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Exact propagators for some degenerate hyperbolic operators

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Abstract. Exact propagators are obtained for the degenerate second order hyperbolic operators $\partial_t^2 - t^{2l}\Delta_x$, $l=1,2,...$, by analytic continuation from the degenerate elliptic operators $\partial_t^2 + t^{2l}\Delta_x$. The partial Fourier transforms are also obtained in closed form, leading to integral transform formulas for certain combinations of Bessel functions and modified Bessel functions.

1. Introduction

Among the methods for obtaining propagators for hyperbolic operators are several that involve analytic continuation from the elliptic case; see, for example [7] and [8]. In this paper we use continuation from the Green's functions [5] for degenerate elliptic operators of the form

$$\sum_{j=1}^{n} \frac{\partial^2}{\partial y_j^2} + |y|^{2k-2} \sum_{j=1}^{m} \frac{\partial^2}{\partial x_j^2} \tag{1.1}$$

with $n=1$ and $t=y_1$ to produce formulas for propagators for the degenerate hyperbolic operators

$$L_m = \frac{\partial^2}{\partial t^2} - t^{2k-2} \sum_{j=1}^{m} \frac{\partial^2}{\partial x_j^2}, \quad k=2,3,\dots. \tag{1.2}$$

(We treat k as fixed and omit it from the notation.) Continuation is carried out in the "space" variables x_j. For even m, the propagators are piecewise algebraic functions (or distribution derivatives of piecewise algebraic functions). This idea was discussed briefly in [4], but the result as stated there is only correct for t sufficiently close to the starting time s.

We remark that continuing in x_m alone as the "time" variable, leads to explicit propagators for the singular hyperbolic operators

$$L=L_{nm}=|y|^{2k-2}\frac{\partial^2}{\partial t^2}-\sum_{j=1}^{n}\frac{\partial^2}{\partial y_j^2}-|y|^{2k-2}\sum_{j=1}^{m-1}\frac{\partial^2}{\partial x_j^2},\quad k=2,3,\dots.$$

This will be carried out in a subsequent paper. The details of the procedure, and the form of the result relative to the original Green's function, are somewhat different from the degenerate case (1.2) that we consider in this paper.

By a propagator for (1.2) we mean a distribution $W_m=W_m(x,t;y,s)$ such that for each test function $f=f(x)$ the function or distribution

$$u(x,t)=\int_{\mathbf{R}^m}W_m(x,t;y,s)f(y)\,dy$$

is the solution of

$$L_m u=0,\ t\neq s;\quad u(x,s)=0,\ \frac{\partial u}{\partial t}(x,s)=f(x).$$

In this section we give an explicit formula for the propagator for m even. The case m odd can be obtained by the method of descent. As in the classical case, the results are easier to state in low dimensions, where the propagator is a locally integrable function. In Theorem 1 the result for the case $m=2$ is stated. The result for the general case is stated in Theorem 2. The support and singular support of the propagator are identified in Theorem 3.

To state the results we introduce functions of two variables

$$\beta_\pm(R,\sigma)=\big(R\pm\sqrt{R^2-\sigma^2}\big)^{1/2k},\quad R\ge\sigma\ge0,$$

which we extend to other real values of R by continuation into the upper half-plane for fixed $\sigma\ge0$:

$$\beta_\pm(R,\sigma)=\begin{cases}\big(R\pm i\sqrt{\sigma^2-R^2}\big)^{1/2k}, & -\sigma\le R\le\sigma,\\ e^{\pm i\pi/2k}\beta_\pm(|R|,\sigma), & R\le-\sigma.\end{cases}\tag{1.3}$$

Using these values we define a locally integrable function W, where

$$W(x,t;y,s)=\frac{\beta_+(R,\sigma)+\operatorname{sgn}(st)\beta_-(R,\sigma)}{\sqrt{R^2-\sigma^2}}\tag{1.4}$$

on the region I defined by the inequality

$$k^2|x-y|^2<(|t|^k-|s|^k)^2$$

and

$$W(x,t;y,s)=\frac{\cos((\alpha+\pi)/2k)\,\sigma^{1/2k}}{\sin(\pi/2k)\,\sqrt{\sigma^2-R^2}} \tag{1.5}$$

on the region II_- defined by the inequalities

$$(|t|^k-|s|^k)^2<k^2|x-y|^2<(|t|^2+|s|^2)^2,\quad st<0;$$

and

$$W(x,t;y,s)=0,\quad \text{otherwise,} \tag{1.6}$$

where

$$R=R(x-y,t,s)=\frac{t^{2k}+s^{2k}-k^2|x-y|^2}{2};$$
$$\sigma=\sigma(t,s)=|st|^k \quad\text{and}\quad \alpha=\alpha(x-y,t,s)=\arccos(R/\sigma).$$

Theorem 1. *The propagator for the operator L_2 with pole at (y,s), $s\neq 0$, is*

$$W_2(x,t;y,s)=\operatorname{sgn}(t-s)\,\frac{k}{4\pi}\,W(x,t;y,s). \tag{1.7}$$

We consider W as a distribution, and take distribution derivatives to obtain the propagator for arbitrary even m.

Theorem 2. *For arbitrary even m, the propagator for the operator L_m is*

$$W_m=c_m\,\sigma^{1/2k}\left(\frac{1}{R}\,\frac{\partial}{\partial\sigma}\right)^{(m-2)/2}(\sigma^{(m-2)/2-1/2k}W). \tag{1.8}$$

where

$$c_m=\left(-\frac{1}{2}\right)^{(m-2)/2}\operatorname{sgn}(t-s)\,\frac{k^{m-1}}{4\pi^{m/2}}. \tag{1.9}$$

Theorem 3. *The singular support of the propagator W_m of Theorem 2, m even, is the union of the set defined by the equality*

$$k^2|x-y|^2=(|t|^k-|s|^k)^2$$

and the set defined by

$$k^2|x-y|^2=(|t|^k+|s|^k)^2,\quad st\le 0.$$

To fix ideas, take $s<0$ here. Then for $t<0$ the operator is strongly hyperbolic and the singular support of the propagator is what it must be: the union of the characteristics from the pole (y,s). At $t=0$ the operator is weakly hyperbolic, and the singular support of the propagator now picks up all forward characteristics from the points $(x,0)$ with $k^2|x-y|^2=s^{2k}$. This is an example of the phenomenon of branching of singularities at multiply characteristic points, introduced and studied extensively in [1], [2], [3], [9], [10], [12], [13], [14], [15], [16] and [17].

Theorem 1 is proved in Section 2 below, Theorem 2 is proved in Section 3, and Theorem 3 is proved in Section 4.

It is natural to take advantage of translation invariance of L_m in the x variables to calculate the partial Fourier transform of the propagator or, in more general situations, to construct a parametrix; this approach is used, in particular, in [12], [16] and [19]. In Section 5 we calculate the partial Fourier transform of the propagator and of the Green's function for the associated degenerate elliptic operator and find the relationship between the two. This provides a check on the results in Sections 2 and 3.

The calculations in Section 5 apply equally well to the generalized degenerate elliptic operator and generalized degenerate wave operator

$$\frac{dt}{dt^2}-t^{2k-2}T \quad \text{and} \quad \frac{dt}{dt^2}+t^{2k-2}T,$$

where T is any nonnegative selfadjoint operator. We obtain a Green's function for the first of these operators and a propagator for the second. In Theorem 4 we relate the two via analytic continuation in the variable t.

Having computed Green's functions and propagators and also their partial Fourier transforms, in Section 6 we obtain formulas (some possibly new) for some integral transforms of certain combinations of Bessel functions and modified Bessel functions.

2. Proof of Theorem 1

We assume throughout this section that $s\neq 0$.

The degenerate elliptic operator

$$L=\frac{\partial^2}{\partial t^2}+t^{2k-2}\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\right) \tag{2.1}$$

has a unique Green's function with pole at (y,s) that is homogeneous with respect to the dilations

$$(x,t;y,s)\longmapsto(\lambda x,\lambda^k t;\lambda y,\lambda^k s),\quad \lambda>0.$$

Up to a multiplicative constant, this Green's function is the algebraic function

$$K_0(x,t;y,s)=\frac{\beta_+(R_0,\sigma)^2-\beta_-(R_0,\sigma)^2}{\sqrt{R_0^2-\sigma^2}\,\left[\beta_+(R_0,\sigma)^2+\beta_-(R_0,\sigma)^2-2st\right]^{1/2}},\tag{2.2}$$

where

$$R_0=R_0(x-y,t;s)=\frac{t^{2k}+s^{2k}+k^2|x-y|^2}{2},\quad \sigma=|st|^k;$$

see [5]. Note that $\beta_+\beta_-=\sigma^{1/k}=|st|$, so

$$\beta_+^2+\beta_-^2-2st=\beta_+^2+\beta_-^2-\operatorname{sgn}(st)\,\beta_+\beta_-=[\beta_+-\operatorname{sgn}(st)\,\beta_-]^2.$$

Therefore K_0 can be written in the form

$$K_0(x,t;y,s)=\frac{\beta_+(R_0(x-y,t,s),\sigma(s,t))+\operatorname{sgn}(st)\,\beta_-(R_0(x-y,t,s),\sigma(t,s))}{\sqrt{R_0(x-y,t,s)^2-\sigma(t,s)^2}}.\tag{2.3}$$

We modify the operator (2.1) by taking x_j to $e^{i\theta/2}x_j$ and $\partial/\partial x_j$ to $e^{-i\theta/2}\partial/\partial x_j$, $0\le\theta\le\pi$. This gives the operator

$$L^\theta=\frac{\partial^2}{\partial t^2}+e^{-i\theta}\,t^{2k-2}\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\right)$$

and the function $K_\theta(t,x;y,s)$ given by (2.3) with R_0 replaced by

$$R_\theta(t,x-y;s)=\frac{t^{2k}+s^{2k}+e^{i\theta}k^2|x-y|^2}{2}.$$

The limit case L^π is the operator L_2 of (1.2).

Lemma 1. *For $0<\theta<\pi$, the function K_θ is smooth except at $t=s$ and $x=y$.*

Proof. Let

$$f_\pm(R,\sigma)=R\pm\sqrt{R^2-\sigma^2},\quad R\ge\sigma>0,$$

and consider the continuations for $\operatorname{Im}R>0$. If $f_\pm(R)=a\in\mathbf{R}$ then $R=(a^2+\sigma^2)/2a$, so R is real. It follows that $\operatorname{Im}f_\pm(R,\sigma)$ has fixed sign on the upper half plane, and it can be seen that the sign of $\pm\operatorname{Im}f_\pm$ is positive. The same is true for $\beta_\pm$, which is defined using the principal branch of the root.

A singularity can occur only where $R^2=\sigma^2$, which does not happen for $x\ne y$, since $x\ne y$ implies that R is in the upper half plane. If $x=y$, then

$$4(R^2-\sigma^2)=(t^{2k}+s^{2k})^2-4(st)^{2k}=(|t|^k+|s|^k)^2(|t|^k-|s|^k)^2.$$

This vanishes only if $|t|=|s|$. But for $st<0$, fix $R\neq 0$ and assume that

$$\varepsilon=\sqrt{1-\sigma^2/R^2}\longrightarrow 0.$$

Then

$$K_\theta=\frac{R^{1/2k}}{R}\,\frac{(1+\varepsilon)^{1/2k}-(1-\varepsilon)^{1/2k}}{\varepsilon}=\frac{R^{1/2k}}{R}\left(\frac{1}{k}+O(\varepsilon)\right).$$

Therefore K_θ is only singular where $x=y$, $|t|=|s|$, and $st>0$. $\square$

To evaluate the limit K_π as $\theta\to\pi$ we distinguish four regions, two of which were identified above:

$$\begin{aligned}
&\text{I:} && k^2|x-y|^2<(|t|^k-|s|^k)^2;\\
&\text{II}_\pm: && (|t|^k-|s|^k)^2<k^2|x-y|^2<(|t|^k+|s|^k)^2, \quad \pm st>0;\\
&\text{III:} && k^2|x-y|^2>(|t|^k+|s|^k)^2.
\end{aligned}$$

As $\theta\to\pi$, continuation from region I leads to limiting values of $R>\sigma$, continuation from regions $\text{II}_\pm$ leads to values $-\sigma<R<\sigma$, and continuation from region III leads to values $R<-\sigma$.

The limiting values of $\sqrt{R^2-\sigma^2}$ are $i\sqrt{\sigma^2-R^2}$ for $-\sigma<R<\sigma$ and $-\sqrt{R^2-\sigma^2}$ for $R<-\sigma$. For $R<-\sigma$ the limiting value of $R\pm\sqrt{R^2-\sigma^2}$ is

$$R\mp\sqrt{R^2-\sigma^2}=-\big(|R|\pm\sqrt{R^2-\sigma^2}\big)$$

Therefore the limiting values of $\beta_\pm$ as $R\to\mathbf{R}$ are as in (1.3):

$$\beta_\pm(R,\sigma)=\begin{cases}(R\pm i\sqrt{\sigma^2-R^2})^{1/2k}, & -\sigma\le R\le\sigma;\\ e^{\pm i\pi/2k}\,\beta_\pm(|R|), & R\le-\sigma.\end{cases}$$

In the region $R^2<\sigma^2$,

$$\beta_+(R,\sigma)=\sigma^{1/2k}[\cos(\alpha/2k)+i\sin(\alpha/2k)],\quad \alpha=\arccos(R/\sigma).$$

It follows that the real parts $\operatorname{Re}K_\pi$ of the limiting values of $\operatorname{Re}K_\theta$ are

$$\begin{aligned}
&\frac{\beta_+(R,\sigma)+\operatorname{sgn}(st)\,\beta_-(R,\sigma)}{\sqrt{R^2-\sigma^2}} && \text{in I;}\\
&0 && \text{in II}_+;\\
&\frac{\sigma^{1/2k}\sin(\alpha/2k)}{\sqrt{\sigma^2-R^2}}, && \text{in II}_-;\\
&-\cos(\pi/2k)\,\frac{\beta_+(|R|,\sigma)+\operatorname{sgn}(st)\,\beta_-(|R|,\sigma)}{\sqrt{R^2-\sigma^2}} && \text{in III.}
\end{aligned}$$

The imaginary parts $\operatorname{Im} K_\pi$ of the limiting values are

$$\begin{array}{ll} 0 & \text{in I;} \\ -\dfrac{\sigma^{1/2k}\cos(\alpha/2k)}{\sqrt{\sigma^2-R^2}}, & \text{in II}_+; \\ 0 & \text{in II}_-; \\ -\sin(\pi/2k)\,\dfrac{\beta_+(|R|,\sigma)-\operatorname{sgn}(st)\,\beta_-(|R|,\sigma)}{\sqrt{R^2-\sigma^2}} & \text{in III.} \end{array}$$

Lemma 2. *The limiting value K_π is locally integrable.*

Proof. It is enough to show that $1/\sqrt{R^2-\sigma^2}$ is locally integrable for $s\neq 0$. At points where $R^2+\sigma^2\neq 0$ the gradient of $R^2-\sigma^2$ is not zero, so $1/\sqrt{R^2-\sigma^2}$ has an integrable singularity. At $R=\sigma=0$ we have $x\neq 0$, so the gradient of R and the gradient of $S=st$ are independent. Therefore the integral of $1/\sqrt{R^2-\sigma^2}$ near such a point is dominated by

$$\int_0^1\int_0^{S^k}\frac{dR\,dS}{\sqrt{S^{2k}-R^2}}+\int_0^1\int_{S^k}^1\frac{dR\,dS}{\sqrt{R^2-S^{2k}}}=\int_0^1\operatorname{arcsin}1\,dS+\int_0^1\operatorname{arcosh}S^{-k}\,dS$$

which is finite, since $\operatorname{arcosh}S^{-k}\sim -k\log S$ as $S\to 0$. □

Lemma 3. $L_2K_\pi=0$ (*as distribution*) *except possibly at $x=0$, $t=s$.*

Proof. It follows from estimates like those in the proof of Lemma 2 that K_θ converges in L^1_{loc} to K_π as $\theta\to\pi$, so K_θ also converges as distribution. The identity $L_\theta K_\theta=0$ holds at $\theta=0$, except possibly at the singularity at $x=y$ and $t=s$. By continuity this identity continues to be valid for $0<\theta<\pi$, and therefore it is valid in the limit $\theta=\pi$. □

Since L_2 has real coefficients, both the real and imaginary parts of K_π are annihilated by L_2 away from the pole, and therefore are candidates for propagator, as are linear combinations of the two. On general principles, the support of the propagator should be the closure of the region I for $st>0$ and lie in the closure of the union of region I and region II$_-$ for $st<0$. However, according to Lemma 2, no nontrivial linear combination of the real and imaginary parts will eliminate region III. To do so we take advantage of the fact that s is simply a parameter in the equation $L_2K_\pi=0$. Replacing s by $-s$ leaves R and σ unchanged. The combination

$$W(x,t;y,s)=\operatorname{Re}K_\pi(x,t;y,s)-\cot(\pi/2k)\,\operatorname{Im}K_\pi(x,t;y,-s) \tag{2.4}$$

is annihilated by L_2 away from $x=y$ and $t=\pm s$. Lemma 2 implies that the support of W is precisely the closure of the union of region I and region II$_-$. It can be checked that the values of W are those given in (1.4)–(1.6).

Proof of Theorem 1. We have observed that $L_2W=0$ except possibly at $x=y$ and $t=\pm s$. However $\operatorname{Im} K_\pi(x,t,y,-s)$ vanishes identically near $x=y$ and $t=-s$, so taking the combination (2.4) does not introduce a new singularity at that point; thus $L_2W=0$ at $x=y$ and $t=-s$.

To complete the proof we need to check the behavior of

$$u(x,t)=\int_{\mathbf{R}^m} W(t,x;y,s)f(y)\,dy.$$

as $t\to s$, where f is a test function. For $st>0$ this is

$$\int_{\sigma<R} \frac{\beta_+(R,\sigma)+\beta_-(R,\sigma)}{\sqrt{R^2-\sigma^2}}\,f(y)\,dy \tag{2.5}$$

with $2R=t^{2k}+s^{2k}-k^2|x-y|^2$ and $\sigma=(st)^k$.

Lemma 4. *As $t\to s\neq 0$, the integral* (2.5) *is*

$$\frac{4\pi}{k}\,|t-s|\,f(x)+O((t-s)^2).$$

Proof. Up to terms of higher order in $t-s$ we have $R\approx s^{2k}$, $R+\sigma\approx 2s^{2k}$, and

$$\begin{aligned} 2(R-\sigma)&=(t^k-s^k)^2-k^2r^2\approx k^2[s^{2k-2}(t-s)^2-r^2],\\ 2\sqrt{R^2-\sigma^2}&\approx k|s|^k[s^{2k-2}(t-s)^2-r^2]^{1/2}, \end{aligned}$$

where $r=|x-y|$. Therefore, on the support of W as $t\to s$, $\beta_+(R,\sigma)\approx|s|$, $\beta_-(R,\sigma)\approx 0$, and

$$W(t,x;y,s)\approx\frac{2|s|}{k|s|^k\sqrt{A^2-r^2}},\quad A=|s|^{k-1}|t-s|.$$

It follows that as $t\to s$, up to higher powers of $t-s$ we have

$$\begin{aligned}\int W(t,x;y,s)\,f(x-y)\,dy&\approx\frac{2f(x)}{k|s|^{k-1}}\int_{r<A}\frac{dy}{\sqrt{A^2-r^2}}=\frac{2f(x)}{k|s|^{k-1}}\int_0^A\frac{2\pi r dr}{\sqrt{A^2-r^2}}\\ &=\frac{2f(x)}{k|s|^{k-1}}\cdot 2\pi A=\frac{4\pi f(x)}{k}\,|t-s|.\end{aligned}$$

Differentiation of (2.5) with respect to t gives $\operatorname{sgn}(t-s)4\pi f(x)/k$. This confirms the normalization in Theorem 1: the propagator is

$$W_2(t,x;s,y)=\operatorname{sgn}(t-s)\,\frac{k}{4\pi}\,W(t,x;y,s).\quad\square$$

3. Proof of Theorem 2

Up to a multiplicative constant, for even m the Green's function for the degenerate elliptic operator (1.1) with pole at (y,s) is

$$K_{m0}(x,t;y,s)=\sigma^{1/2k}\left(\frac{1}{R}\frac{\partial}{\partial\sigma}\right)^{(m-2)/2}[\sigma^{(m-2)/2-1/2k}K_0(x,t;y,s)]; \tag{3.1}$$

see [5]; here K_0 is defined by (2.2) or (2.3).

We showed in Section 2 that the function K_θ converges in the sense of distributions to K_π, considered now as a function of R and σ. Therefore derivatives converge in the sense of distributions. Using the analogous notation we define $K_{m\theta}$, $0<\theta<\pi$, so $K_{m\theta}$ converges as distribution to a limit $K_{m\pi}$ as $\theta\to\pi$.

The degenerate elliptic operator (1.1) annihilates K_{m0} away from the pole, and again this persists under the continuation to the degenerate hyperbolic case. However, because of the division by R the limit appears to have new singularities at $R=0$ in the region II_-. In fact there are no such new singularities in this region.

Lemma 5. *The distribution $K_{m\pi}$ is analytic in* II_-.

Proof. For $\sigma>0$ and $R\approx0$, the function

$$\begin{aligned}&R^{-(m-2)/2}\frac{(R+i\sqrt{\sigma^2-R^2})^{1/2k}+(R-i\sqrt{\sigma^2-R^2})^{1/2k}}{\sqrt{\sigma^2-R^2}}\\&\quad=R^{-(m-2)/2}\sigma^{-1+1/2k}\frac{(\eta+i\sqrt{1-\eta^2})^{1/2k}+(\eta-i\sqrt{1-\eta^2})^{1/2k}}{\sqrt{1-\eta^2}},\quad \eta=\frac{R}{\sigma},\end{aligned}$$

has a convergent expansion of the form

$$R^{-(m-2)/2}\sum_{j=0}^{\infty}a_j\,\sigma^{-1+1/2k}\eta^j=\sum_{j=0}^{\infty}a_j\,\sigma^{-1-j+1/2k}R^{j-(m-2)/2}. \tag{3.2}$$

With the factor $R^{-(m-2)/2}$ removed, the operator that acts on K_0 in (3.1) can be rewritten as

$$\sigma^{1/2k}\left(\frac{\partial}{\partial\sigma}\right)^{(m-2)/2}\sigma^{(m-1)/2}\sigma^{-1/2k}=\sigma^{1/2k}\prod_{j=1}^{(m-2)/2}(D_\sigma+j)\,\sigma^{-1/2k}, \tag{3.3}$$

where $D_\sigma=\sigma\partial/\partial\sigma$. Now $\sigma^{1/2k}D_\sigma\sigma^{-1/2k}=D_\sigma-1/2k$, so the operator (3.3) is

$$\prod_{j=1}^{(m-2)/2}\left(D_\sigma+j-\frac{1}{2k}\right). \tag{3.4}$$

The operator $D+j-1/2k$ kills $\sigma^{-j+1/2k}$. Therefore the operator (3.4) kills the terms up to $j=-1+(m-2)/2$ in the expansion (3.2), which leaves only terms with nonnegative integer powers of R. □

The distribution W_m of (1.8) is, up to the factor c_m of (1.9), obtained by applying (3.3) to W. As t approaches s, W coincides with $K_{m\pi}$ restricted to a neighborhood of the closure of the region I. To complete the proof of Theorem 2, we need to consider the behavior of the formal integral

$$u(x,t)=\int K_{m\pi}(x,t;y,s)\,f(y)\,dy$$

over such a neighborhood. In polar coordinates $r=|x-y|$ and ω,

$$u(x,t)=\int K_{m\pi}(x,t;y,s)\,f(y)\,r^{m-1}\,dr\,d\omega \tag{3.5}$$

Here s is fixed. It will be convenient for the moment to take the independent variables to be R, $\tau=R-\sigma$ and ω in place of t,r and ω. Since

$$k^2r^2=t^k+s^{2k}-2R=(|t|^k-|s|^k)^2-2\tau$$

we have $k^2r\,dr=-d\tau$ and (3.5) for $t\approx s$ is

$$u(x,t)=-\frac{1}{k^2}\int_{\tau>0}R^{-(m-2)/2}\,M[K_\pi(x-y,t;s)]\,f(y)\,r^{m-2}\,d\tau\,d\omega,$$

where M is the differential operator

$$\prod_{j=1}^{(m-2)/2}\left(-\sigma\frac{\partial}{\partial\tau}+j-\frac{1}{2k}\right),$$

with transpose

$$M^t=\prod_{j=1}^{(m-2)/2}\left(\frac{\partial}{\partial\tau}\,\sigma+j-\frac{1}{2k}\right),$$

Thus we may rewrite (3.5) once more as

$$u(x,t)=-\frac{1}{k^2}\int_{\tau>0}R^{-(m-2)/2}K_\pi(x,t;y,s)\,M^t[f(y)\,r^{m-2}]\,d\tau\,d\omega. \tag{3.6}$$

Lemma 6. *The integral* (3.6) *is*

$$(-2)^{(m-2)/2}\,\frac{4\pi^{m/2}}{k^{m-1}}\,|t-s|\,f(x)+O((t-s)^2) \tag{3.7}$$

as $t\to s\neq 0$.

Proof. As in the proof of Theorem 1, on the domain of integration, as $t\to s\neq 0$ we have

$$k^2r^2\le k^2|x-y|^2\le (t^k-s^k)^2\approx k^2s^{2k-2}(t-s)^2$$

so $r=O(t-s)$. Therefore up to terms of higher order in $t-s$, we may assume that all differentiations $\partial/\partial\tau$ in (3.6) fall on

$$r^{m-2}=\left[\frac{(t^k-s^k)^2-2\tau}{k^2}\right]^{(m-2)/2},$$

leading to

$$[(m-2)/2]\,!\left(-\frac{2\sigma}{k^2}\right)^{(m-2)/2}=\Gamma(m/2)\left(-\frac{2\sigma}{k^2}\right)^{(m-2)/2}.$$

On the domain of integration $R\approx\sigma$, so we may replace (3.6) by

$$\begin{aligned}(3.8)\qquad -(-2)^{(m-2)/2}\frac{\Gamma(m/2)}{k^{m-2}}&\int_{\tau>0}W(x,t;y,s)\,d\tau\,d\omega\\ &=(-2)^{(m-2)/2}\frac{\Gamma(m/2)}{k^{m-2}}\int_{\sigma<R}W(x,t;y,s)\,r\,dr\,d\omega\\ &=(-2)^{(m-2)/2}\frac{\Gamma(m/2)\,\sigma_{m-1}}{k^{m-2}}\int_{\sigma<R}W(x,t;y,s)\,r\,dr,\end{aligned}$$

where $\sigma_{m-1}=2\pi^{m/2}/\Gamma(m/2)$ is the volume of the unit $(m-1)$-sphere. As in Lemma 4, up to terms of higher order in $t-s$ the last integral is

$$(3.9)\qquad \frac{2}{k|s|^{k-1}}\int_{r<A}\frac{r\,dr}{\sqrt{A^2-r^2}}=\frac{2A}{k|s|^{k-1}}=\frac{2|t-s|}{k}.$$

Combining (3.8) and (3.9), we obtain (3.7). □

Differentiation with respect to t gives

$$\frac{\partial u}{\partial t}(x,t)=(-2)^{(m-2)/2}\frac{4\pi^{m/2}}{k^{m-1}}\operatorname{sgn}(t-s)\,f(x)+O(t-s).$$

This confirms the normalization in Theorem 2.

4. Proof of Theorem 3

The assertion about the support of the propagator is a consequence of Lemma 2 and the construction of the function W of (2.12). As shown in Section 2, W can only be singular where $R^2=\sigma^2$, which, in the support of W, is precisely the union of the boundary of region I and the boundary of region II$_-$. On the other hand, the singular support clearly contains the boundary of the support, so we only need to examine the common boundary of regions I and II$_-$, i.e. the set where $R=\sigma$ and $st\le 0$. The calculation in Lemma 1 shows that as one approaches the common boundary from region I, $W\approx R^{-1+1/2k}/k$. However the evaluation (1.5) shows that as one approaches the boundary from region II$_-$, $W\to+\infty$.

This proves Theorem 3 for the case $m=2$. According to Lemma 3.1, passage to the propagator for $m>2$ does not introduce any new singularities. Moreover the derivatives that relate W_m to W_2 are transverse to the boundaries of I and II$_-$, so they do not kill the singularities on those boundaries. Therefore the singular support is the same as for $m=2$, and Theorem 3 is valid for general even m.

5. Partial Fourier transform and spectral resolution

As noted in the introduction, a natural way to attack the propagator is to take advantage of translation invariance in the x_j variables and compute the partial Fourier transform

$$U_{|\xi|}(t,s)=\widehat{W}_m(\xi,t;s)=\int_{\mathbf{R}^m} e^{-ix\cdot\xi}W_m(x,t;0,s)\,dx. \tag{5.1}$$

This is to be the solution of the Cauchy problem

$$\begin{gathered}\frac{\partial^2 U_{|\xi|}}{\partial t^2}+t^{2k-2}|\xi|^2U_{|\xi|}=0,\\ U_{|\xi|}(t,\xi;s)\Big|_{t=s}=0 \quad\text{and}\quad \frac{\partial U_{|\xi|}}{\partial t}(t,s)\Big|_{t=s}=1.\end{gathered} \tag{5.2}$$

Fix ξ for the moment, and use the Lommel transformation [11]:

$$U_{|\xi|}(t)=|\tau|^{1/2k}F(\tau),\qquad \tau=\operatorname{sgn} t\,\frac{|\xi||t|^k}{k}. \tag{5.3}$$

The differential equation in (5.2) for $U_{|\xi|}$ is equivalent to Bessel's equation for F:

$$\tau^2F''(\tau)+\tau F'(\tau)+\left(\tau^2-\frac{1}{4k^2}\right)F(\tau)=0.$$

The solutions F are linear combinations of the Bessel functions $J_{\pm 1/2k}$ [6], [18]. Therefore $U_{|\xi|}$ itself is a linear combination of the two functions

$$C_{|\xi|}(t)=\left(\frac{|\tau|}{2}\right)^{1/2k} J_{-1/2k}(\tau)=\sum_{j=0}^{\infty}\frac{(-1)^j}{j!\,\Gamma(j+1-1/2k)}\left(\frac{|\xi|t^k}{2k}\right)^{2j}$$

and

$$S_{|\xi|}(t)=\operatorname{sgn} t\left(\frac{|\tau|}{2}\right)^{1/2k} J_{1/2k}(\tau)=\left(\frac{|\xi|}{2k}\right)^{1/k}\sum_{j=0}^{\infty}\frac{(-1)^j\, t}{j!\,\Gamma(j+1+1/2k)}\left(\frac{|\xi|t^k}{2k}\right)^{2j},$$

which are even and odd with respect to t, respectively. The solution to (5.2) is

$$U_{|\xi|}(t)=\frac{1}{\Delta_{|\xi|}}[C_{|\xi|}(s)S_{|\xi|}(t)-S_{|\xi|}(s)C_{|\xi|}(t)], \tag{5.4}$$
$$\Delta_{|\xi|}=C_{|\xi|}(s)S'(s)-S_{|\xi|}(s)C'(s).$$

The Wronskian $C_{|\xi|}S'_{|\xi|}-S_{|\xi|}C'_{|\xi|}$ is constant, so we may evaluate at $s=0$ to obtain

$$\Delta_{|\xi|}=C_{|\xi|}(0)S'_{|\xi|}(0)=\frac{1}{\Gamma(1-1/2k)}\,\frac{1}{\Gamma(1+1/2k)}\left(\frac{|\xi|}{2k}\right)^{1/k}=\frac{\sin(\pi/2k)}{\pi/2k}\left(\frac{|\xi|}{2k}\right)^{1/k}.$$

To see how this relates to analytic continuation from the Green's function for the degenerate elliptic operator, we note that the partial Fourier transform of the Green's function is the bounded Green's function $G_{|\xi|}(t,s)$ with pole at $t=s$ for the operator

$$\frac{d^2}{dt^2}-t^{2k-2}|\xi|^2.$$

The transformation (5.3) converts this to the modified Bessel's equation

$$\tau^2F''(\tau)+\tau F'(\tau)-\left(\tau^2+\frac{1}{4k^2}\right)F(\tau)=0.$$

The solutions F are linear combinations of the modified Bessel functions $I_{\pm 1/2k}$ [6], [18]. Therefore $G_{|\xi|}$ is a piecewise linear combination of the even and odd functions

$$A_{|\xi|}(t)=\left(\frac{|\tau|}{2}\right)^{1/2k} I_{-1/2k}(\tau)=\sum_{j=0}^{\infty}\frac{1}{j!\,\Gamma(j+1-1/2k)}\left(\frac{|\xi|t^k}{2k}\right)^{2j}$$

and

$$B_{|\xi|}(t)=\operatorname{sgn} t\left(\frac{|\tau|}{2}\right)^{1/2k} I_{1/2k}(\tau)=\left(\frac{|\xi|}{2k}\right)^{1/k}\sum_{j=0}^{\infty}\frac{t}{j!\,\Gamma(j+1+1/2k)}\left(\frac{|\xi|t^k}{2k}\right)^{2j}.$$

The linear combinations $L_{|\xi|}(t)=A_{|\xi|}(t)+B_{|\xi|}(t)$ and $R_{|\xi|}(t)=A_{|\xi|}(t)-B_{|\xi|}(t)$ satisfy

$$\lim_{t\to-\infty} L_{|\xi|}(t)=0=\lim_{t\to\infty} R_{|\xi|}(t),$$

so the bounded Green's function is

$$G_{|\xi|}(t,s)=\begin{cases} \dfrac{1}{\hat{\Delta}_{|\xi|}} R_{|\xi|}(s)L_{|\xi|}(t), & t<s, \\ \dfrac{1}{\hat{\Delta}_{|\xi|}} L_{|\xi|}(s)R_{|\xi|}(t), & t>s, \end{cases} \tag{5.5}$$

where $\hat{\Delta}_{|\xi|}$ is the Wronskian $L_{|\xi|}R'_{|\xi|}-L'_{|\xi|}R_{|\xi|}$. Again this may be evaluated at $t=0$:

$$\hat{\Delta}_{|\xi|}=-2A_{|\xi|}(0)B'_{|\xi|}(0)=-2\,\frac{1}{\Gamma(1-1/2k)}\,\frac{1}{\Gamma(1+1/2k)}\left(\frac{|\xi|}{2k}\right)^{1/k}=-2\Delta_{|\xi|}.$$

The continuation $t\mapsto\omega t$, $\omega=e^{i\pi/2k}$, converts the degenerate elliptic operator to a multiple of the degenerate hyperbolic operator. Note that

$$A_{|\xi|}(\omega t)=C_{|\xi|}(t) \quad\text{and}\quad B_{|\xi|}(\omega t)=S_{|\xi|}(t).$$

Therefore

$$L_{|\xi|}(\omega t)=C_{|\xi|}(t)+\omega S_{|\xi|}(t) \quad\text{and}\quad R_{|\xi|}(\omega t)=C_{|\xi|}(t)-\omega S_{|\xi|}(t).$$

For $t<s$

$$\begin{aligned} G_{|\xi|}(\omega t,\omega s)&=-\frac{1}{2\Delta_{|\xi|}}\,R_{|\xi|}(\omega s)\,L_{|\xi|}(\omega t) \\ &=-\frac{1}{2\Delta_{|\xi|}}[C_{|\xi|}(s)-\omega S_{|\xi|}(s)][C_{|\xi|}(t)+\omega S_{|\xi|}(t)] \\ &=-\frac{1}{2\Delta_{|\xi|}}[C_{|\xi|}(s)C_{|\xi|}(t)-\omega^2 S_{|\xi|}(s)S_{|\xi|}(t)+\omega C_{|\xi|}(s)S_{|\xi|}(t)-\omega S_{|\xi|}(s)C_{|\xi|}(t)], \end{aligned}$$

while for $t>s$

$$\begin{aligned} G_{|\xi|}(\omega t,\omega s)&=-\frac{1}{2\Delta_{|\xi|}}\,L_{|\xi|}(\omega s)\,R_{|\xi|}(\omega t) \\ &=-\frac{2}{\Delta_{|\xi|}}[C_{|\xi|}(s)+\omega S_{|\xi|}(s)][C_{|\xi|}(t)-\omega S_{|\xi|}(t)] \\ &=-\frac{1}{2\Delta_{|\xi|}}[C_{|\xi|}(s)C_{|\xi|}(t)-\omega^2 S_{|\xi|}(s)S_{|\xi|}(t)-\omega C_{|\xi|}(s)S_{|\xi|}(t)+\omega S_{|\xi|}(s)C_{|\xi|}(t)]. \end{aligned}$$

Combining this gives

$$\omega^{-1}G_{|\xi|}(\omega t,\omega s)=-\frac{1}{2\Delta_{|\xi|}}\,[\omega^{-1}C_{|\xi|}(s)C_{|\xi|}(t)-\omega S_{|\xi|}(s)S_{|\xi|}(t)] \\ -\frac{\operatorname{sgn}(t-s)}{2\Delta_{|\xi|}}\,[C_{|\xi|}(s)S_{|\xi|}(t)-S_{|\xi|}(s)C_{|\xi|}(t)].$$

A similar calculation, using the fact that C is even and S is odd, shows that

$$\omega^{-1}G_{|\xi|}(\omega t,-\omega s)=-\frac{1}{2\Delta_{|\xi|}}[\omega^{-1}C_{|\xi|}(s)C_{|\xi|}(t)+\omega S_{|\xi|}(s)S_{|\xi|}(t)] \\ -\frac{\operatorname{sgn}(t+s)}{2\Delta_{|\xi|}}\,[C_{|\xi|}(s)S_{|\xi|}(t)+S_{|\xi|}(s)C_{|\xi|}(t)].$$

Therefore

$$\operatorname{Re}[\omega^{-1}G_{|\xi|}(\omega t,\omega s)]=-\frac{\operatorname{sgn}(t-s)}{2\Delta_{|\xi|}}\,[C_{|\xi|}(s)S_{|\xi|}(t)-S_{|\xi|}(s)C_{|\xi|}(t)] \\ -\frac{\cos(\pi/2k)}{2\Delta_{|\xi|}}\,[C_{|\xi|}(s)C_{|\xi|}(t)-S_{|\xi|}(s)S_{|\xi|}(t)]$$

and

$$\operatorname{Im}[\omega^{-1}G_{|\xi|}(\omega t,-\omega s)]=-\frac{\sin(\pi/2k)}{2\Delta_{|\xi|}}[C_{|\xi|}(s)C_{|\xi|}(t)-S_{|\xi|}(s)S_{|\xi|}(t)].$$

As a consequence we obtain

$$\operatorname{Re}[\omega^{-1}G_{|\xi|}(\omega t,\omega s)]-\cot(\pi/2k)\ \operatorname{Im}[\omega^{-1}G_{|\xi|}(\omega t,-\omega s)]=-\frac{\operatorname{sgn}(t-s)}{2}\,U_{|\xi|}(t,\xi;s).$$

This is in agreement with (2.4), because the normalization constants c_m in Theorems 1 and 2 are twice the negatives of the normalization constants ($n{=}1$) in [5] (once one takes account of the factor $(-1)^{(m-2)/2}$ that should have been included in (4.15) and (1.11) of [5], from differentiating the denominator of F_{n2k} in [5]).

This derivation can be adapted to calculate the propagator and Green's function with the x-Laplacian replaced by any nonpositive selfadjoint operator.

Theorem 4. *Suppose T is a nonnegative selfadjoint operator with spectral resolution*

$$T=\int_0^\infty \lambda^2\,dE_\lambda.$$

Then the (bounded with respect to t) Green's function for the generalized elliptic operator

$$\frac{d^2}{dt^2}-t^{2k-2}T$$

is

$$K_T(t,s)=\int_0^\infty G_\lambda(t,s)\,dE_\lambda.$$

The propagator for the generalized wave operator

$$\frac{d^2}{dt^2}+t^{2k-2}T$$

is

$$W_T(t,s)=\int_0^\infty U_\lambda(t,s)\,dE_\lambda.$$

Here G_λ and U_λ are defined by (5.5) *and* (5.4), *respectively, with $|\xi|=\lambda$. They are related through analytic continuation in t and s by*

$$W_T(t,s)=-2\operatorname{sgn}(t-s)\{\operatorname{Re}[\omega^{-1}K_T(\omega t,\omega s)]-\cot(2\pi/k)\operatorname{Im}[\omega^{-1}K_T(\omega t,\omega s)]\},$$

with $\omega=e^{i\pi/2k}$.

6. Some integral transform formulas

Inverting (5.1) with $m=2$ and taking account of invariance under reflection and rotation in ξ, we have

$$\begin{aligned}\frac{k}{4\pi}W(x,t;0,s)&=\frac{1}{(2\pi)^2}\int_{\mathbf{R}^2}e^{ix\cdot\xi}U_\xi(t,s)\,d\xi\\&=\frac{1}{(2\pi)^2}\int_0^\infty\int_0^{2\pi}e^{i|x|\lambda\cos\theta}U_\lambda(t,s)\,\lambda\,d\theta\,d\lambda\\&=\frac{1}{2\pi}\int_0^\infty J_0(|x|\lambda)\,U_\lambda(t,s)\,\lambda\,d\lambda,\end{aligned}\tag{6.1}$$

where U_λ is given by (5.4) with $|\xi|=\lambda$ and $W(x,t;0,s)$ is defined in (1.4)–(1.6).

Both sides of (6.1) simplify considerably at $s=0$. In fact the region II_- is void. Moreover, in view of homogeneity we may normalize with $t=k^{1/k}$. Then

$$2R(x,t;0)=k^2(1-|x|^2),\quad \beta_+(x,t;0)=R^{1/2k},\quad \sigma=\beta_-(x,t,0)=0,\tag{6.2}$$

so

$$W(x,t;0,0)=\frac{\left[k^2(1-|x|)^2\right]^{1/2k}}{k^2(1-|x|^2)/2}=2k^{-2+1/k}(1-|x|^2)^{-1+1/2k},\quad |x|^2<1.$$

Also

$$\begin{aligned}U_\lambda(t,0)&=\frac{\pi/2k}{\sin(\pi/2k)}\left(\frac{\lambda}{2k}\right)^{-1/k}C_\lambda(0)S_\lambda(t)\\&=\frac{\pi/2k}{\sin(\pi/2k)}\left(\frac{\lambda}{2k}\right)^{-1/k}\frac{1}{\Gamma(1-1/2k)}\left(\frac{\lambda}{2}\right)^{1/2k}J_{1/2k}(\lambda)\\&=\Gamma(1+1/2k)\,k^{1/k}\left(\frac{\lambda}{2}\right)^{-1/2k}J_{1/2k}(\lambda),\end{aligned}$$

and (6.1) becomes a special case of a known formula [6]:

$$\int_0^\infty J_0(r\lambda)\,J_{1/2k}(\lambda)\,\lambda^{1-1/2k}\,d\lambda=\begin{cases}\dfrac{1}{\Gamma(1/2k)}\left(\dfrac{1-r^2}{2}\right)^{-1+1/2k}, & 0\le r<1,\\ 0, & r>1.\end{cases}\tag{6.3}$$

Similarly, we have shown that the partial Fourier transform of the homogeneous Green's function K_{nmk} for the degenerate elliptic operator (5.2) with $n=1$ and $m=2$ is $G_{|\xi|}$ of (5.5), so

$$\begin{aligned}-\frac{k}{8\pi}K_0(t,x;0,s)&=\frac{1}{(2\pi)^2}\int_{\mathbf{R}^2}e^{ix\cdot\xi}G_\xi(t,s)\,d\xi\\&=\frac{1}{2\pi}\int_0^\infty J_0(|x|\lambda)\,G_\lambda(t,s)\,\lambda\,d\lambda\end{aligned}\tag{6.4}$$

where G_λ is given by (5.5) with $|\xi|=\lambda$ and K_0 is defined in (2.2) or (2.3).

As before, both sides of (6.4) simplify when $s=0$. Again we normalize with $t=k^{1/k}$. Using the analogues of (6.2), we obtain

$$K_0(x,t;0,0)=2k^{-2+1/k}(1+|x|^2)^{-1+1/2k},$$

while

$$\begin{aligned}G_\lambda(t,0)&=-\frac{\pi/2k}{2\sin(\pi/2k)}\left(\frac{\lambda}{2k}\right)^{-1/k}A_\lambda(0)[A_\lambda(t)-B_\lambda(t)]\\&=-\frac{\pi/2k}{2\sin(\pi/2k)}\left(\frac{\lambda}{2k}\right)^{-1/k}\frac{1}{\Gamma(1-1/2k)}\left(\frac{\lambda}{2}\right)^{1/2k}[I_{-1/2k}(\lambda)-I_{1/2k}(\lambda)].\end{aligned}$$

In the normalization [6] (which differs from that in [18]),

$$I_{-1/2k} - I_{1/2k} = \frac{2\sin(\pi/2k)}{\pi} K_{1/2k},$$

where K_ν denotes the modified Bessel function of the third kind. Therefore equation (6.4) becomes a special case of a known formula [6]:

$$\int_0^\infty J_0(r\lambda)\, K_{1/2k}(\lambda)\, \lambda^{1-1/2k}\, d\lambda = \frac{1}{2}\,\Gamma(1-1/2k)\left(\frac{1+r^2}{2}\right)^{-1+1/2k}. \tag{6.5}$$

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