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Steady periodic capillary waves with vorticity

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Abstract. We prove the existence of steady periodic capillary water waves on flows with arbitrary vorticity distributions. They are symmetric two-dimensional waves whose profiles are monotone between crest and trough.

1. Introduction

Until recently, the mathematical studies of water waves were mainly restricted to irrotational flows. While the irrotational setting is regarded as appropriate for waves traveling into still water [14], [18], there are many situations in which it is necessary to take vorticity into account. For example, nonuniform currents generate water flows with vorticity [22], [21], [23] and the effect of a wind blowing in one direction results at first in the creation of capillary waves.

In the last few years there has been an increasing amount of research in the area of water waves with vorticity, see [3], [4] regarding the symmetry of rotational water waves, [12], [15], [16], [24] for questions of uniqueness and [1], [2], [5], [6], [25], [26] for existence results. However, in all of these recent investigations the surface tension is neglected. It is therefore an interesting task to study the effects of surface tension in the presence of vorticity. As an approximation, we shall in this paper focus on pure capillary waves, that is, we neglect the force of gravity. Experimental studies show that this approximation is valid for short wavelengths ($L \lesssim 1.7$ cm, cf. [19]).

It is interesting to note that there exist explicit capillary solutions in the irrotational case. In water of infinite depth the explicit solutions are known as Crapper's waves [9], and in the case of finite depth as Kinnersley's waves [17]. These are symmetric regular waves (having one crest and one trough per period, and strictly monotone between crest and trough), which show an interesting feature: as the surface amplitude increases, the waves develop sharp troughs and begin to overhang, until eventually a limiting profile is reached, where an air-bubble is trapped at the trough of the wave. For higher amplitudes the wave profile intersects itself and must therefore be discarded on physical grounds.

On the other hand, there are no known explicit examples of rotational capillary waves. In this paper we prove the existence of symmetric regular capillary waves for arbitrary vorticity distributions, provided that the wavelength is small enough. The proof is inspired by the local bifurcation method used for gravity waves in [5] and [6], but there are several interesting differences. As in [5] and [6] the water wave problem is first transformed into a problem in a fixed domain, cf. (2.6). In the case of pure gravity waves this problem included a first order oblique boundary condition. In the presence of surface tension, this boundary condition involves a second order derivative and requires more consideration. Connected with this is the fact that for pure capillary waves the eigenvalue problem (3.3) involved in the bifurcation analysis has an eigenvalue-dependent boundary condition, and is thus not a standard Sturm–Liouville problem.

2. Preliminaries

In this section we present the governing equations for capillary waves [14]. We consider two-dimensional waves propagating over water with a flat bed. In its undisturbed state the equation for the flat surface is $y=0$ and the flat bottom is given by $y=-d$ for some $d>0$. The x-variable represents the direction of propagation and the wavelength is L. The equations of motion are the equation of mass conservation

$$u_x+v_y=0, \tag{2.1}$$

and Euler's equation

$$\begin{cases} u_t+uu_x+vu_y=-P_x, \\ v_t+uv_x+vv_y=-P_y, \end{cases} \tag{2.2}$$

where $P(t,x,y)$ denotes the pressure. The boundary conditions for capillary waves are the dynamic boundary condition

$$P=P_0-\sigma\frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}} \quad \text{on } y=\eta(t,x), \tag{2.3}$$

P_0 being the constant atmospheric pressure and $\sigma>0$ being the coefficient of surface tension, as well as the kinematic boundary conditions

$$v=\eta_t+u\eta_x \quad \text{on } y=\eta(t,x), \tag{2.4}$$

and

$$v = 0 \quad \text{on } y = -d. \tag{2.5}$$

We are looking for steady periodic waves traveling at speed $c>0$, that is, the space-time dependence of the free surface, the pressure, and the velocity field is of the form $(x-ct)$. The map $x-ct \mapsto x$ transforms (2.2)–(2.5) to the stationary problem

$$\begin{cases} (u-c)u_x + vu_y = -P_x, \\ (u-c)v_x + vv_y = -P_y, \end{cases} \quad \text{on } -d < y < \eta(x),$$

and

$$\begin{cases} v = (u-c)\eta_x & \text{at } y = \eta(x), \\ P = P_0 - \sigma \dfrac{\eta_{xx}}{(1+\eta_x^2)^{3/2}} & \text{at } y = \eta(x), \\ v = 0 & \text{at } y = -d, \end{cases}$$

The equation of mass conservation (2.1) allows us to introduce the (relative) stream function ψ, satisfying $\psi_x = -v$ and $\psi_y = u-c$. The kinematic boundary condition then shows that ψ is constant on the free surface, and we determine it uniquely by requiring that the constant value is zero. Field evidence indicates that for waves not near the spilling or breaking state, the propagation speed c of the surface wave is considerably larger then the horizontal velocity u of each individual water particle [19]. It follows that ψ is a strictly decreasing function of y for each fixed x. Let

$$p_0 = \int_{-d}^{\eta(x)} [u(x,y) - c]\, dy$$

be the relative mass flux – it follows by differentiation, using (2.1) and (2.4), that this expression is independent of $x \in \mathbf{R}$. Then by construction $\psi = -p_0$ on the flat bottom. We can now pose problem (2.1)–(2.5) in terms of ψ:

$$\begin{cases} \psi_y \psi_{xy} - \psi_x \psi_{yy} = -P_x, \\ -\psi_y \psi_{xx} + \psi_x \psi_{xy} = -P_y, \end{cases} \quad \text{on } -d < y < \eta(x),$$

and

$$\begin{cases} \psi = 0 & \text{at } y = \eta(x), \\ P = P_0 - \sigma \dfrac{\eta_{xx}}{(1+\eta_x^2)^{3/2}} & \text{at } y = \eta(x), \\ \psi = -p_0 & \text{at } y = -d, \end{cases}$$

where P, ψ and η are L-periodic in the x-variable and $\psi_y < 0$.

The vorticity ω is defined by $\omega = v_x - u_y$. The assumption $u < c$ guarantees that ω is globally a function of ψ, that is, $\omega = \gamma(\psi)$ (see [6]). Thus $\Delta\psi = -\omega = -\gamma(\psi)$. Introduce the function

$$\Gamma(p) = \int_0^p \gamma(-s)\,ds, \quad p_0 \le p \le 0,$$

and let $\Gamma_{\min} \le 0$ be its minimum value. Using the equations of motion and the properties of ψ we can prove Bernoulli's law, which states that

$$E = \frac{(c-u)^2 + v^2}{2} + P - \Gamma(-\psi)$$

is constant throughout the fluid. On the free surface we have

$$E = \frac{(c-u)^2 + v^2}{2} + P_0 - \sigma\frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}},$$

so that letting $Q = 2(E - P_0)$ we obtain

$$\psi_x^2 + \psi_y^2 - 2\sigma\frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}} = Q$$

on the free surface. This condition is equivalent to the dynamic boundary condition.

We have now obtained the following formulation of the capillary wave problem

$$\begin{cases} \Delta\psi = -\gamma(\psi) & \text{on } -d < y < \eta(x), \\ |\nabla\psi|^2 - 2\sigma\dfrac{\eta_{xx}}{(1+\eta_x^2)^{3/2}} = Q & \text{at } y = \eta(x), \\ \psi = 0 & \text{at } y = \eta(x), \\ \psi = -p_0 & \text{at } y = -d. \end{cases}$$

The main difficulty in this formulation lies in the fact that η is not known a priori. For this purpose, we make a change of variables due to Dubreil-Jacotin [11]. Since ψ is constant on the free surface and on the bottom and strictly decreasing as a function of y, we choose the new variables $q = x$ and $p = -\psi(x,y)$. A domain of one wavelength is then transformed to $R = \{(q,p) \in \mathbf{R}^2 : 0 < q < L \text{ and } p_0 < p < 0\}$. Introducing the height function $h(q,p) = y + d$, we have

$$h_q = \frac{v}{u-c} \quad \text{and} \quad h_p = \frac{1}{c-u}.$$

Thus

$$v = -\frac{h_q}{h_p}, \quad u = c - \frac{1}{h_p}, \quad \partial_x = \partial_q - \frac{h_q}{h_p}\partial_p \quad \text{and} \quad \partial_y = \frac{1}{h_p}\partial_p.$$

Note also that $\omega=\gamma(-p)$ and that $\eta(x)=h(q,0)-d$. We obtain the following formulation of the capillary wave problem:

$$\text{(2.6)} \qquad \begin{cases} (1+h_q^2)h_{pp}-2h_p h_q h_{pq}+h_p^2 h_{qq}=-\gamma(-p)h_p^3 & \text{in } p_0<p<0, \\ 1+h_q^2-Qh_p^2-2\sigma\dfrac{h_p^2 h_{qq}}{(1+h_q^2)^{3/2}}=0 & \text{on } p=0, \\ h=0 & \text{on } p=p_0, \end{cases}$$

where h is L-periodic in the q-variable and $h_p>0$ throughout $\overline{R}$.

So far we have derived (2.6) from (2.1)–(2.5). We shall now see that it is also possible to derive (2.1)–(2.5) starting with (2.6). Denote the fluid domain

$$D_\eta=\{(x,y)\in\mathbf{R}^2: -d<y<\eta(x)\}.$$

For the Hölder-parameter $\alpha\in(0,1)$, let $C_{\text{per}}^{m+\alpha}(\overline{D}_\eta)$ be the space of functions $f:\overline{D}_\eta\to\mathbf{R}$ with Hölder-continuous derivatives of exponent α up to order m, and with period L in the x-variable. Similarly, $C_{\text{per}}^{m+\alpha}(\mathbf{R})$ denotes the space of L-periodic real-valued functions on $\mathbf{R}$ of class $C^{m+\alpha}$. A small modification of the argument in [6] proves the following result.

Proposition 1. *Problem* (2.6) *is equivalent to* (2.1)–(2.5). *Furthermore, if* $h\in C_{\text{per}}^{2+\alpha}(\overline{R})$ *then* $(u,v,\eta)\in C_{\text{per}}^{1+\alpha}(\overline{D}_\eta)\times C_{\text{per}}^{1+\alpha}(\overline{D}_\eta)\times C_{\text{per}}^{2+\alpha}(\mathbf{R})$ *and if h is even in the q-variable, then u and η are even in x while v is odd.*

3. Main result

Our main result is the following.

Theorem 1. *Let the wave speed $c>0$, the relative mass flux $p_0<0$ and the vorticity function $\gamma\in C^\alpha[0,|p_0|]$, $0<\alpha<1$, be given. Then for any wavelength $L<L_0$, where L_0 is given by*

$$\text{(3.1)} \qquad L_0=2\pi\sup_{p_1\in M}\left(\frac{\sigma p_1^2-\int_{p_1}^0(p-p_1)^2(2\Gamma(p)-2\Gamma_{\min})^{1/2}\,dp}{\int_{p_1}^0(2\Gamma(p)-2\Gamma_{\min})^{3/2}\,dp}\right)^{1/2},$$

for $M=\{p_1\in[p_0,0]:\sigma p_1^2>\int_{p_1}^0(p-p_1)^2(2\Gamma(p)-2\Gamma_{\min})^{1/2}\,dp\}$, there exists a C^1 curve $\mathcal{C}$ of small amplitude traveling wave solutions (u,v,η) of (2.1)–(2.5) *in the space $C_{\text{per}}^{1+\alpha}(\overline{D}_\eta)\times C_{\text{per}}^{1+\alpha}(\overline{D}_\eta)\times C_{\text{per}}^{2+\alpha}(\mathbf{R})$, with period L, speed c and relative mass flux p_0, satisfying $u<c$ throughout the fluid. The curve $\mathcal{C}$ contains precisely one trivial flow* (*a parallel shear flow with a flat surface*), *while for each nontrivial solution*

$(u, v, \eta) \in \mathcal{C}$ (i) *the functions* u *and* η *are symmetric around the line* $x=0$ *while* v *is antisymmetric,* (ii) *the function* η *has precisely one maximum* (*crest*) *and one minimum* (*trough*) *per period,* (iii) *the wave profile is strictly monotone between crest and trough.*

Remark. (i) Note that the set M appearing in Theorem 1 is non-empty since

$$\int_{p_1}^{0} \frac{(p-p_1)^2}{p_1^2} (2\Gamma(p) - 2\Gamma_{\min})^{1/2} \, dp \to 0$$

as $p_1 \to 0$ by the dominated convergence theorem. If the denominator in (3.1) is zero (e.g. for $\gamma \equiv 0$), we take (3.1) to mean $L_0 = \infty$. In Section 4 we shall see examples of vorticity distributions for which bifurcation occurs for any wavelength, as well as some for which a restriction of the size of L is needed.

(ii) It is obvious from the definition of L_0 that for a fixed L and γ, the theorem holds true if σ is large enough. Although σ varies with temperature [10], we will consider it as having a fixed value rather than as a parameter.

In the rest of this section we fix $L=2\pi$. The condition $L<L_0$ is then a condition only on γ. Before proving Theorem 1 we shall first prove a number of lemmas.

Lemma 1. *The trivial solutions* $h(q,p)=H(p)$ *are*

$$H(p) = H(p; Q) = \int_{p_0}^{p} \frac{ds}{\sqrt{Q + 2\Gamma(s)}},$$

where $0 \le -2\Gamma_{\min} < Q$.

Proof. A solution of the the form $H(p)$ satisfies the ordinary differential equation

$$H_{pp} = -\gamma(-p) H_p^3.$$

Integrating gives

$$H_p(p) = (\lambda + 2\Gamma(p))^{-1/2}$$

for $\lambda > -2\Gamma_{\min}$. The surface boundary condition yields $\lambda = Q$. Integrating once more and keeping in mind the bottom boundary condition gives the above formula. □

We wish to bifurcate from the curve of trivial solutions parameterized by Q. For a linear operator $\mathcal{L}$, let $\mathcal{N}(\mathcal{L})$ be its null space and let $\mathcal{R}(\mathcal{L})$ be its range. Our main tool will be the Crandall–Rabinowitz bifurcation theorem.

Theorem 2. ([8]) *Let X and Y be Banach spaces, I be an open interval in* $\mathbf{R}$ *containing* λ^*, *and* $\mathcal{F}: I\times X\to Y$ *be a continuous map with the following properties*:

(i) $\mathcal{F}(\lambda,0)=0$ *for all* $\lambda\in I$;

(ii) $\mathcal{F}_\lambda$, $\mathcal{F}_w$ *and* $\mathcal{F}_{\lambda w}$ *exist and are continuous*;

(iii) $\mathcal{N}(\mathcal{F}_w(\lambda^*,0))$ *and* $Y/\mathcal{R}(\mathcal{F}_w(\lambda^*,0))$ *are one-dimensional, with the null space generated by* w^*;

(iv) $\mathcal{F}_{\lambda w}(\lambda^*,0)w^*\notin\mathcal{R}(\mathcal{F}_w(\lambda^*,0))$.

Then there exists a continuous local bifurcation curve $\{(\lambda(s),w(s)):|s|<\varepsilon\}$ *with* ε *sufficiently small such that* $(\lambda(0),w(0))=(\lambda^*,0)$ *and*

$$\{(\lambda,w)\in\mathcal{U}:w\neq 0 \text{ and } \mathcal{F}(\lambda,w)=0\}=\{(\lambda(s),w(s))\in\mathcal{U}:0<|s|<\varepsilon\}$$

for some neighborhood $\mathcal{U}$ *of* $(\lambda^*,0)\in I\times X$. *Moreover, we have*

$$w(s)=sw^*+o(s) \quad \text{in } X,\ |s|<\varepsilon.$$

(v) *If* $\mathcal{F}_{ww}$ *is also continuous, then the curve is of class* C^1.

Let R be the open rectangle $(0,2\pi)\times(p_0,0)$, $T=(0,2\pi)\times\{0\}$ be the top, and $B=(0,2\pi)\times\{p_0\}$ be the bottom of its closure $\overline{R}$, and define the spaces

$$X=\{h\in C^{2+\alpha}_{\text{per}}(\overline{R}):h=0 \text{ on } B\} \quad\text{and}\quad Y=C^\alpha_{\text{per}}(\overline{R})\times C^\alpha_{\text{per}}(T),$$

where the subscript "per" means periodicity and symmetry in the variable q.

We define the nonlinear operator $\mathcal{F}: I\times X\to Y$ by

$$\mathcal{F}(Q,w)=(\mathcal{F}_1(Q,w),\mathcal{F}_2(Q,w))$$

for $w\in X$ and $Q\in I=(-2\Gamma_{\min},\infty)$, where

$$\begin{aligned}\mathcal{F}_1(Q,w) &= (1+w_q^2)(H_{pp}+w_{pp})-2(H_p+w_p)w_qw_{pq}\\ &\quad +(H_p+w_p)^2w_{qq}+\gamma(-p)(H_p+w_p)^3\end{aligned}$$

and

$$\mathcal{F}_2(Q,w)=1+w_q^2-Q(H_p+w_p)^2-2\sigma\frac{(H_p+w_p)^2w_{qq}}{(1+w_q^2)^{3/2}}.$$

We have $\mathcal{F}(Q,0)\equiv 0$ by construction.

The derivative of $\mathcal{F}$ with respect to w at $w=0$ is the pair $\mathcal{F}_w=(\mathcal{F}_{1w},\mathcal{F}_{2w})$, where

$$\begin{aligned}\mathcal{F}_{1w} &= \partial_p^2+H_p^2\partial_q^2+3\gamma(-p)H_p^2\partial_p \quad \text{in } R,\\ \mathcal{F}_{2w} &= -2(Q^{1/2}\partial_p+Q^{-1}\sigma\partial_{qq})|_T,\end{aligned}$$

since $H_p(0)=Q^{-1/2}$, so that the linearization of the problem (2.6) at $w=0$ is $\mathcal{F}_w w=0$, i.e.

$$\begin{cases} w_{pp}+H_p^2 w_{qq}=-3\gamma(-p)H_p^2 w_p & \text{in } p_0<p<0,\\ Q^{3/2}w_p+\sigma w_{qq}=0 & \text{on } p=0,\\ w=0 & \text{on } p=p_0. \end{cases} \tag{3.2}$$

Introduce $a_Q=(Q+2\Gamma(p))^{1/2}$. We can then write (3.2) in the form

$$\begin{cases} (a_Q^3 w_p)_p+(a_Q w_q)_q=0 & \text{in } p_0<p<0,\\ a_Q^3 w_p+\sigma w_{qq}=0 & \text{on } p=0,\\ w=0 & \text{on } p=p_0. \end{cases}$$

To investigate the kernel of $\mathcal{F}_w(Q,0)$, we first look for solutions of the form $w=W(p)\cos kq$. This leads to the equation $(a_Q^3 W_p)_p=k^2 a_Q W$ with the boundary conditions $a_Q^3 W_p=k^2\sigma W$ at $p=0$ and $W=0$ at $p=p_0$. Consider the eigenvalue problem

$$\begin{cases} -(a_Q^3 W_p)_p=\mu a_Q W, & \text{in } p_0<p<0,\\ -a_Q^3(0)W_p(0)=\mu\sigma W(0), & \\ W(p_0)=0. & \end{cases} \tag{3.3}$$

We are looking for a Q such that $\mu(Q)=-k^2$ is an eigenvalue([1]). We restrict our attention to $k=1$.

Lemma 2. *The eigenvalue problem* (3.3) *has precisely one negative eigenvalue,* $\mu_-(Q)$, *and* 0 *is not an eigenvalue.*

Proof. Let $a=a_Q$. We introduce the Pontryagin space $\mathbf{H}=L^2[p_0,0]\times\mathbf{C}$, with the indefinite form

$$[\tilde u_1,\tilde u_2]=\langle a u_1,u_2\rangle_{L^2}-\frac{1}{\sigma}b_1\bar b_2,$$

where $\tilde u_i=(u_i,b_i)\in\mathbf{H}$, $i=1,2$. It is clear that $\mathbf{H}$ is a π_1-space, i.e. any maximal negative definite (or negative semidefinite) subspace of $\mathbf{H}$ has dimension one. On $\mathbf{H}$ there is also an associated Hilbert space inner product, given by $\langle\tilde u,\tilde v\rangle_{\mathbf{H}}=[J\tilde u,\tilde v]$, where

$$J=\begin{pmatrix} I & 0\\ 0 & -1\end{pmatrix}.$$

([1]) For an arbitrary wavelength L the corresponding condition is $\mu(Q)=-4\pi^2k^2/L^2$.

Define the linear operator K by

$$K\tilde{u}=\left(-\frac{1}{a}(a^3u')',-a^3(0)u'(0)\right),$$

where we take as domain of definition

$$D(K)=\{\tilde{u}=(u,b):u\in H^2[p_0,0],\ u(p_0)=0 \text{ and } b=\sigma u(0)\}.$$

The eigenvalues of (3.3) are precisely the eigenvalues of K. It is clear that $D(K)$ is dense in $\mathbf{H}$ and that K is closed. The identity

$$[K\tilde{u},\tilde{u}]=\langle a^3u',u'\rangle_{L^2}>0, \quad \tilde{u}\neq(0,0), \tag{3.4}$$

shows that K is symmetric, that $K>0$, and that zero is not an eigenvalue.

In fact, K is selfadjoint and has discrete spectrum. To see this, notice that $(K-\mu I)\tilde{u}=\tilde{f}=(f,b)$ is equivalent to the system of equations

$$\begin{cases} -(a^3u')'-\mu au=af, \\ B_1(u):=u(p_0)=0, \\ B_2(u):=-a^3(0)u'(0)-\mu\sigma u(0)=b. \end{cases} \tag{3.5}$$

Let u_1 and u_2 be solutions of the equation $-(a^3u')'-\mu au=0$ with initial data $u_1(0)=0, u_1'(0)=1$ and $u_2(0)=1, u_2'(0)=0$, respectively. The characteristic determinant

$$\Delta(\mu)=\begin{vmatrix} B_1(u_1) & B_1(u_2) \\ B_2(u_1) & B_2(u_2) \end{vmatrix}$$

is an entire function of μ. If μ is not a zero of $\Delta(\mu)$, equation (3.5) is solvable by means of the formula

$$u(p)=c_1u_1(p)+c_2u_2(p)+\int_{p_0}^0 G(p,r,\mu)f(r)\,dr,$$

where G is Green's function for (3.5) and c_1 and c_2 are chosen so that u satisfies the boundary conditions. Clearly, $(u,\sigma u(0))\in D(K)$. On the other hand any zero of $\Delta(\mu)$ is an eigenvalue of K. We have already seen that $\mu=0$ is not an eigenvalue of K. Hence $\Delta(\mu)\not\equiv 0$ and $\Delta(\mu)$ has only isolated zeros of finite multiplicity. Since K is symmetric and closed, it is selfadjoint with discrete spectrum.

Since $\mathbf{H}$ is a π_1-space, K has exactly one negative semidefinite eigenvalue (counting multiplicity) cf. [13], which is in fact negative definite due to (3.4). For any eigenvector $\tilde{u}$ corresponding to the eigenvalue μ, we have

$$\mu[\tilde{u},\tilde{u}]=[K\tilde{u},\tilde{u}]>0.$$

It follows therefore that $\mu<0$ if and only if μ is of negative definite type. □

In the following three lemmas, we let K be the operator from Lemma 2 for a fixed Q, and let $\mathcal{R}=\mathcal{R}(K-\mu_-I)$ and $\mathcal{N}=\mathcal{N}(K-\mu_-I)$. The next two lemmas will be needed in the proof of Lemma 9, where the range of $\mathcal{F}_w(Q^*,0)$ is identified.

Lemma 3. *There exists a positive constant C such that for $\mu<\mu_-$,*

$$\|R_\mu\|_{\mathbf{H}} \le \frac{C}{|\mu-\mu_-|},$$

where $R_\mu=(K-\mu I)^{-1}$.

Proof. Note that $\mathcal{N}$ is a maximal negative definite subspace and that it is invariant under K. Letting $\mathcal{N}^{[\perp]}=\{\tilde{u}:[\tilde{u},\mathcal{N}]=0\}$, we have that $\mathbf{H}=\mathcal{N}\,[+]\,\mathcal{N}^{[\perp]}$ (the sum is $[\,\cdot\,,\cdot\,]$-orthogonal, direct and topological) and that $\mathcal{N}^{[\perp]}$ is a positive definite space. Since K is selfadjoint, $\mathcal{N}^{[\perp]}$ is invariant under K in the sense that $K(D(K)\cap\mathcal{N}^{[\perp]})\subseteq\mathcal{N}^{[\perp]}$. Let $\|\cdot\|_*$ be the norm induced by the decomposition $\mathbf{H}=\mathcal{N}\,[+]\,\mathcal{N}^{[\perp]}$. Then $\|\cdot\|_*$ is equivalent to $\|\cdot\|_{\mathbf{H}}$. Let $\tilde{u}=\tilde{u}_1+\tilde{u}_2\in D(K)$ and $\tilde{v}=(K-\mu I)\tilde{u}=\tilde{v}_1+\tilde{v}_2$, where $\tilde{u}_1,\tilde{v}_1\in\mathcal{N}$ and $\tilde{u}_2,\tilde{v}_2\in\mathcal{N}^{[\perp]}$. By invariance, $\tilde{v}_1=(K-\mu I)\tilde{u}_1$ and $\tilde{v}_2=(K-\mu I)\tilde{u}_2$. Using that K is positive with respect to $[\,\cdot\,,\cdot\,]$ we obtain that $[\tilde{u}_2,\tilde{u}_2]\le[\tilde{v}_2,\tilde{u}_2]/|\mu|$, which yields $\|\tilde{u}_2\|_*\le\|\tilde{v}_2\|_*/|\mu|$ by Cauchy–Schwarz inequality. On the other hand, the identity $\tilde{v}_1=(\mu_--\mu)\tilde{u}_1$ shows that $\|\tilde{u}_1\|_*=\|\tilde{v}_1\|_*/(\mu_--\mu)$. Combining these estimates on $\mathcal{N}$ and $\mathcal{N}^{[\perp]}$ gives that there is a $C>0$ such that $\|\tilde{u}\|_*\le C\|\tilde{v}\|_*/|\mu-\mu_-|$ for $\mu<\mu_-$, which proves the statement. □

Lemma 4. *The range $\mathcal{R}$ of $K-\mu_-I$ is closed and thus $\mathbf{H}=\mathcal{N}\,[+]\,\mathcal{R}$.*

Proof. The relation $[(K-\mu_-I)\tilde{u}+\mu_-\tilde{u},\tilde{u}]\ge0$, gives

$$\|\tilde{u}\|_*\le\frac{1}{|\mu_-|}\|(K-\mu_-I)\tilde{u}\|_* \quad \text{for } \tilde{u}\in D(K)\cap\mathcal{N}^{[\perp]}.$$

But $\overline{\mathcal{R}}=\mathcal{N}^{[\perp]}$. It follows that if $\tilde{v}\in\overline{\mathcal{R}}$ and $(K-\mu_-I)\tilde{u}_n\to\tilde{v}$, where $\tilde{u}_n\in D(K)$, then $\tilde{u}_n\to\tilde{u}\in\mathbf{H}$. Since $K-\mu_-I$ is closed, $\tilde{u}\in D(K)$ and $(K-\mu_-I)\tilde{u}=\tilde{v}$. □

The next lemma is a Rayleigh-principle, which will be needed in order to prove the existence of a point Q^* with $\mu_-(Q^*)=-1$ (see Lemma 7).

Lemma 5. *The negative eigenvalue satisfies*

$$\mu_-=\max\frac{[K\tilde{u},\tilde{u}]}{[\tilde{u},\tilde{u}]}, \quad \tilde{u}\in D(K),\ [\tilde{u},\tilde{u}]<0.$$

Proof. We have

$$\begin{aligned}[K\tilde{u},\tilde{u}]&=[K\tilde{u}_1,\tilde{u}_1]+[K\tilde{u}_2,\tilde{u}_2]\geq[K\tilde{u}_1,\tilde{u}_1]\\&=\mu_-[\tilde{u}_1,\tilde{u}_1]\geq\mu_-([\tilde{u}_1,\tilde{u}_1]+[\tilde{u}_2,\tilde{u}_2])=\mu_-[\tilde{u},\tilde{u}],\end{aligned}$$

where $\tilde{u}_1\in\mathcal{N}$ and $\tilde{u}_2\in\mathcal{R}\cap D(K)$. This yields

$$\frac{[K\tilde{u},\tilde{u}]}{[\tilde{u},\tilde{u}]}\leq\mu_-,$$

with equality if and only if $\tilde{u}_2=0$. □

The following lemma shows that the negative eigenvalue is a monotone function of the parameter Q. This will be needed in the proof of Lemma 7.

Lemma 6. *$\mu_-(Q)$ is a strictly decreasing function.*

Proof. Let $\mathcal{B}_1=D(K)$ considered with the $H^2\times\mathbf{C}$-norm, and for a fixed $Q=Q_0$ let $\mathcal{R}$ be the range of $K-\mu_-I$ and $\mathcal{N}=\operatorname{span}\widetilde{W}_0$ its null space. Then $\mathcal{B}_1=\mathcal{N}\dot{+}(\mathcal{R}\cap\mathcal{B}_1)$ by Lemma 4. Define the function $G\colon(\mathcal{B}_1\cap\mathcal{R})\times\mathbf{C}\times(-2\Gamma_{\min},\infty)\to\mathcal{B}_2$, where $\mathcal{B}_2=\mathbf{H}$ endowed with the standard $L^2\times\mathbf{C}$-norm, given by $G(\tilde{u},\mu,Q)=(K(Q)-\mu I)(\widetilde{W}_0+\tilde{u})$, where we consider K as a bounded operator from $\mathcal{B}_1$ to $\mathcal{B}_2$. Then $G_{\tilde{u}}(0,\mu_0,Q_0)=K(Q_0)-\mu_0I$, while $G_\mu(0,\mu_0,Q_0)=-\widetilde{W}_0$, where $\mu_0=\mu_-(Q_0)$. By Lemma 4, $G_{(\tilde{u},\mu)}$ is an isomorphism, and the implicit function theorem guarantees that in a neighborhood of $(0,\mu_0,Q_0)$, $\tilde{u}$ and μ are C^1 functions of Q, where $G(\tilde{u}(Q),\mu(Q),Q)=0$. By uniqueness we must have $\mu(Q)=\mu_-(Q)$. Furthermore, $\widetilde{W}(Q)=\widetilde{W}_0+\tilde{u}(Q)$ is a (local) C^1 curve of generators of the null space of $K(Q)-\mu_-(Q)I$.

The above considerations allow us to differentiate equation (3.3) with respect to Q. Let $Lu=-(a^3u_p)_p$, where $a=a_Q$, and let $W=W(p;Q)$ be defined as a local C^1 curve of solutions to the problem

$$LW=\mu_-aW,\quad W(p_0)=0,\quad W_p(0)=-\mu_-Q^{-3/2}\sigma W(0),$$

where $\mu_-=\mu_-(Q)$ is the negative eigenvalue. Denoting differentiation with respect to Q by $\dot{}$, we have $\dot{a}=1/2a$. Furthermore,

$$L\dot{W}-\frac{3}{2}(aW_p)_p=\dot{\mu}_-aW+\frac{\mu_-}{2a}W+\mu_-a\dot{W},\quad \dot{W}(p_0)=0,$$

and

$$\dot{W}_p(0)=-\dot{\mu}_-Q^{-3/2}\sigma W(0)+\tfrac{3}{2}\mu_-Q^{-5/2}\sigma W(0)-\mu_-Q^{-3/2}\sigma\dot{W}(0).$$

Multiplying the W equation by $\dot{W}$ and vice versa yields after integrating

$$\langle \dot{W}, LW\rangle = \mu_-\langle \dot{W}, aW\rangle$$

and

$$\langle L\dot{W}, W\rangle + \frac{3}{2}\int_{p_0}^0 aW_p^2\,dp - \frac{3}{2}aW_pW\Big|^0 = \dot{\mu}_-\int_{p_0}^0 aW^2\,dp + \int_{p_0}^0 \frac{\mu_-}{2a}W^2\,dp + \mu_-\langle a\dot{W}, W\rangle,$$

where $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the $L^2[p_0,0]$ inner product. On the other hand

$$\begin{aligned}\langle \dot{W}, LW\rangle - \langle L\dot{W}, W\rangle &= \int_{p_0}^0 [-\dot{W}(a^3W_p)_p + (a^3\dot{W}_p)_pW]\,dp\\ &= [-a^3\dot{W}W_p + a^3\dot{W}_pW]|^0.\end{aligned}$$

Combining the last three equations we obtain

$$\frac{3}{2}\int_{p_o}^0 aW_p^2\,dp - \frac{3}{2}aW_pW\Big|^0 = \dot{\mu}_-\int_{p_0}^0 aW^2\,dp + \int_{p_0}^0 \frac{\mu_-}{2a}W^2\,dp + a^3[\dot{W}_pW - \dot{W}W_p]|^0.$$

The boundary terms, evaluated at $p=0$, are

$$\begin{aligned}&a^3(\dot{W}_pW - \dot{W}W_p) + \tfrac{3}{2}aW_pW\\ &\quad= Q^{3/2}\big[-\dot{\mu}_-Q^{-3/2}\sigma W + \tfrac{3}{2}\mu_-Q^{-5/2}\sigma W - \mu_-Q^{-3/2}\sigma\dot{W} + \mu_-Q^{-3/2}\sigma\dot{W}\big]W\\ &\qquad - \mu_-\tfrac{3}{2}Q^{-1}\sigma W^2\\ &\quad= -\dot{\mu}_-\sigma W^2.\end{aligned}$$

We thus have

$$\dot{\mu}_-[\widetilde{W}, \widetilde{W}] = \dot{\mu}_-\left(\int_{p_0}^0 aW^2\,dp - \sigma W^2(0)\right) = -\mu_-\int_{p_0}^0 \frac{1}{2a}W^2\,dp + \frac{3}{2}\int_{p_0}^0 aW_p^2\,dp > 0,$$

so that μ_- is strictly decreasing in view of $[\widetilde{W}, \widetilde{W}] < 0$. □

The next lemma proves the existence of a unique Q^* such that the eigenvalue problem (3.3) has the eigenvalue -1. Theorem 1 will then be proved by applying Theorem 2 with $\lambda^* = Q^*$.

Lemma 7. *If $L_0 > 2\pi$, where L_0 is given by* (3.1), *then there is a unique $Q^* > -2\Gamma_{\min}$ such that $\mu_-(Q^*) = -1$.*

Proof. Let

$$u(p)=\begin{cases} 0, & p_0\le p\le p_1,\\ p-p_1, & p_1\le p\le 0,\end{cases}$$

and $\tilde{u}=(u,\sigma u(0))$. Then

$$[\tilde{u},\tilde{u}]=\int_{p_1}^{0} a_Q(p)(p-p_1)^2\,dp-\sigma p_1^2<0$$

for $p_1\in M$ and Q sufficiently close to $-2\Gamma_{\min}$. Furthermore, as $\lambda\to -2\Gamma_{\min}$,

$$\frac{[K(Q)\tilde{u},\tilde{u}]}{[\tilde{u},\tilde{u}]}=\frac{\int_{p_1}^{0} a_Q^3(p)\,dp}{\int_{p_1}^{0} a_Q(p)(p-p_1)^2\,dp-\sigma p_1^2}\to\frac{\int_{p_1}^{0} a_{-2\Gamma_{\min}}^3(p)\,dp}{\int_{p_1}^{0} a_{-2\Gamma_{\min}}(p)(p-p_1)^2\,dp-\sigma p_1^2}>-1,$$

for some $p_1\in M$. It follows that $\lim_{Q\to -2\Gamma_{\min}}\mu_-(Q)>-1$ by Lemma 5 (the fact that $\tilde{u}\notin D(K)$ can be taken care of by an approximation argument).

On the other hand, let $Q\ge\sigma-2\Gamma_{\min}$. Then $a_Q(p)=\sqrt{Q+2\Gamma(p)}\ge\sqrt{\sigma}$. This yields

$$\int_{p_0}^{0}(a_Q w^2+a_Q^3 w_p^2)\,dp\ge\sqrt{\sigma}\int_{p_0}^{0}(w^2+\sigma w_p^2)\,dp\ge 2\sigma\int_{p_0}^{0} w w_p\,dp=\sigma w^2(0)$$

for $\widetilde{w}\in D(K)$. It follows that if $[\widetilde{w},\widetilde{w}]<0$, then $[K\widetilde{w},\widetilde{w}]/[\widetilde{w},\widetilde{w}]\le -1$. Hence $\mu_-(Q)\le -1$ by Lemma 5. By continuity there is a Q^* such that $\mu_-(Q^*)=-1$ and Q^* is unique by Lemma 6. □

The next lemma identifies the null space of $\mathcal{F}_w(Q^*,0)$ and is needed in order to prove property (iii) in Theorem 2.

Lemma 8. *The null space of $\mathcal{F}_w(Q^*,0)$ is one-dimensional.*

Proof. Expanding an arbitrary function $w\in X$ in a cosine series $w(q,p)=\sum_{k=0}^{\infty} W_k(p)\cos kq$, we obtain that $w\in\mathcal{N}(\mathcal{F}_w(Q^*,0))$ if and only if W_k solves (3.3) with $\mu=-k^2$. Since the only nonpositive eigenvalue of (3.3) for $Q=Q^*$ is -1, it follows that the null space is one-dimensional. □

In the following lemma the range of $\mathcal{F}_w(Q^*,0)$ is identified. This is also needed in order to prove property (iii) in Theorem 2.

Lemma 9. *Let φ generate the kernel of $\mathcal{F}_w(Q^*,0)$. The range of $\mathcal{F}_w(Q^*,0)$ consists of $(\mathcal{A},\mathcal{B})\in Y$ such that*

$$\iint_R \mathcal{A}a^3\varphi\,dq\,dp+\frac{1}{2}\int_T \mathcal{B}a^2\varphi\,dq=0.$$

Proof. The vector $(\mathcal{A},\mathcal{B})$ belongs to the range if and only if $\mathcal{A}=a^{-3}(a^3v_p)_p+a^{-2}v_{qq}$ in R and $\mathcal{B}=-2(av_p+\sigma a^{-2}v_{qq})$ on T, where $a=a_{Q^*}$. We have

$$\begin{aligned}\iint_R \mathcal{A}a^3\varphi\,dq\,dp &= \iint_R ((a^3v_p)_p+av_{qq})\varphi\,dq\,dp\\ &= \iint_R ((a^3\varphi_p)_p+a\varphi_{qq})v\,dq\,dp+\int_T a^3(v_p\varphi-v\varphi_p)\,dq\\ &= \int_T a^3(v_p\varphi-v\varphi_p)\,dq,\end{aligned}$$

where the integral over R vanishes because of the equation satisfied by φ.

On the top we have

$$2(v_p\varphi-v\varphi_p)=2v_p\varphi+2v(\sigma a^{-3}\varphi_{qq})=-a^{-1}\mathcal{B}\varphi+2\sigma a^{-3}(v\varphi_{qq}-v_{qq}\varphi).$$

Thus the last integral equals

$$-\frac{1}{2}\int_T a^2\mathcal{B}\varphi\,dq+\sigma\int_T (v\varphi_{qq}-v_{qq}\varphi)\,dq,$$

where, integrating by parts, the last term disappears. This proves the necessity.

To prove the sufficiency, we expand in cosine series:

$$\mathcal{A}(q,p)=\sum_{k=0}^{\infty}\mathcal{A}_k(p)\cos kq,\quad \mathcal{B}(q)=\sum_{k=0}^{\infty}\mathcal{B}_k\cos kq,$$

where $\sum_{k=0}^{\infty}\|\mathcal{A}_k\|_{L^2}^2<\infty$ and $\sum_{k=0}^{\infty}\mathcal{B}_k^2<\infty$. Letting K be the operator from Lemma 2, we obtain the following sequence of problems

$$(K+k^2)\tilde{u}_k=\left(-a^2\mathcal{A}_k,\frac{a^2(0)}{2}\mathcal{B}_k\right):=\tilde{v}_k,\quad k\ge 0.$$

For $k\neq 1$, there is a unique solution $\tilde{u}_k=R_{-k^2}(v_k)\in D(K)$. For $k=1$, we have to use the orthogonality condition. Let $\varphi(q,p)=W(p)\cos q$. Then it is easy to see that the orthogonality condition means precisely that

$$\int_{p_0}^0 a^3\mathcal{A}_1W\,dp+\frac{a^2(0)}{2}\mathcal{B}_1W(0)=0.$$

But by Lemma 4, $\mathcal{R}(K+I)=(\mathcal{N}(K+I))^{[\perp]}$, so that $\tilde{v}_1\in\mathcal{R}(K+I)$. Hence the equation for $k=1$ is also solvable.

We now let $u=\sum_{k=0}^{\infty} u_k(p)\cos kq$. By Lemma 3, we have for $k\geq 2$ the estimate

$$\|k^2\tilde{u}_k\|_{\mathbf{H}}^2 \leq \frac{C^2k^4}{(k^2-1)^2\|\tilde{v}_k\|_{\mathbf{H}}^2} \leq C_1(\|\mathcal{A}_k\|_{L^2}^2+|\mathcal{B}_k|^2)$$

for some $C_1>0$. From the relation $[K\tilde{u}_k,\tilde{u}_k]=-k^2[\tilde{u}_k,\tilde{u}_k]+[\tilde{v}_k,\tilde{u}_k]$ and (3.4) we obtain

$$k^2\langle a^3u_k',u_k'\rangle_{L^2}=k^2[K\tilde{u}_k,\tilde{u}_k]\leq \|k^2\tilde{u}_k\|_{\mathbf{H}}^2+\tfrac{1}{2}(\|\tilde{v}_k\|_{\mathbf{H}}^2+\|k^2\tilde{u}_k\|_{\mathbf{H}}^2),$$

where we have used that $|[\tilde{u},\tilde{v}]|=|\langle J\tilde{u},\tilde{v}\rangle_{\mathbf{H}}|\leq\|\tilde{u}\|_{\mathbf{H}}\|\tilde{v}\|_{\mathbf{H}}$, since J is unitary. It follows that for some $C_2>0$,

$$\|ku_k'\|_{L^2}^2\leq C_2(\|\mathcal{A}_k\|_{L^2}^2+|\mathcal{B}_k|^2).$$

Using the equation satisfied by u_k we obtain also that $\|u_k''\|_{L^2}^2\leq C_3(\|\mathcal{A}_k\|_{L^2}^2+|\mathcal{B}_k|^2)$ for some $C_3>0$. Combining all the estimates yields that the sum defining u converges in $H^2_{\mathrm{per}}(R)$ and in $H^2_{\mathrm{per}}(T)$. The limit function u is a strong solution of the problem

$$\begin{cases} (a^3w_p)_p+(a\ w_q)_q=a^3\mathcal{A} & \text{in } p_0<p<0,\\ a^3w_p+\sigma w_{qq}=-\tfrac{1}{2}a^2\mathcal{B} & \text{on } p=0,\\ w=0 & \text{on } p=p_0. \end{cases} \tag{3.6}$$

Let $\psi\in C_c^\infty(\mathbf{R})$ be a nonnegative even function satisfying $\int_{\mathbf{R}}\psi(s)\,ds=1$, and let $\psi_\varepsilon(s)=\psi(s/\varepsilon)/\varepsilon$. By mollifying u in the q-direction, we obtain a family $\{u^{(\varepsilon)}\}_{0<\varepsilon\leq 1}$, $u^{(\varepsilon)}(q,p)=\int_{\mathbf{R}}u(q-s,p)\psi_\varepsilon(s)\,ds$, satisfying

$$\begin{cases} (a^3u_p^{(\varepsilon)})_p+(au_q^{(\varepsilon)})_q=\psi_\varepsilon*(a^3\mathcal{A}) & \text{in } p_0<p<0,\\ a^3u_p^{(\varepsilon)}+\sigma u_{qq}^{(\varepsilon)}=-\tfrac{1}{2}\psi_\varepsilon*(a^2\mathcal{B}) & \text{on } p=0,\\ u^{(\varepsilon)}=0 & \text{on } p=p_0, \end{cases}$$

where convolution with ψ_ε is in the q-variable. A priori $u^{(\varepsilon)}\in H^2_{\mathrm{per}}(R)$. However, $u^{(\varepsilon)}(\,\cdot\,,0)\in C^\infty_{\mathrm{per}}(T)$, so that in fact $u^{(\varepsilon)}\in C^{2+\alpha}_{\mathrm{per}}(\overline{R})$ (note that $\psi_\varepsilon*(a^3\mathcal{A}),\psi_\varepsilon*(a^2\mathcal{B})\in C^\alpha_{\mathrm{per}}(\overline{R})$). Furthermore, since $\psi_\varepsilon*(a^3\mathcal{A})$ and $\psi_\varepsilon*(a^2\mathcal{B})$ are bounded in $C^\alpha_{\mathrm{per}}(\overline{R})$, and $u^{(\varepsilon)}\to u$ uniformly in $\overline{R}$, the Schauder estimates in [20] show that $u^{(\varepsilon)}$ is bounded in $C^{2+\alpha}_{\mathrm{per}}(\overline{R})$. It follows that there is some subsequence, $\{u^{(\varepsilon_n)}\}_{n=1}^\infty$, converging in $C^2_{\mathrm{per}}(\overline{R})$ to a solution of (3.6). Since $u^{(\varepsilon_n)}$ converges uniformly to u, it follows that $u\in C^2_{\mathrm{per}}(\overline{R})$. From the boundary condition on T it follows that $u|_T\in C^{2+\alpha}_{\mathrm{per}}(T)$ and hence $u\in C^{2+\alpha}_{\mathrm{per}}(\overline{R})$. □

Before proving Theorem 1 we need to verify property (iv) of Theorem 2. The proof uses the characterization of the range of $\mathcal{F}_w(Q^*,0)$ given in Lemma 9.

Lemma 10. *$\mathcal{F}_{wQ}(Q^*,0)\varphi\notin\mathcal{R}(\mathcal{F}_w(Q^*,0))$, where φ generates $\mathcal{N}(\mathcal{F}_w(Q^*,0))$.*

Proof. Throughout the proof we let $a=a_{Q^*}$. Let us first calculate $\mathcal{F}_{wQ}$. We have

$$\mathcal{F}_{wQ}(Q^*,0)=\big(-a^{-4}\partial_q^2-3\gamma a^{-4}\partial_p, 2\big(a^{-4}\sigma\partial_q^2-\tfrac{1}{2}a^{-1}\partial_p\big)\big|_T\big).$$

By Lemma 9 we must check that $\mathcal{I}\neq 0$, where

$$\mathcal{I}=\iint_R a^3\varphi(-a^{-4}\varphi_{qq}-3\gamma(-p)a^{-4}\varphi_p)\,dq\,dp+\int_T a^2\varphi(a^{-4}\sigma\varphi_{qq}-(2a)^{-1}\varphi_p)\,dq.$$

The first term equals

$$\iint_R a^{-1}\varphi^2\,dq\,dp$$

due to the cosine. The second term equals

$$-\frac{3}{2}\iint_R a\varphi_p^2\,dq\,dp-\frac{3}{2}\iint_R a^{-1}\varphi^2\,dq\,dp+\frac{3}{2}\int_T a\varphi\varphi_p\,dq. \tag{3.7}$$

Indeed, using $a_p=\gamma(-p)a^{-1}$ (by definition) and $(a^3\varphi_p)_p=a\varphi$ throughout R, we have that

$$\begin{aligned}\iint_R \gamma(-p)a^{-1}\varphi\varphi_p\,dq\,dp&=\iint_R a_p\varphi\varphi_p\,dq\,dp\\&=-\iint_R(a\varphi_p^2+a\varphi\varphi_{pp})\,dq\,dp+\int_T a\varphi\varphi_p\,dq-\int_B a\varphi\varphi_p\,dq\\&=-\iint_R a\varphi_p^2\,dq\,dp-\iint_R a^{-1}\varphi^2\,dq\,dp\\&\quad+3\iint_R\gamma(-p)a^{-1}\varphi\varphi_p\,dq\,dp+\int_T a\varphi\varphi_p\,dq\end{aligned}$$

since $\varphi=0$ on B. We now obtain

$$\iint_R\gamma(-p)a^{-1}\varphi\varphi_p\,dq\,dp=\frac{1}{2}\iint_R a^{-1}\varphi^2\,dq\,dp+\frac{1}{2}\iint_R a\varphi_p^2\,dq\,dp-\frac{1}{2}\int_T a\varphi\varphi_p\,dq,$$

proving (3.7). The total contribution of the third and fourth terms is

$$-\frac{3}{2}\int_T a\varphi\varphi_p\,dq,$$

due to the boundary condition satisfied by φ on T.

Adding up all terms we find that

$$\mathcal{I} = -\frac{1}{2}\iint_R a^{-1}\varphi^2\,dq\,dp - \frac{3}{2}\iint_R a\varphi_p^2\,dp < 0. \quad \square$$

Proof of Theorem 1. We verify the conditions of Theorem 2 for $\lambda^*=Q^*$. Condition (i) follows by construction, while the regularity conditions (ii) and (v) are obviously satisfied. Condition (iii) follows from Lemma 8 and Lemma 9, while (iv) is a consequence of Lemma 10. By Theorem 2, we deduce the existence of a local bifurcation curve of class C^1 for the problem $\mathcal{F}(Q,w)=0$. Since $h=H+w$ and $H_p>0$ throughout $\overline{R}$ it follows that $h_p>0$ in $\overline{R}$ sufficiently close to $(Q^*,0)$ in $I\times X$. Furthermore, we have that $h_q(q,0)=-sW(0)\sin q+o(s)$ in $C^{1+\alpha}_{\text{per}}(T)$, while $h_{qq}(q,0)=-sW(0)\cos q+o(s)$ in $C^{\alpha}_{\text{per}}(T)$. Choosing $q_0\in(0,\pi/2)$, we can find an ε such that for $0<s<\varepsilon$, $h_q(q,0)<0$ for $q\in(q_0,\pi-q_0)$, $h_{qq}(q,0)<0$ for $q\in[0,q_0)$, while $h_{qq}(q,0)>0$ for $q\in(\pi-q_0,\pi]$. Since $h(q,0)$ is even and 2π-periodic, it follows that $h_q(0,0)=h_q(\pi,0)=0$ and thus by construction $h_q<0$ in $(0,\pi)$ for $0<s<\varepsilon$. Due to the antisymmetry of h_q with respect to $q=\pi$, it follows that $h_q>0$ in $(\pi,2\pi)$. By restricting the bifurcation curve, we may assume that $h_p>0$ holds throughout $\overline{R}$ and that $h|_T$ is strictly decreasing in $(0,\pi)$ and strictly increasing in $(\pi,2\pi)$. For small $s<0$, we have instead that $h|_T$ is strictly increasing in $(0,\pi)$ and strictly decreasing in $(\pi,2\pi)$. Proposition 1 allows us to pass from solutions of the problem $\mathcal{F}(Q,w)=0$ to solutions of the water wave problem (2.1)–(2.5). Since $h=H+w$ with $H\in C^{2+\alpha}(\overline{R})$, and $u=c-1/h_p$, we obtain the desired regularity of the solution (u,v,η) of (2.1)–(2.5), and that $u<c$ throughout the fluid region. The nodal property follows from $\eta=h|_T$.

Although we assumed that $L=2\pi$, a similar method as in Lemma 7 shows that $\mu_-(Q)\to-\infty$ as $Q\to\infty$, so that one can find a Q^* with $\mu_-(Q^*)=-4\pi^2/L^2$ as long as $L<L_0$. A careful examination shows that nothing is essentially changed in the rest of the proofs. $\square$

Remark. We proved bifurcation for the lowest wave number $k=1$. Since $\lim_{Q\to\infty}\mu_-(Q)=-\infty$, there are always waves of any sufficiently large wave number. From the uniqueness assertion of the Crandall–Rabinowitz theorem, it follows that close to the trivial curve, the waves of mode k are identical to those of mode 1 and wavelength L/k.

4. Examples

In this section we look at a few particular examples of vorticity distributions. We no longer assume that $L=2\pi$.

Example 1. In the case of irrotational flow ($\gamma\equiv 0$), the eigenvalue problem (3.3) for $\mu=-4\pi^2/L^2$ is simply

$$\begin{cases} W_{pp}=\dfrac{4\pi^2}{L^2Q}W, & p_0<p<0,\\ Q^{3/2}W_p(0)=\dfrac{4\pi^2\sigma}{L^2}W(0), \\ W(p_0)=0. \end{cases}$$

The solution is in this case $W(p)=\sinh(2\pi(p-p_0)/L\sqrt{Q^*})$, where Q^* is determined uniquely by $Q^*=(2\pi\sigma)/L\tanh(2\pi|p_0|/L\sqrt{Q^*})$. The uniform flow corresponding to the bifurcation point has the velocity components $(c-u^*,v^*)=(\sqrt{Q^*},0)$ since

$$H^*(p)=\frac{p-p_0}{\sqrt{Q^*}}.$$

From the definition of p_0, we infer that $d\sqrt{Q^*}=|p_0|$. Hence, we have $(c-u^*)^2=(2\pi\sigma/L)\tanh(2\pi d/L)$, so that we obtain the dispersion relation [14], [19],

$$c-u^*=\sqrt{\frac{2\pi\sigma}{L}\tanh\left(\frac{2\pi d}{L}\right)}.$$

Note that the intrinsic wave speed $c-u^*$ decreases with L.

Example 2. In the case of constant vorticity $\gamma\neq 0$, the substitution

$$W(p)=\frac{2\gamma}{\sqrt{Q+2\gamma p}}W_0\left(\frac{\sqrt{Q+2\gamma p}}{\gamma}\right)$$

transforms (3.3) with $\mu=-4\pi^2/L^2$ into $W_0''=4\pi^2W_0/L^2$. Since $W_0(p_0)=0$, we obtain that

$$W(p)=\frac{1}{\sqrt{Q+2\gamma p}}\sinh\left(\frac{2\pi(\sqrt{Q+2\gamma p}-\sqrt{Q+2\gamma p_0})}{L\gamma}\right).$$

The boundary condition $Q^{3/2}W'(0)=4\pi^2\sigma W(0)/L^2$ then determines Q^* as the unique solution of the equation

$$\tanh\left(\frac{2\pi(\sqrt{Q}-\sqrt{Q+2\gamma p_0})}{L\gamma}\right)=\frac{2\pi Q}{L}\frac{1}{4\pi^2\sigma/L^2+\gamma\sqrt{Q}}. \tag{4.1}$$

The trivial flow corresponding to Q^* has the velocity components $(c-u^*,v^*)=(\sqrt{Q^*}+\gamma y,0)$ because $u_y=\gamma$. Moreover,

$$H^*(p)=\frac{\sqrt{Q^*+2\gamma p}-\sqrt{Q^*+2\gamma p_0}}{\gamma}.$$

From the definition of p_0 we infer that $\sqrt{Q^*}d-\frac{1}{2}\gamma d^2=|p_0|$, and thus

$$d=\frac{\sqrt{Q^*}-\sqrt{Q^*+2\gamma p_0}}{\gamma}.$$

Thus Q^* is the solution of $\tanh(2\pi d/L)=(2\pi Q^*/L)/(4\pi^2\sigma/L^2+\gamma\sqrt{Q^*})$. Solving for $\sqrt{Q^*}=c-u_0^*$, u_0^* being the velocity of the trivial flow at the surface, we obtain the dispersion relation

$$c-u_0^*=\frac{L\gamma}{4\pi}\tanh\left(\frac{2\pi d}{L}\right)+\frac{1}{2}\sqrt{\frac{L^2\gamma^2}{4\pi^2}\tanh^2\left(\frac{2\pi d}{L}\right)+\frac{8\pi\sigma}{L}\tanh\left(\frac{2\pi d}{L}\right)}.$$

Let

$$f(Q)=\tanh\left(\frac{4\pi|p_0|}{L(\sqrt{Q}+\sqrt{Q+2\gamma p_0})}\right)-\frac{2\pi Q}{L}\frac{1}{4\pi^2\sigma/L^2+\gamma\sqrt{Q}}.$$

Then equation (4.1) has a solution $Q^*>-2\Gamma_{\min}$ if and only if f has a zero in the same interval. For $\gamma<0$, this function is well-defined on $[0,(4\pi^2\sigma/|\gamma|L^2)^2)$. Note that

$$f(0)=\tanh\left(\frac{2\pi}{L}\sqrt{\frac{2p_0}{\gamma}}\right)>0, \quad \text{while} \quad \lim_{Q\to(4\pi^2\sigma/|\gamma|L^2)^2} f(Q)=-\infty.$$

Hence bifurcation always occurs for negative constant vorticity, regardless of the size of L.

For $\gamma>0$, the function f is well-defined and decreasing on $[2|p_0|\gamma,\infty)$. Noting that $\lim_{Q\to\infty} f(Q)=-\infty$, we find that bifurcation occurs if and only if

$$\tanh\left(\frac{2\pi}{L}\sqrt{\frac{2|p_0|}{\gamma}}\right)>\frac{2\pi}{L}\frac{2|p_0|\gamma}{4\pi^2\sigma/L^2+\gamma\sqrt{2|p_0|\gamma}}.$$

Since tanh never takes values larger than 1, bifurcation does not occur if $4\pi|p_0|\gamma>L(4\pi^2\sigma/L^2+\gamma\sqrt{2|p_0|\gamma})$. Fixing γ and L, this inequality holds for $|p_0|$ sufficiently large. By Lemma 6, for this γ and $|p_0|$ bifurcation does not occur for any larger L.

Remark. The fact that bifurcation always occurs for constant negative vorticity can be generalized to an arbitrary nonpositive vorticity distribution by a similar argument as in [26].

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