked at the beginning of Section 4, this happens if and only if E is a Carleson set (i.e. (11) holds). In this case, Carleson's construction actually yields an outer function f_0 such that $f_0''\in A(\mathbf{D})$ and $Z(f)=E$.

The idea will now be to take such an f_0, and modify it to construct functions $f\in I_E$ for which $\|f\|_{\mathcal{D}}/|f(w)|$ is relatively small, so that Lemma 7.4 yields good estimates for $|\phi(w)|$. For this, we need estimates both for $\|f\|_{\mathcal{D}}$ and $|f(w)|$ when f is an outer function. We begin with $\|f\|_{\mathcal{D}}$.

Lemma 7.5. *Let* $\rho\colon\mathbf{T}\to[0,1]$ *be a* C^1 *function such that, for some* $\beta\in(0,1)$,

$$I_\beta:=\int_{\mathbf{T}}\rho(\zeta)^{-\beta}\,|d\zeta|<\infty.$$

Define $f\colon\mathbf{D}\to\mathbf{C}$ *by*

$$f(z):=\exp\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\log\rho(e^{i\theta})\,d\theta\right),\quad z\in\mathbf{D}. \tag{24}$$

Then $f\in\mathcal{D}$ *and*

$$\|f\|_{\mathcal{D}}^2\leq 1+\frac{\pi e^2}{8}Mm\left(\log M+\frac{2}{\beta}\log\frac{I_\beta}{m}\right)^2,$$

where $M:=\max(e^{4/\beta},\|\rho'\|_\infty)$ *and* $m:=|\{\zeta:\rho(\zeta)<1\}|$.

Proof. We have

$$\|f\|_{\mathcal{D}}^2=\|f\|_{H^2}^2+\frac{1}{\pi}\int_{\mathbf{D}}|f'(z)|^2\,dA(z).$$

Since $|f^*|=\rho\le 1$ a.e., it follows that $\|f\|_{H^2}^2\le 1$. The problem is to estimate the Dirichlet integral. For this, we use a formula of Carleson [6] for the Dirichlet integral of an outer function: for f defined as above,

(25)

$$\frac{1}{\pi}\int_{\mathbf{D}}|f'(z)|^2\,dA(z)=\left(\frac{1}{2\pi}\right)^2\int_{\mathbf{T}}\int_{\mathbf{T}}\frac{(\rho(\zeta)^2-\rho(\zeta')^2)(\log|\rho(\zeta)|-\log|\rho(\zeta')|)}{|\zeta-\zeta'|^2}\,|d\zeta|\,|d\zeta'|.$$

Let J denote the right-hand side of (25). By the mean-value theorem, $|\rho(\zeta)-\rho(\zeta')|\le Md(\zeta,\zeta')$, where d denotes arclength distance on $\mathbf{T}$. Since $d(\zeta,\zeta')\le(\pi/2)|\zeta-\zeta'|$, we obtain

$$J\le\frac{M}{8}\int_{\mathbf{T}}\int_{\mathbf{T}}\frac{|\log\rho(\zeta)-\log\rho(\zeta')|}{d(\zeta,\zeta')}\,|d\zeta|\,|d\zeta'|.$$

Now, if $\rho(\zeta)\le\rho(\zeta')$, then

$$|\log\rho(\zeta)-\log\rho(\zeta')|\le\frac{M}{\rho(\zeta)}d(\zeta,\zeta')\quad\text{and}\quad|\log\rho(\zeta)-\log\rho(\zeta')|\le\log\frac{1}{\rho(\zeta)}.$$

Hence, for each $\varepsilon\in(0,1)$,

$$\begin{aligned}\iint\limits_{\rho(\zeta)\le\rho(\zeta')}\frac{|\log\rho(\zeta)-\log\rho(\zeta')|}{d(\zeta,\zeta')}\,|d\zeta|\,|d\zeta'|&\le\int_{\mathbf{T}}\int_{\mathbf{T}}\left(\frac{M}{\rho(\zeta)}\right)^{\varepsilon}\left(\frac{\log(1/\rho(\zeta))}{d(\zeta,\zeta')}\right)^{1-\varepsilon}|d\zeta|\,|d\zeta'|\\&\le\int_{\mathbf{T}}\left(\frac{M}{\rho(\zeta)}\right)^{\varepsilon}\left(\log\frac{1}{\rho(\zeta)}\right)^{1-\varepsilon}\frac{2\pi}{\varepsilon}\,|d\zeta|.\end{aligned}$$

A similar estimate holds on the set $\{\zeta:\rho(\zeta)\ge\rho(\zeta')\}$. Adding the two, we obtain

$$J\le\frac{\pi}{2}\frac{M^{1+\varepsilon}}{\varepsilon}\int_{\mathbf{T}}\frac{1}{\rho(\zeta)^{\varepsilon}}\left(\log\frac{1}{\rho(\zeta)}\right)^{1-\varepsilon}|d\zeta|\le\frac{\pi}{2}\frac{M^{1+\varepsilon}}{\varepsilon^2}\int_{\{\rho<1\}}\frac{1}{\rho(\zeta)^{2\varepsilon}}\,|d\zeta|.$$

Now, provided that $\varepsilon\le\beta/2$, we have

$$\int_{\{\rho<1\}}\frac{1}{\rho^{2\varepsilon}}\,|d\zeta|=m\int_{\{\rho<1\}}\frac{1}{\rho^{2\varepsilon}}\frac{|d\zeta|}{m}\le m\left(\int_{\{\rho<1\}}\frac{1}{\rho^{\beta}}\frac{|d\zeta|}{m}\right)^{2\varepsilon/\beta}=m^{1-2\varepsilon/\beta}I_{\beta}^{2\varepsilon/\beta}.$$

Hence

$$J\le\frac{\pi}{2}Mm\frac{K^{\varepsilon}}{\varepsilon^2},\quad\text{where } K=M(I_\beta/m)^{2/\beta}.$$

This is minimized by taking $\varepsilon=2/\log K$, which gives the result. □

Lemma 7.6. *Let ρ, f, I_β and m be as in the preceding lemma. Then*

$$|f(z)| \geq \exp\left(-\frac{2}{\pi\beta}\frac{m\log(I_\beta/m)}{1-|z|}\right), \quad z \in \mathbf{D}.$$

Proof. For $z\in\mathbf{D}$, we have

$$\log\frac{1}{|f(z)|} = \frac{1}{2\pi}\int_{\mathbf{T}}\frac{1-|z|^2}{|\zeta-z|^2}\log\frac{1}{\rho(\zeta)}\,|d\zeta| \leq \frac{1}{\pi}\frac{1}{1-|z|}\int_{\mathbf{T}}\log\frac{1}{\rho(\zeta)}\,|d\zeta|.$$

Now, for each $\lambda\in(0,1)$,

$$\int_{\{\rho<\lambda\}}\log\frac{1}{\rho(\zeta)}\,|d\zeta| \leq \lambda^\beta\log(1/\lambda)\int_{\mathbf{T}}\frac{1}{\rho(\zeta)^\beta}\,|d\zeta| = I_\beta\lambda^\beta\log(1/\lambda),$$

and

$$\int_{\{\lambda\leq\rho<1\}}\log\frac{1}{\rho(\zeta)}\,|d\zeta| \leq \int_{\{\rho<1\}}\log\frac{1}{\lambda}\,|d\zeta| = m\log(1/\lambda).$$

Taking $\lambda=(m/I_\beta)^{1/\beta}$, it follows that

$$\int_{\mathbf{T}}\log\frac{1}{\rho(\zeta)}\,|d\zeta| \leq 2\frac{m}{\beta}\log\frac{I_\beta}{m}.$$

Inserting this estimate into the first line of the proof gives the result. $\square$

The final ingredient is a maximum principle due to Solomyak [14, p. 366].

Lemma 7.7. *Let E be a closed subset of $\mathbf{T}$ and let u be a subharmonic function on $\mathbf{C}\setminus E$. Suppose that*

$$u(z) \leq \psi(\operatorname{dist}(z,\mathbf{T})), \quad z \in \mathbf{C}\setminus\mathbf{T},$$

where $\psi\colon \mathbf{R}^+\to\mathbf{R}^+$ is a decreasing function such that $\sup_{t>0}\psi(t)/\psi(2t)<\infty$. Then there exists a constant C such that

$$u(z) \leq C\psi(\operatorname{dist}(z,E)), \quad z \in \mathbf{C}\setminus E.$$

Proof of Theorem 7.1. Using Proposition A.1, we see that the hypothesis (22) implies

$$\int_0^{2\pi}\frac{1}{d(e^{i\theta},E)(\log(\pi/d(e^{i\theta},E)))^{2\alpha+2}}\,d\theta < \infty. \tag{26}$$

In particular, E is a Carleson set, so there exists an outer function f_0 such that $f_0''\in A(\mathbf{D})$ and $Z(f)=E$. We briefly recall the construction of f_0 given in [5, Theorem 1]. Define $\rho_0\colon \mathbf{T}\to\mathbf{R}^+$ by setting $\rho_0=0$ on E, and on each complementary arc $(e^{i\theta_1},e^{i\theta_2})$ setting

$$\rho_0(e^{i\theta}):=\left(\frac{\theta-\theta_1}{2\pi}\right)^3\left(\frac{\theta_2-\theta}{2\pi}\right)^3, \quad \theta_1<\theta<\theta_2.$$

Then f_0 is the outer function obtained by taking $\rho=\rho_0$ in (24).

We now modify f_0 as follows. For each $\delta\in(0,1)$, we can find $\rho_\delta\in C^3(\mathbf{T})$ such that

$$\begin{cases} \rho_\delta=\rho_0 & \text{on } E_{\delta/2},\\ \rho_\delta=1 & \text{on } \mathbf{T}\setminus E_\delta,\\ \rho_0\le\rho_\delta\le 1 & \text{on } \mathbf{T},\\ \|\rho_\delta'\|_\infty\le 1/\delta. \end{cases}$$

Since $\rho_\delta(e^{i\theta})\ge\rho_0(e^{i\theta})\ge d(e^{i\theta},E)^6/(2\pi)^6$, condition (26) implies that the integrals

$$I_{1/7,\delta}:=\int_{\mathbf{T}}\frac{1}{\rho_\delta(\zeta)^{1/7}}\,|d\zeta|$$

are bounded above independently of δ. Hence, if we define f_δ by

$$f_\delta(z):=\exp\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\log\rho_\delta(e^{i\theta})\,d\theta\right), \quad z\in\mathbf{D},$$

then, by Lemmas 7.5 and 7.6, there exist constants $C_1,C_2>0$, independent of δ, such that

$$\|f_\delta\|_{\mathcal{D}}\le C_1\frac{|E_\delta|^{1/2}}{\delta^{1/2}}\left(\log\frac{1}{\delta}+\log\frac{1}{|E_\delta|}\right)$$
$$|f_\delta(z)|\ge\exp\left(-C_2\frac{|E_\delta|\log(1/|E_\delta|)}{1-|z|}\right), \quad z\in\mathbf{D}.$$

Note also that, since $\rho_\delta=\rho_0$ on a neighbourhood of E, the proof of [5, Theorem 1] shows that we still have $f_\delta''\in A(\mathbf{D})$ and $Z(f)=E$. In particular, $f_\delta\in I_E$, where I_E is as defined in Lemma 7.4. Hence,

$$\inf_{f\in I_E}\frac{\|f\|_{\mathcal{D}}}{|f(z)|}\le\inf_{\delta>0}C_1\frac{|E_\delta|^{1/2}}{\delta^{1/2}}\left(\log\frac{1}{\delta}+\log\frac{1}{|E_\delta|}\right)\exp\left(C_2\frac{|E_\delta|\log(1/|E_\delta|)}{1-|z|}\right), \quad z\in\mathbf{D}.$$

Clearly $|E_\delta|\ge\delta$, and from (22) we have $|E_\delta|=O(\delta(\log 1/\delta)^{2\alpha})$ as $\delta\to0^+$. Hence, there exist constants C_1' and C_2' such that

$$\inf_{f\in I_E}\frac{\|f\|_{\mathcal{D}}}{|f(z)|}\le\inf_{\delta>0}C_1'\left(\log\frac{1}{\delta}\right)^{\alpha+1}\exp\left(C_2'\frac{\delta(\log(1/\delta))^{2\alpha+1}}{1-|z|}\right), \quad z\in\mathbf{D}.$$

Choosing $\delta=(1-|z|)^2$, it follows that, for some constant C_3,

$$\inf_{f\in I_E}\frac{\|f\|_{\mathcal{D}}}{|f(z)|}\le C_3\left(\log\frac{1}{1-|z|}\right)^{\alpha+1},\quad z\in\mathbf{D}. \tag{27}$$

Now let $\phi\in\mathcal{H}_E(\mathcal{N}^+,\mathcal{B}_e)$. Substituting (27) into Lemma 7.4, we obtain

$$|\phi(z)|\le\frac{C'}{1-|z|}\left(\log\frac{1}{1-|z|}\right)^{\alpha+1},\quad z\in\mathbf{D}.$$

Recall also from (9) that, on the exterior of the unit disk, we have the elementary estimate

$$|\phi(z)|\le\frac{C''}{|z|-1},\quad z\in\mathbf{D}_e.$$

Combining these and applying Lemma 7.7 with $u=|\phi|$, we finally arrive at the desired conclusion. □

8. Completion of the proof

We now examine the implications of Theorem 7.1. The first result is a refinement of Theorem 3.1.

Theorem 8.1. *Let $\alpha>0$, and let E be a closed subset of $\mathbf{T}$ satisfying*

$$\int_0\frac{|E_t|}{t^2(\log(1/t))^{2\alpha}}\,dt<\infty. \tag{28}$$

Let $f\in\mathcal{D}$, and suppose that there exists a constant C such that

$$|f(z)|\le C\left(\log\frac{2}{\operatorname{dist}(z,E)}\right)^{-(3\alpha+1)},\quad z\in\mathbf{D}. \tag{29}$$

Then $\mathcal{H}_E(\mathcal{N}^+,\mathcal{B}_e)\subset[f]^{\perp}_{\mathcal{D}}$.

Proof. Let $\phi\in\mathcal{H}_E(\mathcal{N}^+,\mathcal{B}_e)$. Condition (28) implies that (22) holds, so by Theorem 7.1 there is a constant C' such that

$$|\phi(z)|\le\frac{C'}{\operatorname{dist}(z,E)}\left(\log\frac{4}{\operatorname{dist}(z,E)}\right)^{\alpha+1},\quad 1<|z|<2.$$

Combining this with (29), we deduce that there exists a constant C'' such that

$$|f(re^{i\theta})\phi(e^{i\theta}/r)|\le\frac{C''}{d(e^{i\theta},E)(\log(\pi/d(e^{i\theta},E)))^{2\alpha}},\quad \tfrac12<r<1.$$

By Proposition A.1, condition (28) implies that

$$\int_0^{2\pi} \frac{d\theta}{d(e^{i\theta},E)(\log(\pi/d(e^{i\theta},E)))^{2\alpha}} < \infty.$$

It follows that the family of functions

$$e^{i\theta} \longmapsto f(re^{i\theta})\phi(e^{i\theta}/r), \quad \tfrac{1}{2} < r < 1,$$

is uniformly integrable. Lemma 3.4 therefore applies, and we conclude that $\phi \in [f]_{\mathcal{D}}^{\perp}$, as desired. □

The next result extends Theorem 4.1.

Theorem 8.2. *Let E be a closed subset of $\mathbf{T}$ satisfying* (28). *Suppose that there exists $f \in \mathcal{D}$ such that*

$$\text{(30)} \qquad \operatorname{Re} f(z) \geq \log\log\log \frac{2e}{\operatorname{dist}(z,E)} \quad \textit{and} \quad |\operatorname{Im} f(z)| \leq 1, \quad z \in \mathbf{D}.$$

Then E is a Bergman–Smirnov exceptional set.

Proof. Define g_λ as in Lemma 4.2. Using (30), we see that if λ is real and sufficiently large, then

$$|g_\lambda(z)| \leq C\left(\log \frac{2}{\operatorname{dist}(z,E)}\right)^{-(3\alpha+1)}, \quad z \in \mathbf{D},$$

and so by the preceding theorem $\mathcal{H}_E(\mathcal{N}^+, \mathcal{B}_e) \subset [g_\lambda]_{\mathcal{D}}^{\perp}$. The rest of the proof is identical to that of Theorem 4.1. □

Finally, we can complete the proof of Theorem 2.3, and hence of our main result, Theorem 1.3.

Proof of Theorem 2.3. Let E be a closed subset of $\mathbf{T}$ such that

$$\int_0 \frac{|E_t|}{(t\log(1/t)\log\log(1/t))^2}\,dt < \infty.$$

This condition implies that (28) holds for all $\alpha > 1$. By Proposition A.4, it also guarantees that

$$\int_0 c(E_t)\,d\eta^2(t) > -\infty,$$

where $\eta(t) = \log\log\log(1/t)$ for t close to zero, and so by Theorem 5.1 there exists $f \in \mathcal{D}$ such that (30) holds. Invoking Theorem 8.2, we conclude that E is a Bergman–Smirnov exceptional set. □

9. An example: Cantor sets

We give a sufficient condition for a Cantor set to be Bergman–Smirnov exceptional, thereby justifying the claim made in (2).

Theorem 9.1. *Let* $(l_n)_{n\ge 0}$ *be a positive sequence such that* $\sup_n l_{n+1}/l_n < \frac{1}{2}$, *and* E *be the circular Cantor set constructed from this sequence. Then* E *is a Bergman–Smirnov exceptional set provided that*

$$\sum_n \frac{2^n}{n^2\log(1/l_n)} < \infty. \tag{31}$$

Proof. Let $\omega\colon (0,\pi]\to\mathbf{R}$ be a positive, smooth, decreasing function such that, for t close to zero,

$$\omega(t) = \frac{1}{t(\log(1/t))^2(\log\log(1/t))^2}.$$

Then, for t close to zero,

$$|\omega'(t)| \asymp \frac{1}{t^2(\log(1/t))^2(\log\log(1/t))^2} \quad \text{and} \quad \int_0^t \omega(s)\,ds \asymp \frac{1}{\log(1/t)(\log\log(1/t))^2}.$$

Thus equivalence (ii)⇔(iii) of Proposition A.1 shows that (4) holds if and only if $|E|=0$ and

$$\sum_{n\ge 0} 2^n \int_0^{(l_n-2l_{n+1})/2} \omega(t)\,dt < \infty,$$

i.e. if and only if $2^n l_n\to 0$ and

$$\sum_{n\ge 0} \frac{2^n}{\log(1/d_n)(\log\log(1/d_n))^2} < \infty,$$

where $d_n=(l_n-2l_{n+1})/2$. Both of these conditions are implied by (31). □

A. Conditions of Carleson type

In this appendix we gather a few results about measure and capacity that are used in the rest of the paper. Most, if not all, of these results are well known, but they are scattered about the literature, and not necessarily in the precise form in which we need them.

We begin by fixing some notation. Let E be a subset of the unit circle $\mathbf{T}$. If E is measurable, we denote by $|E|$ the (circular) Lebesgue measure of E. Given $t\in(0,\pi]$, we write

$$E_t := \{e^{i\theta}\in\mathbf{T} : d(e^{i\theta},E)\le t\},$$

where d denotes arclength distance on $\mathbf{T}$. Also, we define $N_E(t)$ to be the smallest number of closed arcs of length $2t$ that cover E.

Our first result establishes the equivalence between various forms of 'Carleson-type' conditions. The classical condition (11) for Carleson sets corresponds to taking $\omega(t)=\log^+(1/t)$. The conditions in several of our results, in particular Theorem 2.3, are also of this general type.

Proposition A.1. *Let E be a closed subset of* $\mathbf{T}$. *Let $\omega\colon [0,\pi]\to(0,\infty]$ be a continuous, decreasing function such that $\omega(0)=\infty$ and $\lim_{t\to 0^+} t\omega(t)=0$. Then the following are equivalent*:

(i) $\int_0^{2\pi} \omega(d(e^{i\theta},E))\,d\theta<\infty$;

(ii) $|E|=0$ *and* $\sum_j \int_0^{|I_j|/2} \omega(t)\,dt<\infty$, *where $(I_j)_j$ are the components of* $\mathbf{T}\setminus E$;

(iii) $\int_0^\pi |E_t|\,d\omega(t)>-\infty$;

(iv) $\int_0^\pi tN_E(t)\,d\omega(t)>-\infty$.

For the proof, we need two lemmas. We are grateful to A. Bourhim for telling us about the first of these.

Lemma A.2. *Let E,ω be as in Proposition* A.1, *and suppose, in addition, that $|E|=0$. Then*

$$\int_0^{2\pi} \omega(d(e^{i\theta},E))\,d\theta = 2\sum_j \int_0^{|I_j|/2} \omega(t)\,dt = 2\pi\omega(\pi)-\int_0^\pi |E_t|\,d\omega(t).$$

Proof. We have

$$\int_0^{2\pi} \omega(d(e^{i\theta},E))\,d\theta = \sum_j \int_{e^{i\theta}\in I_j} \omega(d(e^{i\theta},\partial I_j))\,d\theta = \sum_j 2\int_0^{|I_j|/2} \omega(t)\,dt,$$

which gives the first equality. To prove the second, we begin by noting that $|E_t|=\sum_j \min(2t,|I_j|)$. Now, for each j, integration by parts gives

$$\int_0^\pi \min(2t,|I_j|)\,d\omega(t) = |I_j|\omega(\pi)-2\int_0^{|I_j|/2} \omega(t)\,dt.$$

Hence, summing over j, we get

$$\int_0^\pi |E_t|\,d\omega(t) = \sum_j |I_j|\omega(\pi)-2\sum_j \int_0^{|I_j|/2} \omega(t)\,dt = 2\pi\omega(\pi)-2\sum_j \int_0^{|I_j|/2} \omega(t)\,dt,$$

as desired. □

Lemma A.3. *Let E be a subset of* $\mathbf{T}$. *Then*

$$tN_E(t)\le|E_t|\le 4tN_E(t), \quad 0<t\le\pi.$$

Proof. Since E is covered by $N_E(t)$ arcs of length $2t$, it follows that E_t is covered by $N_E(t)$ arcs of length $4t$. This gives the right-hand inequality.

To prove the left-hand inequality, let $J_1, ..., J_n$ be a maximal collection of disjoint, closed arcs of length t, each of which meets E. Then $J_1 \cup ... \cup J_n \subset E_t$, so $nt \leq |E_t|$. Also, by maximality of the collection, each point of E lies within a distance $t/2$ of one of the J_k. So, after doubling the J_k, they cover E, and hence $N_E(t) \leq n$. The result follows. □

Proof of Proposition A.1. Since $\omega(0)=\infty$, both (i) and (iii) clearly imply that $|E|=0$, and the equivalence of (i), (ii) and (iii) then follows immediately from Lemma A.2. The equivalence of (iii) and (iv) is obvious from Lemma A.3. □

Several results in the paper, notably Theorem 5.1, give rise to a Carleson-type condition expressed in terms of the capacities $c(E_t)$ rather than the measures $|E_t|$. (The capacity $c(E)$ is defined in Section 5.) The following proposition provides a link between the two types of condition.

Proposition A.4. *Let E be a proper closed subset of $\mathbf{T}$, and let $\eta\colon (0,\pi] \to \mathbf{R}^+$ be a C^1 function such that $\eta'(t)<0$ for all t. If*

$$\int_0 |E_t| \eta'(t)^2 \, dt < \infty, \tag{32}$$

then

$$\int_0 c(E_t) \, d\eta^2(t) > -\infty. \tag{33}$$

To prove this, we need an elementary inequality.

Lemma A.5. *Let $a>0$, and let $h\colon [0,a] \to \mathbf{R}$ be a C^1 function such that $h(x)>0$ and $h'(x)>0$ for all $x \in [0,a]$. Then*

$$\int_0^a \frac{x}{h(x)} \, dx \leq 2 \int_0^a \frac{1}{h'(x)} \, dx.$$

Proof. By Schwarz's inequality,

$$\left(\int_0^a \frac{x}{h(x)} \, dx \right)^2 \leq \left(\int_0^a \frac{x^2 h'(x)}{h(x)^2} \, dx \right) \left(\int_0^a \frac{1}{h'(x)} \, dx \right).$$

Integrating by parts, we have

$$\int_0^a \frac{x^2 h'(x)}{h(x)^2} \, dx = \left[\frac{-x^2}{h(x)} \right]_0^a + \int_0^a \frac{2x}{h(x)} \, dx \leq 2 \int_0^a \frac{x}{h(x)} \, dx.$$

The result follows. □

Proof of Proposition A.4. By a well-known calculation, which can be found for example in [7, pp. 30–31], there exists a constant $C>0$ such that, for all closed subsets E of $\mathbf{T}$,

$$\frac{1}{c(E)} \geq C \int_0^\pi \frac{ds}{|E_s|}.$$

Replacing E by E_t, we obtain

$$\frac{1}{c(E_t)} \geq C \int_0^{\pi-t} \frac{ds}{|E_{t+s}|} = C \int_t^\pi \frac{ds}{|E_s|}, \quad 0 < t \leq \pi.$$

Thus, if we define

$$g(t) := \int_t^\pi \frac{ds}{|E_s|}, \quad 0 < t \leq \pi,$$

then (33) will hold provided that

$$\int_0^\pi \frac{d\eta^2(t)}{g(t)} > -\infty. \tag{34}$$

We can suppose, without loss of generality, that $\eta(\pi)=0$. Making the change of variable $x=\eta(t)$, the condition (34) is then equivalent to

$$\int_0^\infty \frac{x}{h(x)}\, dx < \infty,$$

where $h:=g\circ\eta^{-1}$. Using Lemma A.5, this will hold provided that

$$\int_0^\infty \frac{1}{h'(x)}\, dx < \infty.$$

Undoing the change of variable, this last condition is equivalent to

$$\int_0^\pi \frac{\eta'(t)^2}{g'(t)}\, dt > -\infty,$$

which in turn is equivalent to (32), since $g'(t)=-1/|E_t|$. Thus (32) implies (33). □

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