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An interpolation theorem between one-sided Hardy spaces

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Abstract. In this paper we generalize an interpolation result due to J.-O. Strömberg and A. Torchinsky to the case of one-sided Hardy spaces. This generalization is important in the study of the weak type (1,1) for lateral strongly singular operators. We shall need an atomic decomposition in which for every atom there exists another atom supported contiguously at its right. In order to obtain this decomposition we have developed a rather simple technique to break up an atom into a sum of others atoms.

1. Definitions and prerequisites

Let $f(x)$ be a Lebesgue measurable function defined on $\mathbf{R}$. The one-sided Hardy–Littlewood maximal functions $M^+f(x)$ and $M^-f(x)$ are defined as

$$M^+f(x)=\sup_{h>0}\frac{1}{h}\int_x^{x+h}|f(t)|\,dt \quad \text{and} \quad M^-f(x)=\sup_{h>0}\frac{1}{h}\int_{x-h}^{x}|f(t)|\,dt.$$

As usual, a weight ω is a measurable and non-negative function. If $E\subset\mathbf{R}$ is a Lebesgue measurable set, we denote its ω-measure by $\omega(E)=\int_E\omega(t)\,dt$. A function $f(x)$ belongs to L^s_ω, $0<s<\infty$, if $\|f\|_{L^s_\omega}=(\int_{-\infty}^{\infty}|f(x)|^s\omega(x)\,dx)^{1/s}$ is finite.

A weight ω belongs to the class A^+_s, $1<s<\infty$, defined by E. Sawyer in [6], if there exists a constant $C_{\omega,s}$ such that if $-\infty<a<b<c<\infty$ then

$$\left(\int_a^b\omega(t)\,dt\right)\left(\int_b^c\omega(t)^{-1/(s-1)}\,dt\right)^{s-1}\le C_{\omega,s}(c-a)^s.$$

In the limit case of $s=1$ we say that ω belongs to the class A^+_1 if, and only if, $M^-\omega(x)\le C_{\omega,1}\omega(x)$ almost everywhere. We also consider the class $A^+_\infty=\bigcup_{s\ge1}A^+_s$.

In general a weight belonging to A^+_∞ can be equal to zero in a set with positive measure. For simplicity in this paper we shall assume all weights being positive almost everywhere.

As usual, $C_0^\infty(\mathbf{R})$ denotes the set of all functions with compact support having derivatives of all orders. We denote by $\mathcal{D}$ the space of all functions in $C_0^\infty(\mathbf{R})$ equipped with the usual topology and by $\mathcal{D}'$ the space of distributions on $\mathbf{R}$.

Given a positive integer γ and $x\in\mathbf{R}$, we shall say that a function ψ in $C_0^\infty(\mathbf{R})$ belongs to the class $\Phi_\gamma(x)$ if there exists a bounded interval $I_\psi=[x,b]$ containing the support of ψ such that $D^\gamma\psi$ satisfies

$$|I_\psi|^{\gamma+1}\|D^\gamma\psi\|_\infty\le 1.$$

For $f\in\mathcal{D}'$ we consider the one-sided maximal function

$$f^*_{+,\gamma}(x)=\sup\{|\langle f,\psi\rangle|:\psi\in\Phi_\gamma(x)\}.$$

Let $\omega\in A_s^+$, $0<p<\infty$ and that γ be a positive integer such that $p(\gamma+1)>s$. We say that a distribution f in $\mathcal{D}'$ belongs to $H_+^p(\omega)$ if the "p-norm" $\|f\|_{H_+^p(\omega)}=(\int_{-\infty}^\infty f^*_{+,\gamma}(x)^p\omega(x)\,dx)^{1/p}$ is finite. These spaces have been defined by L. de Rosa and C. Segovia in [4]. They also prove in [5] that the definition does not depend on γ.

Let N be an integer. A function $a(x)$ defined on $\mathbf{R}$ is called an (∞,N)-atom if there exists an interval I containing the support of $a(x)$, such that

(i) $\|a\|_\infty\le 1$;

(ii) the identity $\int_I a(y)y^k\,dy=0$ holds for every integer k, $0\le k\le N$.

The following theorem gives an atomic decomposition of the $H_+^p(\omega)$ spaces.

Theorem 1. *Let $\omega\in A_s^+$ and $0<p<\infty$. Then there is an integer $N(p,\omega)$ with the following property: given any $f\in H_+^p(\omega)$ and $N\ge N(p,\omega)$, we can find a sequence $\{\lambda_k\}_{k=1}^\infty$ of positive coefficients and a sequence $\{a_k\}_{k=1}^\infty$ of (∞,N)-atoms with support contained in intervals $\{I_k\}_{k=1}^\infty$ respectively, such that the sum $\sum_{k=1}^\infty\lambda_k a_k$ converges unconditionally to f both in the sense of distributions and in the $H_+^p(\omega)$-norm. Moreover*

$$C_1\Big\|\sum_{k=1}^\infty\lambda_k\chi_{I_k}\Big\|_{L^p_\omega}\le\|f\|_{H_+^p(\omega)},$$

holds with $C_1>0$ not depending on f.

Conversely, if we have a sequence $\{\lambda_k\}_{k=1}^\infty$ of positive coefficients and a sequence $\{a_k\}_{k=1}^\infty$ of (∞,N)-atoms with support contained in intervals $\{I_k\}_{k=1}^\infty$, respectively, such that $\|\sum_{k=1}^\infty\lambda_k\chi_{I_k}\|_{L^p_\omega}<\infty$, then $\sum_{k=1}^\infty\lambda_k a_k$ converges unconditionally both in the sense of distributions and in the $H_+^p(\omega)$-norm to an element

$f \in H^p_+(\omega)$ and

$$\|f\|_{H^p_+(\omega)} \le C_2 \left\| \sum_{k=1}^{\infty} \lambda_k \chi_{I_k} \right\|_{L^p_\omega},$$

holds with C_2 not depending on f.

Moreover, in case $f \in L^1_{\text{loc}}(\mathbf{R})$ then

$$\left| \sum_{k=1}^{\infty} \lambda_k^r \chi_{I_k}(x) \right| \le C_r f^*_{+,N}(x)^r, \tag{1}$$

holds for every $r>0$ with $C_r = c^r 2^r/(2^r-1)$, where c is an absolute constant.

For $0<p\le 1$, this result is essentially proved in [4], and it can be generalized to the case $p>1$ following the ideas of Theorem 1 of Chapter VII in [7].

Through this paper the letters c and C will mean positive finite constants not necessarily the same at each occurrence.

2. Statement of the main results

We shall denote by Ω the strip in the complex plane $\{z \in \mathbf{C} : 0 \le \text{Re}(z) \le 1\}$.

For $0<p_0, p_1<\infty$ and $z\in\Omega$ we define $p(z)$ as

$$\frac{1}{p(z)} = \frac{1-z}{p_0} + \frac{z}{p_1}.$$

Let ω and ν be weights in A^+_∞ such that $\omega(x)>0$ and $\nu(x)>0$ for almost $x\in\mathbf{R}$.

Given $0\le u\le 1$ we define

$$\mu(u) = \frac{u\, p(u)}{p_1},$$

and, using the same notation,

$$\mu(u)(x) = \omega(x)^{1-\mu(u)} \nu(x)^{\mu(u)}. \tag{2}$$

It will be proved in Lemma 4 that $\mu(u) \in A^+_\infty$.

In the same way, we define $q(z)$ and $\tilde{\mu}(u)$, for $0<q_0, q_1<\infty$, $\widetilde{\omega}$ and $\tilde{\nu}$ weights in A^+_∞.

Let s be a fixed real number, $0<s<1$, we shall put $p=p(s)$, $q=q(s)$, $\mu=\mu(s)$, $\tilde{\mu}=\tilde{\mu}(s)$.

With this notation we have the following result.

Theorem 2. *Let* $T_z\colon H_+^{p_0}(\omega)+H_+^{p_1}(\nu)\to H_+^{q_0}(\widetilde{\omega})+H_+^{q_1}(\tilde{\nu})$ *be a family of linear operators for* $z\in\Omega$ *such that if* $f\in H_+^{p_0}(\omega)+H_+^{p_1}(\nu)$ *then* $T_z f*\phi_t(x)$ *is uniformly continuous and bounded for* $z\in\Omega$ *and* (x,t) *in any compact subset of* $\{(x,t):x\in\mathbf{R}$ *and* $t>0\}$ *and analytic for* z *in the interior of* Ω.

If, in addition, for some constants $k,\beta>0$,

$$\|T_{it}f\|_{H_+^{q_0}(\widetilde{\omega})}\le ke^{\beta|t|}\|f\|_{H_+^{p_0}(\omega)}\quad \textit{and}\quad \|T_{1+it}f\|_{H_+^{q_1}(\tilde{\nu})}\le ke^{\beta|t|}\|f\|_{H_+^{p_1}(\nu)},$$

then there exists a constant C *such that*

$$\|T_s f\|_{H_+^q(\tilde{\mu})}\le C\|f\|_{H_+^p(\mu)}$$

for all $f\in H_+^p(\mu)$.

This type of result in complex interpolation theory was introduced by A. P. Calderón in [1]. By standard arguments, the previous theorem is a consequence of the following result.

Theorem 3. *Let* $\phi\in C_0^\infty(\mathbf{R})$ *with* $\operatorname{supp}(\phi)\subset(-\infty,0]$ *and* $\int_{\mathbf{R}}\phi\neq0$.

(a) *Let* $f(z)$ *be a function of the complex variable* $z\in\Omega$ *which takes values in* $H_+^{p_0}(\omega)+H_+^{p_1}(\nu)$, *such that* $F(x,t,z)=f(z)*\phi_t(x)$ *is uniformly continuous and bounded for* $z\in\Omega$ *and* (x,t) *in any compact subset of* $\{(x,t):x\in\mathbf{R}$ *and* $t>0\}$ *and analytic for* z *in the interior of* Ω. *If* $f(it)\in H_+^{p_0}(\omega)$, $\sup_t\|f(it)\|_{H_+^{p_0}(\omega)}<\infty$, $f(1+it)\in H_+^{p_1}(\nu)$ *and* $\sup_t\|f(1+it)\|_{H_+^{p_1}(\nu)}<\infty$, *then* $f(s)\in H_+^p(\mu)$ *and*

$$\|f(s)\|_{H_+^p(\mu)}\le c\Big(\sup_t\|f(it)\|_{H_+^{p_0}(\omega)}\Big)^{1-s}\Big(\sup_t\|f(1+it)\|_{H_+^{p_1}(\nu)}\Big)^{s}.$$

(b) *If* $f\in H_+^p(\mu)$, *there exists a function* $f(z)$ *such that* $F(x,t,z)$ *has the properties stated in the first part,* $f(s)=f$ *and*

$$\|f(u+it)\|^{p(u)}_{H_+^{p(u)}(\mu(u))}\le c\|f\|^p_{H_+^p(\mu)} \tag{3}$$

holds for $z=u+it\in\Omega$ *and* c *does not depend on* f.

S. Chanillo in [2] proved that strongly singular operators associated with kernels of the type $e^{i|x|^{-b}}/|x|$ for $0<b<\infty$ satisfy a weighted weak type $(1,1)$ inequality for weights in Muckenhoupt's class A_1. Chanillo solves a key point of his proof with a version of Theorem 3 for weights in the class A_∞, due to J.-O. Strömberg and

A. Torchinsky ([7]). To obtain Chanillo's result for one-sided strongly singular operators and for weights in the class A_1^+ that key point is solved with Theorem 3.

3. Some preliminary results

In the following lemmas we state some results we shall need to prove Theorem 3. They correspond to Lemma 7 and Lemma 8 of Chapter XII of [7].

Lemma 4. *Let* $\omega \in A_p^+$, $\nu \in A_q^+$, $0<\delta<1$ *and* $\eta>0$. *If* $s=p(1-\delta)+q\delta$ *then*

$$\omega(x)^{1-\delta}\nu(x)^{\delta} \in A_s^+,$$

and there exists a constant $C=C(\eta,\delta,\omega,\nu)$ *such that*

$$\left(\frac{1}{|I^-|}\int_{I^-}\omega(x)\,dx\right)^{1-\delta}\left(\frac{1}{|I^-|}\int_{I^-}\nu(x)\,dx\right)^{\delta} \le C\left(\frac{1}{|I^+|}\int_{I^+}\omega(x)^{1-\delta}\nu(x)^{\delta}\,dx\right)$$

holds for every pair of intervals $I^-=(a,b)$ *and* $I^+=(b,c)$ *with* $b-a=\eta(c-b)$.

Proof. Assume that $p>1$ and $q>1$. By applying Hölder's inequality with exponents $\alpha=(s-1)/(p-1)(1-\delta)$ and $\beta=(s-1)/(q-1)\delta$, it is easy to see that $\omega(x)^{1-\delta}\nu(x)^{\delta}\in A_s^+$ with constant $C_{\omega,p}^{1-\delta}C_{\nu,q}^{\delta}$.

Let us prove the rest of the lemma. Since $\omega\in A_p^+$, $\nu\in A_q^+$ and by Hölder's inequality with exponents α and β, we have

$$\begin{aligned}
&\left(\int_{I^-}\omega(x)\,dx\right)^{1-\delta}\left(\int_{I^-}\nu(x)\,dx\right)^{\delta}\\
&\le C_{\omega,p}^{1-\delta}C_{\nu,q}^{\delta}(c-a)^s\left[\left(\int_{I^+}\omega(x)^{-1/(p-1)}\,dx\right)^{(p-1)(1-\delta)}\left(\int_{I^+}\nu(x)^{-1/(q-1)}\,dx\right)^{(q-1)\delta}\right]^{-1}\\
&\le C_{\omega,p}^{1-\delta}C_{\nu,q}^{\delta}(c-a)^s\left(\int_{I^+}(\omega(x)^{1-\delta}\nu(x)^{\delta})^{-1/(s-1)}\,dx\right)^{(s-1)(-1)}\\
&\le C_{\omega,p}^{1-\delta}C_{\nu,q}^{\delta}\left(\frac{c-a}{|I^+|}\right)^{s}\int_{I^+}\omega(x)^{1-\delta}\nu(x)^{\delta}\,dx.
\end{aligned}$$

If $b-a=\eta(c-b)$ then

$$\left(\frac{1}{|I^-|}\int_{I^-}\omega(x)\,dx\right)^{1-\delta}\left(\frac{1}{|I^-|}\int_{I^-}\nu(x)\,dx\right)^{\delta} \le C\frac{1}{|I^+|}\int_{I^+}\omega(x)^{1-\delta}\nu(x)^{\delta}\,dx,$$

where $C=C_{\omega,p}^{1-\delta}C_{\nu,q}^{\delta}(1+\eta)^s/\eta$. □

Lemma 5. *Given $0<p<\infty$ and $\eta>0$ there exists a constant $C=C(p,\eta)$ such that if δ is a weight on the real line then*

$$\left\|\sum_{k=1}^{\infty}\lambda_k\chi_{I_k}\right\|_{L^p_\delta}\le C\left\|\sum_{k=1}^{\infty}\lambda_k\chi_{E_k}\right\|_{L^p_\delta}$$

holds for every $\lambda_k>0$ and for all intervals I_k and all δ-measurable E_k, $E_k\subset I_k$ with $\delta(E_k)\ge\eta\delta(I_k)$.

Proof. The main ideas of the proof can be found on p. 116 of [7]. For the sake of completeness we give a proof here.

For the case $1\le p<\infty$, let $g\in L^{p'}_\delta$, where $pp'=p+p'$. We assume that $g\ge 0$ and that $\|g\|_{L^{p'}_\delta}=1$. Since $M_\delta(g)(y)=\sup_{I\ni y}(1/\delta(I))\int_I g(t)\delta(t)\,dt$ then we have

$$\begin{aligned}\int_{\mathbf{R}}\chi_{I_k}(y)g(y)\delta(y)\,dy&\le\delta(I_k)\inf_{y\in E_k}M_\delta(g)(y)\\&\le\frac{\delta(E_k)}{\eta}\inf_{y\in E_k}M_\delta(g)(y)\le\frac{1}{\eta}\int_{\mathbf{R}}\chi_{E_k}(y)M_\delta(g)(y)\delta(y)\,dy.\end{aligned}$$

Thus,

$$\begin{aligned}\int_{\mathbf{R}}\left(\sum_{k=1}^{\infty}\lambda_k\chi_{I_k}(y)\right)g(y)\delta(y)\,dy&\le\frac{1}{\eta}\int_{\mathbf{R}}\left(\sum_{k=1}^{\infty}\lambda_k\chi_{E_k}(y)\right)M_\delta(g)(y)\delta(y)\,dy\\&\le\frac{1}{\eta}\left\|\sum_{k=1}^{\infty}\lambda_k\chi_{E_k}\right\|_{L^p_\delta}\|M_\delta(g)\|_{L^{p'}_\delta}.\end{aligned}$$

Now, since we are working on the real line, we have that $\|M_\delta(g)\|_{L^{p'}_\delta}\le C_p\|g\|_{L^{p'}_\delta}$ holds for every weight δ, with $C_p^{p'}=2^{p'+1}p$ and therefore

$$\int_{\mathbf{R}}\left(\sum_{k=1}^{\infty}\lambda_k\chi_{I_k}(y)\right)g(y)\delta(y)\,dy\le C_p\frac{1}{\eta}\left\|\sum_{k=1}^{\infty}\lambda_k\chi_{E_k}\right\|_{L^p_\delta}.$$

From this inequality we obtain immediately the conclusion for $p\ge1$.

Now, let us prove the case $0<p<1$. By denoting $\Psi=\sum_{k=1}^{\infty}\lambda_k\chi_{E_k}$ and $\Phi=\sum_{k=1}^{\infty}\lambda_k\chi_{I_k}$, for a fixed $t>0$, we define

$$\mathcal{E}=\{x\in\mathbf{R}:\Psi(x)>t\}\quad\text{and}\quad\mathcal{O}=\left\{x\in\mathbf{R}:M_\delta\chi_{\mathcal{E}}(x)>\frac{\eta}{2}\right\}.$$

Since we are working on the real line, M_δ is of weak type $(1,1)$ with constant 2 with respect to the weight δ. So, we have

$$\delta(\mathcal{O})\le\frac{4}{\eta}\delta(\mathcal{E}).\tag{4}$$

If $\mathcal{O}^c \cap I_k \neq \varnothing$ then $\delta(\mathcal{E}\cap E_k) \le \delta(\mathcal{E}\cap I_k) \le \frac{1}{2}\eta\delta(I_k) \le \frac{1}{2}\delta(E_k)$ and consequently,

$$\delta(E_k) \le \tfrac{1}{2}\delta(E_k) + \delta(\mathcal{E}^c \cap E_k).$$

Therefore

(5) $$\eta\delta(\mathcal{E}^c \cap I_k) \le \delta(E_k) \le 2\delta(\mathcal{E}^c \cap E_k), \quad \text{if } \mathcal{O}^c \cap I_k \neq \varnothing.$$

Taking $r>1$ and using that $\mathcal{O}^c \subseteq \mathcal{E}^c$ we obtain

$$\int_{\mathcal{O}^c} \Phi(x)^r \delta(x)\,dx \le \int_{\mathcal{E}^c} \biggl(\sum_{\mathcal{O}^c \cap I_k \neq \varnothing} \lambda_k \chi_{I_k}(x)\biggr)^r \delta(x)\,dx.$$

Since (5) holds, we can apply, with the weight $\delta(x)\chi_{\mathcal{E}^c}(x)$, the case $r>1$ we have just proved, to estimate the last term by

$$\frac{C_r}{\eta} \int_{\mathcal{E}^c} \biggl(\sum_{\mathcal{O}^c \cap I_k \neq \varnothing} \lambda_k \chi_{E_k}(x)\biggr)^r \delta(x)\,dx.$$

So, we have

(6) $$\int_{\mathcal{O}^c} \Phi(x)^r \delta(x)\,dx \le \frac{C_r}{\eta} \int_{\mathcal{E}^c} \Psi(x)^r \delta(x)\,dx.$$

From $\delta(\{x:\Phi(x)>t\}) \le \delta(\mathcal{O}) + \delta(\mathcal{O}^c \cap \{x:\Phi(x)>t\})$ and by (4) we have

$$\delta(\{x:\Phi(x)>t\}) \le \frac{4}{\eta}\delta(\mathcal{E}) + \frac{1}{t^r}\int_{\mathcal{O}^c} \Phi(x)^r \delta(x)\,dx.$$

By (6) we obtain

$$\delta(\{x:\Phi(x)>t\}) \le \frac{4}{\eta}\delta(\mathcal{E}) + \frac{C_r}{\eta t^r}\int_{\mathcal{E}^c} \Psi(x)^r \delta(x)\,dx.$$

From the estimation above, we get

$$\begin{aligned}\biggl\|\sum_{k=1}^{\infty} \lambda_k \chi_{I_k}\biggr\|_{L^p_\delta}^p &= \int_0^\infty p t^{p-1} \delta(\{x:\Phi(x)>t\})\,dt \\ &\le \frac{4}{\eta}\int_0^\infty p t^{p-1}\delta(\mathcal{E})\,dt + \frac{C_r}{\eta}\int_0^\infty p t^{p-1-r}\int_{\mathcal{E}^c} \Psi(x)^r \delta(x)\,dx\,dt.\end{aligned}$$

So, the lemma follows since for $0<p<1$ the last term equals

$$\biggl(\frac{4}{\eta} + \frac{C_r}{\eta}\frac{p}{r-p}\biggr)\biggl\|\sum_{k=1}^{\infty} \lambda_k \chi_{E_k}\biggr\|_{L^p_\delta}^p. \quad \square$$

The next lemma is contained in Theorem 1 of [3].

Lemma 6. *Let* $\delta\in A^+_\infty$.

(a) *There exists* $\beta>0$ *such that the following implication holds*: *given* $\lambda>0$ *and an interval* (a,b) *such that* $\lambda\le\delta(a,x)/(x-a)$ *for all* $x\in(a,b)$, *then*

$$|\{x\in(a,b):\delta(x)>\beta\lambda\}|>\tfrac{1}{2}(b-a).$$

(b) *There exists* $\gamma>0$ *such that the following implication holds*: *given* $\lambda>0$ *and an interval* (a,b) *such that* $\lambda\ge\delta(x,b)/(b-x)$ *for all* $x\in(a,b)$, *then*

$$\delta(\{x\in(a,b):\delta(x)<\gamma\lambda\})>\tfrac{1}{2}\delta(a,b).$$

4. An appropriate atomic decomposition

In this section we give an atomic decomposition of a distribution $f\in H^p_+(\omega)$ with additional properties that we shall need.

We shall say that an interval J *follows* the interval I if $I=[c,d]$ and $J=[d,e]$.

Our goal is to prove that given $f\in H^p_+(\omega)$ there is an atomic decomposition as stated in Theorem 1 such that for every atom a_k supported in an interval I_k there is another atom a_j supported in an interval I_j following I_k.

First we shall need a couple of lemmas in order to 'break up' an atom.

Lemma 7. *Let* $r>0$. *There exists a sequence* $\{\eta_j\}_{j=-\infty}^{\infty}$ *of* C_0^∞ *functions such that*

(a) $0\le\eta_j\le1$ *and* $\sum_{j=1}^{\infty}\eta_j(x)=\chi_{(-\infty,r)}(x)$.

(b) $\operatorname{supp}(\eta_j)\subset I_j=[r-2^{-j}r,r-2^{-j-2}r]$.

(c) *If we let* $r_j=r/2^j$ *and* $x\in I_j$ *then* $\frac{1}{4}r_j\le r-x\le r_j$.

(d) *Each* x *belongs to at most three intervals* I_j.

(e) *For every non-negative integer* m *there exists a positive constant* c_m *such that* $|D^m\eta_j(x)|\le c_m r_j^{-m}$.

Proof. Let

$$h(y)=\int_y^{y/2}\rho(t)\,dt,$$

where ρ is a non-negative $C_0^\infty(\mathbf{R})$ function with support contained in $[-2,-1]$ and $\int_{\mathbf{R}}\rho(t)\,dt=1$. We define

$$\eta_j(x)=h\Big(\frac{x-r}{2^{-j-2}r}\Big).$$

It is not hard to see that $\{\eta_j\}_{j=-\infty}^{\infty}$ satisfy the five conditions. For details see [4]. □

Lemma 8. *Let $a(y)$ be an (∞, N)-atom with support contained in an interval I. There exists a sequence $\{a_j(x)\}_{j=1}^{\infty}$ of (∞, N)-atoms with associated intervals $\{I_j\}_{j=1}^{\infty}$ such that $a(x)=c\sum_{j=1}^{\infty} a_j(x)$ almost everywhere, with c being a positive constant independent of $a(x)$. In addition $I=\bigcup_{j=1}^{\infty} I_j$ and no point $x\in I$ belongs to more than three intervals I_j.*

Proof. Without lost of generality, we can assume that $I=[0,r]$. Since $a(y)$ is bounded with compact support,

$$A(x)=\frac{1}{N!}\int_x^{\infty}(y-x)^N a(y)\,dy$$

is well defined.

The vanishing moments condition of $a(x)$ implies that $\operatorname{supp}(A)\subset[0,r]$. Moreover, it is not hard to see that $D^{N+1}A(x)=(-1)^{N+1}a(x)$ for almost every x.

Let $\{\eta_j\}_{j=-\infty}^{\infty}$ be the functions of Lemma 7 associated to the interval $(-\infty, r)$. Then, by condition (a) of Lemma 7, we have

$$A(x)=\sum_{j=-1}^{\infty} A(x)\eta_j(x). \tag{7}$$

If we let $A_j(x)=A(x)\eta_j(x)$ and $b_j(x)=(-1)^{N+1}D^{N+1}A_j(x)$, we have that $\operatorname{supp}(b_j)\subset[0,r]\cap[r-2^{-j}r, r-2^{-j-2}r]=I_j$, and, since $\operatorname{supp}(A_j)$ is bounded, it can be shown by integration by parts that $b_j(x)$ has N vanishing moments.

We claim that $\|b_j\|_\infty\le c$, where c only depends on N. By Leibniz's formula we have

$$D^{N+1}A_j(x)=\sum_{k=0}^{N+1} c_{k,N}D^kA(x)D^{N+1-k}\eta_j(x). \tag{8}$$

For $x\in\operatorname{supp}(\eta_j)$ and $k\le N$, since $\|a\|_\infty\le 1$, we obtain

$$|D^kA(x)|\le c\int_x^r (t-x)^{N-k}|a(t)|\,dt\le c(r-x)^{N+1-k}.$$

Thus, from (8) and conditions (e) and (c) of Lemma 7, we get

$$\begin{aligned}|b_j(x)|=|D^{N+1}A_j(x)|&\le c_N\sum_{k=0}^{N+1}(r-x)^{N+1-k}|D^{N+1-k}\eta_j(x)| \\ &\le c_N\sum_{k=0}^{N+1}\frac{(r-x)^{N+1-k}}{r_j^{N+1-k}}\le c_N(N+2)=c.\end{aligned} \tag{9}$$

Furthermore, as a consequence of (7), we have,

$$a(x)=c\sum_{j=-1}^{\infty} a_j(x)$$

for a.e. x, where $a_j(x)=b_j(x)/c$ with c as in (9) are (∞, N) atoms. $\square$

Remark 9. We observe that I_{j+2} follows I_j and that $|I_{j+2}|\leq|I_j|\leq 4|I_{j+2}|$.

Taking into account this remark and as a consequence of Theorem 1 and the previous lemma we have the following result.

Theorem 10. *Let $\omega\in A_s^+$ and $0<p<\infty$. Then there is an integer $N(p,\omega)$ with the following property: given any $f\in H_+^p(\omega)$ and $N\geq N(p,\omega)$, we can find a sequence $\{\lambda_k\}_{k=1}^{\infty}$ of positive coefficients and a sequence $\{a_{k,j}(x)\}_{k,j=1}^{\infty}$ of (∞,N)-atoms with support contained in intervals $\{I_{k,j}\}_{k,j=1}^{\infty}$ respectively such that the sum $\sum_{k,j=1}^{\infty}\lambda_k a_{k,j}$ converges unconditionally to f both in the sense of distributions and in the $H_+^p(\omega)$-norm. Moreover,*

$$\|f\|_{H_+^p(\omega)}\sim\left\|\sum_{k,j=1}^{\infty}\lambda_k\chi_{I_{k,j}}\right\|_{L_\omega^p}$$

and, for every j and k, $I_{k,j+2}$ follows $I_{k,j}$, and $|I_{k,j+2}|\leq|I_{k,j}|\leq 4|I_{k,j+2}|$.

5. Proof of Theorem 3

The first part of the theorem can be obtained as in the proof of Theorem 3 in Chapter XII of [7] using the maximal function $M_1^+(f,\phi,x)$ defined on [5]. For the second part it is enough to define $f(z)$ and to prove inequality (3).

Let $f\in H_+^p(\mu)$ then there exists an atomic decomposition $f=\sum_{k,j=1}^{\infty}\lambda_k a_{k,j}$ as the one given in Theorem 10. With the notation introduced in Section 2 and for $z\in\Omega$, we define

$$f(z)=\sum_{k,j=1}^{\infty}\lambda_{k,j}(z)a_{k,j},$$

where

$$\lambda_{k,j}(z)=\lambda_k^{p/p(z)}\left(\frac{\omega(I_{k,j+2})}{\nu(I_{k,j+2})}\right)^{(z-s)p/p_0p_1}.$$

By Theorem 1 there exists a constant C such that

$$\|f(u+it)\|_{H_+^{p(u)}(\mu(u))}\leq C\left\|\sum_{k,j=1}^{\infty}|\lambda_{k,j}(u+it)|\chi_{I_{k,j}}\right\|_{L_{\mu(u)}^{p(u)}},$$

as long as the right-hand side is finite. Then we shall prove that

$$\left\|\sum_{k,j=1}^{\infty}|\lambda_{k,j}(u+it)|\chi_{I_{k,j}}\right\|_{L^{p(u)}_{\mu(u)}}^{p(u)}\leq C\|f\|^{p}_{H^p_+(\mu)}<\infty.$$

First we claim that there exist $\beta_{k,j}\in I_{k,j+2}$ and a constant c such that, for every $x\in I_{k,j}\cup I_{k,j+2}$,

$$\frac{\mu(u)(x,\beta_{k,j})}{\beta_{k,j}-x}\leq 4c\frac{\mu(u)(I_{k,j+2})}{|I_{k,j+2}|}. \tag{10}$$

In fact, since M^- is of weak type $(1,1)$ with constant 2 with respect to the Lebesgue measure, we have, for every k and j, that

$$\left|\left\{x\in I_{k,j+2}:M^-(\mu(u)\chi_{I_{k,j}\cup I_{k,j+2}})(x)>4\frac{\mu(u)(I_{k,j}\cup I_{k,j+2})}{|I_{k,j+2}|}\right\}\right|\leq\frac{|I_{k,j+2}|}{2}.$$

So, there exists $\beta_{k,j}\in I_{k,j+2}$ such that

$$M^-(\mu(u)\chi_{I_{k,j}\cup I_{k,j+2}})(\beta_{k,j})\leq 4c\frac{\mu(u)(I_{k,j+2})}{|I_{k,j+2}|},$$

where c is the left doubling constant of $\mu(u)$. This implies (10).
We denote by $\alpha_{k,j}$ the left end point of $I_{k,j}$ and

$$E_{k,j}=\left\{x\in(\alpha_{k,j},\beta_{k,j}):\mu(u)(x)<4\gamma c\frac{\mu(u)(I_{k,j+2})}{|I_{k,j+2}|}\right\},$$

where γ is the constant given in Lemma 6(b). By (10) we can apply Lemma 6(b) with $\lambda=4c(\mu(u)(I_{k,j+2}))/|I_{k,j+2}|$ to obtain

$$\mu(u)(E_{k,j})\geq\frac{\mu(u)(\alpha_{k,j},\beta_{k,j})}{2}. \tag{11}$$

Since $I_{k,j}\subset(\alpha_{k,j},\beta_{k,j})$, we have

$$\left\|\sum_{k,j=1}^{\infty}|\lambda_{k,j}(u+it)|\chi_{I_{k,j}}\right\|_{L^{p(u)}_{\mu(u)}}\leq\left\|\sum_{k,j=1}^{\infty}|\lambda_{k,j}(u+it)|\chi_{(\alpha_{k,j},\beta_{k,j})}\right\|_{L^{p(u)}_{\mu(u)}}.$$

From (11) we can apply Lemma 5, and estimate the last term by

$$C\left\|\sum_{k,j=1}^{\infty}|\lambda_{k,j}(u+it)|\chi_{E_{k,j}}\right\|_{L^{p(u)}_{\mu(u)}}=C\left\|\sum_{k,j=1}^{\infty}|\lambda_{k,j}(u+it)|\mu(u)(\,\cdot\,)^{1/p(u)}\chi_{E_{k,j}}\right\|_{L^{p(u)}}.$$

By the definition of $\lambda_{k,j}(u+it)$, taking into account that if $x\in E_{k,j}$ then

$$\mu(u)(x) < 4\gamma c \frac{\mu(u)(I_{k,j+2})}{|I_{k,j+2}|}$$

and from (2), the last term is bounded by

$$C\Bigg\| \sum_{k,j=1}^{\infty} \lambda_k^{p/p(u)} \left[\frac{\omega(I_{k,j+2})}{\nu(I_{k,j+2})}\right]^{(u-s)p/p_0p_1} \times \left[\frac{1}{|I_{k,j+2}|}\int_{I_{k,j+2}} \omega(x)^{1-\mu(u)}\nu(x)^{\mu(u)}\,dx\right]^{1/p(u)} \chi_{E_{k,j}}\Bigg\|_{L^{p(u)}}.$$

Since $(u-s)p/p_0p_1=(\mu(u)-\mu)/p(u)$ and using Hölder's inequality the last expression is bounded by

$$C\Bigg\| \sum_{k,j=1}^{\infty} \lambda_k^{p/p(u)} \left(\frac{\omega(I_{k,j+2})}{\nu(I_{k,j+2})}\right)^{(\mu(u)-\mu)/p(u)} \left[\frac{\omega(I_{k,j+2})^{1-\mu(u)}\nu(I_{k,j+2})^{\mu(u)}}{|I_{k,j+2}|}\right]^{1/p(u)} \chi_{E_{k,j}}\Bigg\|_{L^{p(u)}}$$

$$\leq C\Bigg\| \sum_{k,j=1}^{\infty} \lambda_k^{p/p(u)} \left(\frac{\omega(I_{k,j+2})^{1-\mu}\nu(I_{k,j+2})^{\mu}}{|I_{k,j+2}|}\right)^{1/p(u)} \chi_{I_{k,j}\cup I_{k,j+2}\cup I_{k,j+4}\cup I_{k,j+6}}\Bigg\|_{L^{p(u)}}.$$

Therefore, by Lemma 4 and Lemma 5, we get

$$\|f(u+it)\|_{H_+^{p(u)}(\mu(u))} \leq C\Bigg\| \sum_{k,j=1}^{\infty} \lambda_k^{p/p(u)} \left(\frac{\mu(I_{k,j+4})}{|I_{k,j+4}|}\right)^{1/p(u)} \chi_{I_{k,j+6}}\Bigg\|_{L^{p(u)}}$$

$$\leq C\Bigg\| \sum_{k,j=1}^{\infty} \lambda_k^{p/p(u)} \left(\frac{\mu(I_{k,j+2})}{|I_{k,j+2}|}\right)^{1/p(u)} \chi_{I_{k,j+4}}\Bigg\|_{L^{p(u)}}.$$

Now we shall consider $M_\mu^+ g(x)=\sup_{h>0}(1/\mu(x,x+h))\int_x^{x+h} g(t)\mu(t)\,dt$. Since M_μ^+ is of weak type $(1,1)$ with constant 2 with respect to the weight μ we have

$$\mu\left(\left\{x\in I_{k,j+2} : M_\mu^+(\mu^{-1}\chi_{I_{k,j+2}\cup I_{k,j+4}})(x) > 4\frac{|I_{k,j+2}\cup I_{k,j+4}|}{\mu(I_{k,j+2})}\right\}\right) \leq \frac{\mu(I_{k,j+2})}{2},$$

so we can choose $c_{k,j}\in I_{k,j+2}$ such that

$$M_\mu^+(\mu^{-1}\chi_{I_{k,j+2}\cup I_{k,j+4}})(c_{k,j}) \leq 8\frac{|I_{k,j+2}|}{\mu(I_{k,j+2})}. \tag{12}$$

Since, for every $x\in I_{k,j+2}\cup I_{k,j+4}$, $x>c_{k,j}$,

$$\frac{x-c_{k,j}}{\mu(c_{k,j},x)} \leq M_\mu^+(\mu^{-1}\chi_{I_{k,j+2}\cup I_{k,j+4}})(c_{k,j}),$$

we have from (12) that

$$\frac{\mu(I_{k,j+2})}{8\,|I_{k,j+2}|} \leq \frac{\mu(c_{k,j},x)}{x-c_{k,j}}.$$

Then if we denote by $d_{k,j}$ the right end point of $I_{k,j+4}$ and

$$F_{k,j}=\left\{x\in(c_{k,j},d_{k,j}):\mu(x)>\frac{\beta\mu(I_{k,j+2})}{4|I_{k,j+2}|}\right\},$$

we can apply Lemma 6(a) to obtain

$$|F_{k,j}|\geq\frac{d_{k,j}-c_{k,j}}{2}.$$

So, by Lemma 5 and the definition of $F_{k,j}$,

$$\begin{aligned}\Bigg\|\sum_{k,j=1}^{\infty}\lambda_k^{p/p(u)}\left(\frac{\mu(I_{k,j+2})}{|I_{k,j+2}|}\right)^{1/p(u)}\chi_{I_{k,j+4}}\Bigg\|_{L^{p(u)}}&\\ \leq C\Bigg\|\sum_{k,j=1}^{\infty}\lambda_k^{p/p(u)}\left(\frac{\mu(I_{k,j+2})}{|I_{k,j+2}|}\right)^{1/p(u)}\chi_{F_{k,j}}\Bigg\|_{L^{p(u)}}&\\ \leq C\Bigg\|\sum_{k,j=1}^{\infty}\lambda_k^{p/p(u)}\mu(\,\cdot\,)^{1/p(u)}\chi_{F_{k,j}}\Bigg\|_{L^{p(u)}}\leq C\Bigg\|\sum_{k,j=1}^{\infty}\lambda_k^{p/p(u)}\chi_{I_{k,j+4}}\Bigg\|_{L_\mu^{p(u)}}.&\end{aligned}$$

Therefore,

$$\|f(u+it)\|_{H_+^{p(u)}(\mu(u))}\leq C\Bigg\|\sum_{k,j=1}^{\infty}\lambda_k^{p/p(u)}\chi_{I_{k,j}}\Bigg\|_{L_\mu^{p(u)}}.$$

By a density argument such as the one given on p. 187 of [7], we can assume that $f\in L^1_{\mathrm{loc}}(\mathbf{R})$. Thus, using (1) in the last inequality, we have

$$\|f(u+it)\|_{H_+^{p(u)}(\mu(u))}^{p(u)}\leq C\|f_N^{*\,p/p(u)}\|_{L_\mu^{p(u)}}^{p(u)}=C\|f\|_{H_+^p(\mu)}^p,$$

as we wanted to prove.

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