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Two separation theorems of Andreotti–Vesentini type

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1. Introduction

Let X be a complex space with countable topology. Let $\mathcal{F}$ be a coherent analytic sheaf on X. By $\nu(\mathcal{F})$ we denote the largest non-negative integer m such that $\operatorname{prof} \mathcal{F}_x \geq m$ for every point x outside a compact subset of X; see also Section 2.

It is a standard fact that for each positive integer i, the cohomology module $H^i(X, \mathcal{F})$ becomes in a natural way a topological complex vector space. Also it is known that this topology is separated whenever $H^i(X, \mathcal{F})$ has finite dimension. Although in general this is not the case, there are certain settings when the separation still holds. See [7], [4], [15], [21], and especially [19].

In this paper we give two situations when separation holds (for definitions see Section 2).

Theorem 1. *Let q be a positive integer. If X equals an increasing union of q-concave open subsets, then the space $H^i(X, \mathcal{F})$ is separated for all non-negative integers $i \leq \nu(\mathcal{F}) - q$.*

In particular, if X is a complex manifold of pure dimension n and $\mathcal{F}$ is locally free, then $\nu(\mathcal{F}) = n$ so that the Dolbeault cohomology groups $H^{\bullet,s}(X, \mathcal{F})$ are separated for $s \leq n - q$. We remark that this has been considered in [18] under restrictive conditions, namely for $q < n/2$.

Theorem 2. *Let p and q be positive integers with $p+q \leq m := \operatorname{prof}_{\partial K}(\mathcal{F})$ and $K \subset X$ be a compact set. Then the following statements hold.*

(a) *If X is cohomologically p-convex and K admits a base of q-convex open sets, then the space $H^i(X \setminus K, \mathcal{F})$ is separated for $p \leq i \leq m - q$.*

(b) *If X is cohomologically p-complete and K admits a base of q-complete open sets, then $H^i(X \setminus K, \mathcal{F}) = 0$ for $p \leq i < m - q$.*

A brief account of the proofs is as follows. First we need a separation criterion for cohomology with coefficients in a coherent sheaf for an increasing union of open subspaces (viz., Theorem 4). Then we shall prove the separation theorems for q-concave and q-convex spaces, see Theorems 5 and 6, respectively. Theorem 2 is more involved and here we need two more facts which we now briefly recall.

Let i and j be non-negative integers. From a standard exact sequence in cohomology we retain the exact portion

$$H^i(X,\mathcal{F}) \longrightarrow H^i(X\setminus K,\mathcal{F}) \xrightarrow{\alpha_i} H^{i+1}_K(X,\mathcal{F}).$$

Also there is a canonical morphism (given by restriction)

$$(*) \qquad \beta_j\colon H^j_K(X,\mathcal{F}) \longrightarrow \varprojlim_{U\supset K} H^j_c(U,\mathcal{F}).$$

Due to the hypothesis, the projective limit may be indexed over a countable base of relatively compact q-convex neighborhoods U of K. Now the key points in proof are:

- Let $i\geq p$. The subsequent Lemma 2 reduces the separation of $H^i(X\setminus K,\mathcal{F})$ to the existence of a separated topology on $H^{i+1}_K(X,\mathcal{F})$ for which α_i is continuous.
- Take $j\leq m-q+1$. Then we show, viz. Proposition 8, that β_j is injective and the projective limit in $(*)$ admits a separated topology so that the induced topology on $H^j_K(X,\mathcal{F})$ is separated.

2. Preliminaries

Let $X=(X,\mathcal{O}_X)$ be a complex space with countable topology and $\mathcal{F}$ be a coherent analytic sheaf on X. For any point $x\in X$ there exists an embedding $\iota\colon U\to \widehat{U}\subset \mathbf{C}^{m(x)}$ of an open neighborhood $U\ni x$ into the Zariski tangent space $\mathbf{C}^{m(x)}$ of X at x. Let $\widehat{\mathcal{F}}$ be the trivial extension of $\iota_\star(\mathcal{F}|_U)$; it is a coherent analytic sheaf on $\widehat{U}$. Let

$$0\longrightarrow \mathcal{O}^{p_d}\longrightarrow \mathcal{O}^{p_{d-1}}\longrightarrow \ldots \longrightarrow \mathcal{O}^{p_0}\longrightarrow \widehat{\mathcal{F}}\longrightarrow 0$$

be a resolution of $\widehat{\mathcal{F}}$ on a neighborhood of $\iota(x)$ of minimal length. It can be shown that $d\leq m(x)$ and the number $\operatorname{prof}_x(\mathcal{F}):=m(x)-d$ does not depend on the embedding ι. Moreover, the function from X to $\mathbf{N}$ given by $x\mapsto \operatorname{prof}_x(\mathcal{F})$ is lower semi-continuous.

If $A\subset X$ is a set, we put

$$\operatorname{prof}_A(\mathcal{F}) := \min_{x\in A} \operatorname{prof}(\mathcal{F}_x).$$

An open set $U \subset X$ is called a neighborhood of the boundary of X if $X \setminus U$ is compact. Then we denote

$$\nu(\mathcal{F}) := \sup\{\operatorname{prof}_U(\mathcal{F})\,; U \text{ neighborhood of the boundary of } X\}.$$

Obviously $\operatorname{prof}_X(\mathcal{F}) \le \nu(\mathcal{F})$. For instance, if X is a complex manifold of pure dimension n and $\mathcal{F}$ is locally free outside a compact set of X, then $\nu(\mathcal{F})=n$ so that $\operatorname{prof}_X(\mathcal{F})$ could be less than n.

Now let $\varphi\colon X \to \mathbf{R}$ be a continuous function and q be a positive integer. Then φ is said to be *q-convex* (in the sense of Andreotti–Grauert [3]) if there exists a covering of X by open patches A_λ isomorphic to closed analytic sets in open sets $D_\lambda \subset \mathbf{C}^{N_\lambda}$, $\lambda \in \Lambda$, such that each restriction $\varphi|_{A_\lambda}$ admits a smooth extension $\widehat{\varphi}_\lambda$ to D_λ which is q-convex, i.e. $i\partial\overline{\partial}\widehat{\varphi}_\lambda$ has at most $q-1$ negative or zero eigenvalues at each point of D_λ. The q-convexity property is easily shown not to depend on the covering nor on the embeddings $A_\lambda \hookrightarrow D_\lambda$.

We say that X is *q-convex* (resp., *q-concave*) if there exists a proper function $\varphi\colon X \to [0,\infty)$ (resp., $\varphi\colon X \to (0,1]$) which is q-convex on $X \setminus K$, for a compact set $K \subset X$. The space X is called *q-complete* if X is q-convex and the corresponding compact set is empty. (Note that in the definition of q-concavity we cannot take the special compact set to be empty; at least if $q \le \dim(X)$ with X irreducible).

On the other hand, in the above definitions for $q=1$ it is sufficient to require that "φ is continuous on X and strictly plurisubharmonic on $X \setminus K$" instead of "φ is smooth on X and 1-convex on $X \setminus K$".

By a (p,q)-*corona*(1) we mean a complex space X on which there exists a smooth proper function $\varphi\colon X \to (0,\infty)$ with the following property:

($\sharp$) There exist positive numbers ε_0 and M_0 such that the function φ is p-convex on $\{x; \varphi(x) < \varepsilon_0\}$ and is q-convex on $\{x; M_0 < \varphi(x)\}$.

If in ($\sharp$) we can choose $M_0 < \varepsilon_0$, then X is called a *complete* (p,q)-*corona*. For practical purposes, we shall employ the term "corona" instead of "$(1,1)$-corona".

We remark that if X has pure dimension n and X is a complete (p,q)-corona, then $p+q \le n+1$. This can be easily verified using the maximum principle for q-convex functions.

A standard example of a (p,q)-corona can be obtained in the following way. Let Z be a q-convex space and $K \subset Z$ be a compact set. Then $X := Z \setminus K$ is a (p,q)-corona if K is *strictly p-convex* in the sense that there exist an open set $U \supset K$ and $\psi \in C^\infty(U, \mathbf{R}_+)$ with $K = \{x; \psi(x) = 0\}$ and ψ being p-convex on $U \setminus K$. On the other hand, if K admits a fundamental system of p-complete neighborhoods, then X is

(1) Sometimes the more suggestive label (p,q)-*concave-convex space* term is used. Besides one also requires that $0 = \inf_X \varphi$ and $\sup_X \varphi = \infty$. See, for instance [21].

merely an increasing union of (p,q)-coronae. (We leave the simple verification as an exercise!)

Theorem 3. *Assume that X is a (p,q)-corona. Then the space $H^i(X,\mathcal{F})$ has finite dimension (a fortiori it is separated) for $q\leq i<\nu(\mathcal{F})-p$.*

The proof is a straightforward application of the bumping method from [3]. Notice that the separation of $H^i(X,\mathcal{F})$ for $i=\nu(\mathcal{F})-p$ is stated by Ramis [21]. However, it seems that his proofs have some gaps. See the note in [19] in which it is shown that the example due to Rossi of a complex smooth surface X, which is a complete corona X and cannot be "filled in", has $H^1(X,\mathcal{O}_X)$ non-separated.

3. A separation criterion for increasing unions

Before stating the separation criterion due to Cassa [9] for an increasing union of open subsets in complex spaces, let us recall a few of his definitions.

Let $F=\{F_n,\rho_{m,n}\}$ be a projective system of locally convex topological vector spaces and continuous linear maps. We say that F satisfies a *topological Mittag-Leffler* condition (or, briefly, that F is a *tML*-system) if, for every $n\geq 1$ and for every convex, circled neighborhood U of $0\in F_n$, there exists an integer $n^*\geq n$ (n^* depends on U and n) such that, for any $k\geq n^*$ we have

$$\overline{\rho_{k,n}(F_k)}^U=\overline{\rho_{n^*,n}(F_{n^*})}^U,$$

where the closure is taken in the topology of F_n defined by the Minkowski seminorm of U.

The system F satisfies a *closed Mittag-Leffler* condition (or F is a *cML*-system) if, for any $n\geq 1$, there exists an integer $n^*\geq n$ such that, for any $k\geq n^*$ we have

$$\overline{\rho_{k,n}(F_k)}=\overline{\rho_{n^*,n}(F_{n^*})}\quad\text{in } F_n.$$

A special case of cML-system is what is usually called a *Runge system*, i.e. a projective system $F=\{F_n,\rho_{m,n}\}$ such that, for every m and n with $m\geq n$ the map $\rho_{m,n}$ has dense image in F_n.

Obviously, a cML-system is a tML-system. (Note that for a projective system of normed spaces these two conditions coincide, since by definition the topology is generated by exactly one seminorm.)

As a straightforward but useful observation we mention the following: If each F_n has finite dimension (as a vector space), then the projective system F is a cML-system. Indeed, fix $n_0\geq 1$. Since for $k\geq m\geq n_0$ we have $\rho_{k,n_0}(F_k)\subseteq\rho_{m,n_0}(F_m)\subseteq F_{n_0}$, and F_{n_0} has finite dimension, there exists $n_1>n_0$ such that, for all $k,m\geq n_1$ we have $\rho_{k,n_0}(F_k)=\rho_{m,n_0}(F_m)$.

Similarly, if each of the canonical mappings

$$\varprojlim_{n\mapsto\infty} F_n \longrightarrow F_k, \quad k=0,1,...,$$

has finite codimension, then the projective system F is a cML-system.

For the rest of this section we consider the following setting: X is a complex space which is exhausted by an increasing sequence of open sets $\{X_n\}_n$,

$$X_0\subset ...\subset X_n\subset X_{n+1}\subset ...,$$

and $\mathcal{F}$ is a coherent sheaf on X. Fix an integer $q\geq 1$. One has canonical restrictions

$$\rho_{m,n}\colon H^{q-1}(X_m,\mathcal{F})\longrightarrow H^{q-1}(X_n,\mathcal{F}), \quad m\geq n.$$

Here is the separation theorem due to Cassa [9].

Theorem 4. *Suppose that each space $H^q(X_n,\mathcal{F})$ is separated. Then $H^q(X,\mathcal{F})$ is separated if and only if the projective system $\{H^{q-1}(X_n,\mathcal{F}),\rho_{m,n}\}$ satisfies the topological Mittag-Leffler condition.*

Now we say a few more words when $q=1$. The projective system $\{\mathcal{F}(X_n),\rho_{m,n}\}$ fulfils the topological Mittag-Leffler condition if and only if for any compact set $K\subset X$ there exists a positive integer j with $K\subset X_j$ such that $\mathcal{F}(X)$ approximates $\mathcal{F}(X_j)$ uniformly on K (cf. [25], p. 190).

It is important to observe that for $\mathcal{F}=\mathcal{O}_X$ the above topological Mittag-Leffler condition can be reformulated by saying that

$$X_0\subset ...\subset X_n\subset X_{n+1}\subset ...$$

is a Runge family according to [20], p. 118; see also [16].

To give an example we consider a non-Stein complex manifold Z of dimension $N\geq 2$ which is an increasing union of Stein open subsets $Z_n, n=0,1,...$ (see [12]). By [20] and [25] it follows that $\{Z_n\}_n$ is not a Runge family. Now fix a point $z_0\in Z_0$. Put $X_n:=Z_n\setminus\{z_0\}$. Obviously each X_n is a complete corona. It is not difficult to check that $\{X_n\}_n$ is not a Runge family. Obviously $X:=Z\setminus\{z_0\}$ is exhausted by the increasing family $\{X_n\}_n$. Then the space $H^1(X,\mathcal{O}_X)$ is not separated (use the separation theorem presented in (i) below).

In this circle of ideas we give the following result.

Proposition 1. *Suppose that each X_n is a complete corona, $\nu(\mathcal{O}_X)\geq 3$, and $\mathcal{F}^{[1]}=\mathcal{F}$, where $\mathcal{F}^{[1]}$ is the 1^{st}-absolute gap sheaf of $\mathcal{F}$. Then $H^1(X,\mathcal{F})$ is separated if and only if $\{\mathcal{F}(X_n),\rho_{m,n}\}$ satisfies the topological Mittag-Leffler condition.*

Proof. Recall that for a non-negative integer p, the p^{th}-*absolute gap sheaf* of $\mathcal{F}$ is defined as the canonical sheaf associated to the presheaf

$$U \longmapsto \varinjlim \mathcal{F}(U \setminus A),$$

where the inductive limit is taken over all analytic sets $A \subset U$ of dimension $\leq p$. The equality $\mathcal{F}^{[p]} = \mathcal{F}$ means that the canonical morphism $\mathcal{F} \to \mathcal{F}^{[p]}$ is an isomorphism.

Then the proof of the proposition concludes readily in a standard way from Theorem 4 and the following facts:

(i) Let Y be a Stein space, $K \subset Y$ be a holomorphically convex compact set, and $\mathcal{G}$ be a coherent analytic sheaf on Y. Then $H^\bullet(Y \setminus K, \mathcal{G})$ are separated. See [8].

(ii) Let Y be a complete corona defined by a function $\varphi\colon Y \to (0,\infty)$. Suppose that $\operatorname{prof}(\mathcal{O}_X) \geq 3$ on $\{x; \varphi(x) < \varepsilon_0\}$ for some $\varepsilon_0 > 0$. Then there exists a Stein space $\widetilde{Y}$ containing Y as an open set such that, for any $\varepsilon > 0$ the set $K_\varepsilon := \{x; \varphi(x) \leq \varepsilon\} \cup (\widetilde{Y} \setminus Y)$ is compact; in fact it is even holomorphically convex. Such a space $\widetilde{Y}$ is called a Stein completion of Y (see [5]). Furthermore, if $\mathcal{G}$ is a coherent analytic sheaf on Y with $\mathcal{G}^{[1]} = \mathcal{G}$, then there exists a coherent sheaf $\widetilde{\mathcal{G}}$ on $\widetilde{Y}$ that extends $\mathcal{G}$, that is $\widetilde{\mathcal{G}}|_Y = \mathcal{G}$. Then, using (i), for any $\varepsilon > 0$ sufficiently small, the spaces $H^\bullet(\{\varphi < \varepsilon\}, \mathcal{G})$ are separated. □

4. Proof of Theorem 1

This is a straightforward consequence of the discussion in the previous section and the following theorem.

Theorem 5. *Let X be a q-concave space and $\mathcal{F}$ be a coherent analytic sheaf on X. Then the space $H^i(X, \mathcal{F})$ has finite dimension if $0 \leq i < \nu(\mathcal{F}) - q$, and it is separated if $i = \nu(\mathcal{F}) - q$.*

Proof. The finiteness part is standard (by the bumping method of [3]). The separation in question is proved in [4], p. 240, for a complex manifold X with methods specific to the smooth case. The more general singular case is a standard consequence of spectral sequences arguments and the following proposition, which is a particular case of a theorem in [2], p. 1040. More concretely we proceed as follows.

Proposition 2. *Let X be a complex space of finite dimension. Then for each coherent analytic sheaf $\mathcal{F}$ on X there exists a spectral sequence $\{E_r^{i,j}\}_r$ with*

$$E_2^{i,-j} = H_c^i(X, \mathcal{D}^j \mathcal{F}) \Rightarrow H_{j-i}(X, \mathcal{F}_\star).$$

Note. Here $\mathcal{F}_\star$ denotes the dual sheaf of $\mathcal{F}$ (see [4], pp. 207–208) and $H_\bullet(X,\mathcal{F}_\star)$ the homology groups with closed supports and coefficients in $\mathcal{F}_\star$. The sheaf $\mathcal{D}^j\mathcal{F}$ is defined as the canonical sheaf associated with the presheaf defined on the family of all open subsets $U \Subset X$ such that U is Stein and its closure $\overline{U}$ has a Stein neighborhood basis by the rule:

$$U \longmapsto \mathcal{D}^j\mathcal{F} := \mathrm{Homcont}(H^i_c(U,\mathcal{F}),\mathbf{C}).$$

By [4], Proposition 18, $\mathcal{D}^j\mathcal{F}$ are coherent sheaves on X which have compact (analytic) support for $j<\nu(\mathcal{F})$.

Now we return to the proof of Theorem 5. From [4] (see Theorem II and the remark on pp. 214–215) one has that, if Y is a complex space and $\mathcal{G}$ a coherent sheaf on Y, then $H^{r+1}(Y,\mathcal{G})$ is separated provided that $H_r(Y,\mathcal{G}_\star)$ is separated.

On the other hand, again by [4] (see Theorem 8) it follows that $H^i_c(X,\mathcal{F})$ has finite dimension (as a complex vector space) for all $i>q$. Therefore, by Proposition 2 we obtain that the homology group $H_l(X,\mathcal{F}_\star)$ has finite dimension for $l<\nu(\mathcal{F})-q$.

Finally, granting the open mapping theorem for continuous surjections of Souslin(2) spaces and the way the topology of $H_j(X,\mathcal{F}_\star)$ is defined (see [4]), we deduce that $H_j(X,\mathcal{F}_\star)$ is separated whenever it has finite dimension. The proof of the theorem concludes immediately. □

In the remaining part of this section we say a few more words on q-concavity. As a simple consequence of [30] one shows that a finite union of 1-concave open subsets of X is still 1-concave.

Now we give a positive result for an increasing union.

Proposition 3. *Let X be the union of an increasing sequence of 1-concave open subsets X_n. Let φ_n define the 1-concavity of X_n and K_n be the exceptional compact set. If $K_{n+1}\subset X_n$ for all n, then X is 1-concave.*

Proof. Clearly, we may arrange things such that $K_{n+1}(\subset X_n)$ is a neighborhood of K_n and $\{K_n\}_n$ exhausts X. Then select a sequence of positive numbers $\{\varepsilon_n\}_n$ strictly decreasing to 0 such that $\varepsilon_n\varphi_{n+1}<\varepsilon_{n-1}\varphi_n$ on K_{n+1} (with $\varepsilon_0=1$). For $x\in X$, put $N(x):=\{n; x\in X_{n+1}\}$. Define a function $\varphi\colon X\to(0,1)$ by setting

$$\varphi(x)=\sup\{\varepsilon_n\varphi_{n+1}(x)\,;n\in N(x)\},\quad x\in X.$$

It can be checked that φ is continuous and exhaustive from below. To conclude the proposition, we show that φ is strictly plurisubharmonic on $X\setminus K_1$. Indeed,

(2) A Souslin space is a topological space which is the continuous image of a complete, metric, separable space. See [4], p. 191.

let $x_0 \in X \setminus K_1$, and let j be maximal such that $x_0 \in K_{j+1}$. Then, on a suitable neighborhood W of x_0 in the definition of $\varphi|_W$ only functions from $\varepsilon_0\varphi_1, ..., \varepsilon_{j-1}\varphi_j$ appear and, moreover, those involved are strictly plurisubharmonic. $\square$

A class of examples of 1-concave spaces is obtained by removing special Stein compact sets from a given 1-concave space. Toward this aim, let us recall a few notions. Let X be a complex space and K be a compact set in X. Then

• the set K is called *Stein* if K admits a fundamental system of Stein open neighborhoods;

• the set K is called *pseudoconvex* if there exist an open set $U \supset K$ and a non-negative plurisubharmonic function ψ on U vanishing precisely on K and such that ψ is strictly plurisubharmonic on $U \setminus K$.

It follows easily that if X is Stein, then a compact $K \subset X$ is pseudoconvex if and only if there exists a Stein open neighborhood U of K such that K is holomorphically convex with respect to $\mathcal{O}(U)$ (see [30]); a fortiori pseudoconvex compact sets in Stein spaces are Stein. It is worth noticing that, while any compact set in $\mathbf{C}$ is Stein, there are examples of non-pseudoconvex compact sets. For instance, using the maximum principle for subharmonic functions we show easily that the compact set $M \subset \mathbf{C}$ is not pseudoconvex, where

$$M := \{0\} \cup \bigcup_{n=1}^{\infty} \partial\Delta(1/n).$$

Here $\Delta(r) := \{z \in \mathbf{C}; |z| < r\}$ for $r > 0$ and $\Delta = \Delta(1)$. Furthermore, if K_i, $i = 1, ..., n$, are compact sets in $\mathbf{C}$, then their product $K_1 \times ... \times K_n$ is pseudoconvex in $\mathbf{C}^n$ if and only if each K_i is pseudoconvex in $\mathbf{C}$.

By [29] we get a slight improvement of [30], Proposition 4.1.

Proposition 4. *Let X be a 1-concave space. Then, for any pseudoconvex compact set $K \subset X$, the complement $X \setminus K$ is 1-concave.*

Let $K \subset \mathbf{C}^n$ be a compact set and consider $K \subset \mathbf{P}^n$ via the standard open embedding $\mathbf{C}^n \subset \mathbf{P}^n$. Taking into account the well-known fact (see [13] and [26]) that a locally Stein proper open subset of $\mathbf{P}^n$ is Stein, we obtain the following result.

Proposition 5. *The space $\mathbf{P}^n \setminus K$ is an increasing union of 1-concave open subsets* (*resp.*, *$\mathbf{P}^n \setminus K$ is 1-concave*) *if and only if K is a Stein* (*resp.*, *pseudoconvex*) *compact set.*

It is important to notice that every irreducible complex space of dimension n is n-concave [11] so that maximal concavity is not very interesting.

Examples of q-concave spaces can be obtained by removing analytic sets in compact complex spaces, more precisely we have: If Z is a compact complex space and $A\subset Z$ is an analytic set of dimension k, then $Z\setminus A$ is $(k+1)$-concave. (See [29], Proposition 9.)

Now we show the following result.

Proposition 6. *For each pair* (n,q) *of integers with* $1\leq q<n$ *there exists a complex manifold* X *of dimension* n *such that*:

(i) X *is an increasing union of* q*-concave open subsets*;

(ii) X *is not* q*-concave.*

Proof. We consider $X:=\mathbf{P}^n\setminus K$ for $K=M\times\overline{\Delta^{n-q}}$, where

$$M:=\{0\}\cup\bigcup_{n=1}^{\infty}\partial\Delta^q(1/n).$$

Since an arbitrary open set in $\mathbf{C}^q$ is q-complete (see [14]), K admits a fundamental system of q-complete open neighborhoods. From this we infer readily that X satisfies (i). By using the maximum principle for q-convex functions, one derives property (ii). □

Remark. For $q=1$ one gets another kind of example using the "discrete hat" in $\mathbf{C}^2$, namely

$$K=\left(\bigcup_{n=1}^{\infty}\{1/n\}\times\partial\Delta\right)\cup(\{0\}\times\overline{\Delta}).$$

Observe that K is not pseudoconvex (as follows readily using the maximum principle for plurisubharmonic functions) but K is Stein. For this it suffices to show that K is meromorphically convex (see [22], p. 479); this condition is a straightforward consequence of the fact that $\mathbf{C}^2\setminus K$ is a union of complex lines. (For instance, if $z_0=(1/n,w_0)$ with $|w_0|<1$, then we consider L given by $\{(1/n+t,w_0+\lambda t);t\in\mathbf{C}\}$ for $\lambda\in\mathbf{C}$. We shall require $|w_0-\lambda/n|<1$ and $|w_0+\lambda(1/m-1/n)|\neq 1$ for all $m=1,2,\dots$. Clearly this can be satisfied if $|\lambda|\neq 0$ is small enough. The other cases are done in a similar way and we omit their simple verification.)

In the circle of ideas presented here, we relate q-concavity with *pseudoconcavity* in the sense of Andreotti [1]. Let X be a complex space and $\Omega\subset X$ be an open set. A point $x_0\in\partial\Omega$ is a *pseudoconcave boundary point* of Ω if x_0 has a fundamental system of neighborhoods $\{U_\nu\}_\nu$ in X such that for each ν,

$$x_0\in\operatorname{int}(\widehat{U_\nu\cap\Omega}),$$

where the hull of $U_\nu\cap\Omega$ is with respect to $\mathcal{O}(U_\nu)$. We say that X is *pseudoconcave* if a non-empty, relatively compact open subset $\Omega\subset X$ is given such that the following properties hold:

• Ω meets any irreducible component of X (hence X has finitely many irreducible components);

• each point of $\partial\Omega$ is a pseudoconcave boundary point (i.e. Ω has a pseudoconcave boundary).

The relation with q-concavity is as follows (we cite Proposition 10 from [1]).

Proposition 7. *Let X be an irreducible complex space of dimension n. If X is $(n-1)$-concave, then X is pseudoconcave.*

Note that pseudoconcavity of X does not guarantee $(n-1)$-concavity of X. To exhibit a counterexample, let $a\in\mathbf{P}^n$ $(n\geq 2)$ and consider a sequence $\{\xi_\nu\}_\nu\subset\mathbf{P}^n\setminus\{a\}$ that converges to a. Let $\pi\colon X\to\mathbf{P}^n\setminus\{a\}$ be the blowing-up of this sequence. It follows easily that X is pseudoconcave; in fact, if B is a small ball around a in $\mathbf{P}^n$ such no ξ_ν lies on ∂B, then $\Omega:=X\setminus\pi^{-1}(B\setminus\{a\})$ displays the pseudoconcavity of X. On the other hand, if X would be $(n-1)$-concave, then there would exist a function $\varphi\colon X\to(0,\infty)$, exhaustive from below, and $(n-1)$-convex on $\{x;0<\varphi(x)<c\}$ for a suitable $c>0$; hence for sufficiently large ν, φ would be $(n-1)$-convex on $\pi^{-1}(\xi_\nu)$ which is false by the maximum principle. Therefore X is not $(n-1)$-concave, as desired.

5. Proof of Theorem 2

First we prepare a few general facts.

Lemma 1. *Let T be a paracompact space with countable basis, $K\subset T$ be a compact set and $\mathcal{G}$ be a sheaf of abelian groups on T. Then the canonical morphism*

$$H^r(T\setminus K,\mathcal{G})\longrightarrow \varprojlim_{U\supset K} H^r(T\setminus U,\mathcal{G})$$

is an epimorphism, for any non-negative integer r. If, moreover, we assume that there exists a decreasing sequence $\{U_\nu\}_\nu$ to K of open subsets of T such that the restrictions

$$H^{r-1}(T\setminus U_{\nu+1},\mathcal{G})\longrightarrow H^{r-1}(T\setminus U_\nu,\mathcal{G})$$

are surjective, then that morphism is an isomorphism.

The proof is based on considering a resolution $\mathcal{C}^\bullet$ of $\mathcal{G}$ by injective sheaves which allow us to compute the invariants $H^\bullet(X\setminus K,\mathcal{F})$ and $H^\bullet(X\setminus U,\mathcal{G})$, open neighborhoods U of K. The applications $\Gamma(X\setminus U_{\nu+1},\mathcal{G})\to\Gamma(X\setminus U_\nu,\mathcal{G})$ are surjective, $\nu\geq 1$. The conclusion of the lemma follows elementarly by a standard argument on projective systems and suitable diagrams.

Lemma 2. *Let X be a complex space and $\mathcal{F}$ be a coherent analytic sheaf on X. Let q be a positive integer. If the closure of $\{0\}\subset H^q(X,\mathcal{F})$ has finite dimension (over $\mathbf{C}$), then the space $H^q(X,\mathcal{F})$ is separated.*

Proof. Let $\mathcal{U}=\{U_i\}_i$ be a locally finite Stein open covering of X. It is known that the canonical map $H^q(\mathcal{U},\mathcal{F})\to H^q(X,\mathcal{F})$ is a topological isomorphism. Now consider the natural surjection $\rho\colon Z^q(\mathcal{U},\mathcal{F})\to H^q(\mathcal{U},\mathcal{F})$. Then ρ is continuous (and open).

Let $\xi^{(1)},...,\xi^{(m)}\in Z^q(\mathcal{U},\mathcal{F})$ be such that $\rho(\xi^{(1)}),...,\rho(\xi^{(m)})$ form a basis for the closure $\overline{\{0\}}$ of $\{0\}$ in $H^q(\mathcal{U},\mathcal{F})$. Let $G\subset Z^q(\mathcal{U},\mathcal{F})$ be the complex subspace spanned by $\xi^{(1)},...,\xi^{(m)}$. Note that $G\cap B^q(\mathcal{U},\mathcal{F})=\{0\}$. Let $T:=\rho^{-1}(\overline{\{0\}})$. Then T is a Fréchet space (because it is a closed subspace of the Fréchet space $Z^q(\mathcal{U},\mathcal{F})$). Note that $T=B^q(\mathcal{U},\mathcal{F})\oplus G$. Consider the continuous surjective map

$$\begin{aligned}\theta\colon C^{q-1}(\mathcal{U},\mathcal{F})\times\mathbf{C}^m &\longrightarrow T,\\ (\xi,g)&\longmapsto \delta(\xi)+\lambda_1\xi^{(1)}+...+\lambda_m\xi^{(m)},\end{aligned}$$

where δ is the ordinary coboundary map. By the open mapping theorem, θ is an open map. This gives easily that $B^q(\mathcal{U},\mathcal{F})$ is closed in T because it equals the complement in T of the open set $\theta(C^{q-1}(\mathcal{U},\mathcal{F})\times(\mathbf{C}^m\setminus\{0\}))$. Therefore $H^q(\mathcal{U},\mathcal{F})$ is separated, whence the lemma. $\square$

We shall employ this lemma in the following setting. Let X be a complex space and $\mathcal{F}$ be a coherent analytic sheaf on X. Let $K\subset X$ be a compact set. Suppose that there is a positive integer j such that $H^j(X,\mathcal{F})$ has finite dimension and we can endow $H^{j+1}_K(X,\mathcal{F})$ with a topology for which the canonical map

$$H^j(X\setminus K,\mathcal{F})\longrightarrow H^{j+1}_K(X,\mathcal{F})$$

is continuous. Then $H^j(X\setminus K,\mathcal{F})$ is separated. (This follows readily by the above lemma if we consider the exact sequence

$$H^j(X,\mathcal{F})\longrightarrow H^j(X\setminus K,\mathcal{F})\longrightarrow H^{j+1}_K(X,\mathcal{F}).)$$

Below we recall some facts concerning the topology of cohomology groups with compact supports. Let X be a complex space and $\mathcal{F}$ a coherent analytic sheaf on X. By a "special covering" of X we mean a locally finite Stein open covering $\mathcal{U}=\{U_i\}_{i\in I}$ (hence I is an at most countable set of indices so there is no loss in generality to take $I=\mathbf{N}$) such that each $\overline{U}_i$ is a Stein compactum (that is it admits a neighborhood system of Stein open sets). It is clear that for each open covering $\mathcal{V}$ of X there exists a finer special covering. Now let $\mathcal{U}$ be a special covering of X.

The cohomology of the topological complex of finite cochains

$$C^q_\star(\mathcal{U},\mathcal{F}) := \bigoplus_{i_0,\dots,q} \mathcal{F}(U_{i_0}\cap\ldots\cap U_{i_q}), \quad q\geq 0,$$

endowed with the direct sum topology becomes a complex of topological vector spaces of LF-type, whose cohomology is $H^i_c(\mathcal{U},\mathcal{F})$. If $\mathcal{V}$ is another special covering of X, finer than $\mathcal{U}$, then we get a canonical topological isomorphism $H^\bullet_c(\mathcal{V},\mathcal{F})\to H^\bullet_c(\mathcal{U},\mathcal{F})$. In this way we get the canonical topology on $H^\bullet_c(X,\mathcal{F})$. It is not difficult to see that $H^q_c(X,\mathcal{F})$ is separated if it has finite dimension.

Lemma 3. *Let D be a relatively compact open subset of X. Then the natural connecting morphisms*

$$\delta^q\colon H^q(\partial D,\mathcal{F}) \longrightarrow H^{q+1}_c(D,\mathcal{F}), \quad q=0,1,\dots,$$

are continuous.

Proof. Recall that if $A\subset X$ is a closed set, then on

$$H^i(A,\mathcal{F}) := \varinjlim_{U\supset A} H^i(U,\mathcal{F}).$$

we put the inductive limit topology.

Now fix a non-negative integer q. One has to show that, for every open neighborhood U of ∂D, the morphism $\eta^q\colon H^q(U,\mathcal{F})\to H^{q+1}_c(D,\mathcal{F})$ obtained by composing δ^q and the restriction $H^q(U,\mathcal{F})\to H^q(\partial D,\mathcal{F})$ is continuous. In order to check this, choose special coverings $\mathcal{U}=\{U_i\}_i$ and $\mathcal{D}=\{D_j\}_j$ of U and D respectively both indexed over the set $\mathbf{N}$ of non-negative integers and such that, for some $n_0\in\mathbf{N}$ there is a function $\rho\colon \{n_0,n_0+1,\dots\}\to\mathbf{N}$ with $\overline{D}_j\subset U_{\rho(j)}$ for all $j\geq n_0$. The desired continuity follows now simply from the following description. There is a natural morphism θ^q for which the next diagram commutes:

$$\begin{array}{ccc} Z^q(\mathcal{U},\mathcal{F}) & \xrightarrow{\theta^q} & Z^{q+1}_c(\mathcal{D},\mathcal{F}) \\ {\scriptstyle r}\downarrow & & \downarrow \\ H^q(U,\mathcal{F}) & \xrightarrow{\eta^q} & H^{q+1}_c(D,\mathcal{F}), \end{array}$$

where the vertical arrows are the canonical (open) surjections. Now, to define θ^q, we let $\xi\in Z^q(\mathcal{U},\mathcal{F})$; then set $\tilde{\xi}\in C^q(\mathcal{D},\mathcal{F})$ by

$$\tilde{\xi}_{j_0\dots j_q} = \xi_{\rho(j_0)\dots\rho(j_q)}|_{D_{j_0}\cap\ldots\cap D_{j_q}},$$

if all $j_0,\dots,j_q\geq n_0$, and 0 otherwise. Then put $\theta^q(\xi)=\delta(\tilde{\xi})$, where $\delta\colon C^q(\mathcal{D},\mathcal{F})\to C^{q+1}(\mathcal{D},\mathcal{F})$ is the coboundary map. We have that $\theta^q(\xi)$ belongs to $Z^{q+1}_c(\mathcal{D},\mathcal{F})$. Indeed, for some $N\in\mathbf{N}$ large enough, $D_j\cap D_l=\varnothing$ for $j\geq N$ and $l<n_0$. Thus $N\geq n_0$.

Let $(j_0, ..., j_{q+1})$ be in the nerve of $\mathcal{D}$ such that at least one index is $\geq N$; then the remaining indices are $\geq n_0$. Obviously $\theta^q(\xi)_{j_0 \dots j_{q+1}} = 0$. Therefore $\theta^q(\xi) \in C_c^{q+1}(\mathcal{D}, \mathcal{F})$ and thus it belongs also to $Z_c^{q+1}(\mathcal{D}, \mathcal{F})$. Finally, it is straightforward but a little bit tedious to check that θ^q induces $\eta^q \circ \gamma$ assuring the commutativity of the above diagram. □

Now fix for the moment a non-negative integer j. Let U be a relatively compact open neighborhood of K. There exists a canonical commutative diagram

$$\begin{array}{ccc} H^j(X \setminus K, \mathcal{F}) & \xrightarrow{\quad\alpha\quad} & H_K^{j+1}(X, \mathcal{F}) \\ & {\scriptstyle\beta_U}\searrow \quad \swarrow{\scriptstyle\gamma_U} & \\ & H_c^{j+1}(U, \mathcal{F}) & \end{array}$$

with β_U continuous by the above lemma. Let $\{U_\nu\}_\nu$ be a countable base of relatively compact open neighborhoods of K. From the above diagram, we obtain another commutative diagram

$$\begin{array}{ccc} H^j(X \setminus K, \mathcal{F}) & \xrightarrow{\quad\alpha\quad} & H_K^{j+1}(X, \mathcal{F}) \\ & {\scriptstyle\beta}\searrow \quad \swarrow{\scriptstyle\gamma} & \\ & \varprojlim_{\nu} H_c^{j+1}(U_\nu, \mathcal{F}), & \end{array}$$

where $\beta = \varprojlim_\nu \beta_\nu$ is continuous and $\gamma = \varprojlim_\nu \gamma_\nu$.

Suppose we may choose the base $\{U_\nu\}_\nu$ such that each $H_c^{j+1}(U_\nu, \mathcal{F})$ is separated; then the projective limit inherits a separated topology as a closed subspace of the product of $H_c^{j+1}(U_\nu, \mathcal{F})$. If, moreover, γ is injective, then we may put a separated topology on $H_K^{j+1}(X, \mathcal{F})$ such that α becomes continuous. Therefore, if $H^j(X, \mathcal{F})$ has finite dimension, then the space $H^j(X \setminus K, \mathcal{F})$ is separated. This idea is used for the proof of Theorem 2. To reach this setting we prepare a few more facts.

Theorem 6. *Let X be a complex space and $\mathcal{F}$ be a coherent analytic sheaf on X. Then the following statements hold*:

(a) *If X is q-convex, then $H_c^i(X, \mathcal{F})$ is separated for $i \leq \nu(\mathcal{F}) - q + 1$ and has finite dimension for $i \leq \nu(\mathcal{F}) - q$.*

(b) *If X is q-complete, then $H_c^i(X, \mathcal{F}) = 0$ for $i \leq \nu(\mathcal{F}) - q$.*

Proof. For the definition of $\nu(\mathcal{F})$ see the beginning of Section 2. We consider only the q-convex case. Let φ: $X \to \mathbf{R}$ be the function displaying the q-convexity of X. The bumping method from [3] gives the following. For each $\lambda\in\mathbf{R}$ put $X(\lambda)=\{x;\varphi(x)<\lambda\}$. Let $c_0\in\mathbf{R}$ be so large that φ is q-convex on $\{x;\varphi(x)>c_0\}$ and for all $x\in X$ one has $\mathrm{prof}_x(\mathcal{F})\geq\nu(\mathcal{F})$. Then for all $\lambda,\mu\in\mathbf{R}$ with $c_0\leq\lambda<\mu$ the extension mappings

$$H^j_c(X(\lambda),\mathcal{F})\longrightarrow H^j_c(X(\mu),\mathcal{F})$$

are bijective for $j\leq\nu(\mathcal{F})-q$ and injective for $j=\nu(\mathcal{F})-q+1$. Now the theorem follows easily from the following closeness criterion due to Ramis–Ruget–Verdier (see [2], p. 1012). □

Theorem 7. *Let X be a complex space with countable topology, $\mathcal{F}$ be a coherent analytic sheaf on X and q be an integer. Then $H^q_c(X,\mathcal{F})$ is separated provided that the following condition is fulfilled: For every compact set $K\subset X$, there is a compact set $K'\supset K$ such that*

$$\mathrm{Ker}(H^q_K(X,\mathcal{F})\to H^q_c(X,\mathcal{F}))=\mathrm{Ker}(H^q_K(X,\mathcal{F})\to H^q_{K'}(X,\mathcal{F})).$$

Proposition 8. *Let Z be a complex space and $K\subset Z$ be a compact set for which there exists a smooth function $\varphi\colon Z\to\mathbf{R}$ such that $K=\{x\in Z;\varphi(x)\leq 0\}$ and φ is q-convex on $Z\setminus K$. Let $\mathcal{F}$ be a coherent analytic sheaf $\mathcal{F}$ on Z. Then the canonical map*

$$H^j_K(Z,\mathcal{F})\longrightarrow \varprojlim_{W\supset K} H^j_c(W,\mathcal{F})$$

is injective for $j\leq\mathrm{prof}_{\partial K}(\mathcal{F})-q+1$.

Proof. Put $m=\mathrm{prof}_{\partial K}(\mathcal{F})-q$. Then let U and V be open neighborhoods of K of the form $U=\{x\in Z;\varphi(x)<\varepsilon'\}$ and $V=\{x\in Z;\varphi(x)<\varepsilon''\}$ with $0<\varepsilon'<\varepsilon''$ such that $\mathrm{prof}_V(\mathcal{F})\geq m$. The bumping method of [3] gives that the extension

$$H^j_c(U,\mathcal{F})\longrightarrow H^j_c(V,\mathcal{F})$$

is bijective for $j\leq m$ and injective for $j=m+1$. Then, for each integer $l\geq 0$ there is a canonical commutative diagram with exact rows

$$\begin{array}{ccccccc}
H^l(Z,\mathcal{F}) & \longrightarrow & H^l(Z\setminus U,\mathcal{F}) & \longrightarrow & H^{l+1}_c(U,\mathcal{F}) & \longrightarrow & H^{l+1}(Z,\mathcal{F})\\
\downarrow & & \downarrow & & \downarrow & & \downarrow\\
H^l(Z,\mathcal{F}) & \longrightarrow & H^l(Z\setminus V,\mathcal{F}) & \longrightarrow & H^{l+1}_c(V,\mathcal{F}) & \longrightarrow & H^{l+1}(Z,\mathcal{F})
\end{array}$$

which, granting the five lemma, implies the bijectivity of the restrictions

$$H^j(Z\setminus V,\mathcal{F})\longrightarrow H^j(Z\setminus U,\mathcal{F}),\quad j<m.$$

Thus applying Lemma 1 we obtain the bijectivity of the canonical morphisms

$$(\flat)\qquad H^j(Z\backslash K,\mathcal{F})\longrightarrow \varprojlim_{W\supset K} H^j(Z\backslash W,\mathcal{F}),\quad j\le m-q.$$

Now there exists the following natural commutative diagram with exact rows

$$\begin{array}{ccccccc} H^{j-1}(Z,\mathcal{F}) & \longrightarrow & H^{j-1}(Z\backslash K,\mathcal{F}) & \longrightarrow & H^j_K(Z,\mathcal{F}) & \longrightarrow & H^j(Z,\mathcal{F}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^{j-1}(Z,\mathcal{F}) & \longrightarrow & H^{j-1}(Z\backslash U,\mathcal{F}) & \longrightarrow & H^j_c(U,\mathcal{F}) & \longrightarrow & H^j(Z,\mathcal{F}) \end{array}$$

from which we infer readily the injectivity of $H^j_K(Z,\mathcal{F})\to H^j_c(U,\mathcal{F})$ for $j\le m$, whence the proposition for $j\le\operatorname{prof}(\mathcal{F})-q$.

Now we treat the case $j=m+1$. First note the exact sequence (as follows by standard algebraic facts on projective systems)

$$\varprojlim_{W\supset K} H^m_c(W,\mathcal{F})\longrightarrow H^m(Z,\mathcal{F})\longrightarrow \varprojlim_{W\supset K} H^m(Z\backslash W,\mathcal{F})\longrightarrow \varprojlim_{W\supset K} H^{m+1}_c(W,\mathcal{F}).$$

Then, by ($\flat$), a natural commutative diagram, and the five lemma again we derive the injectivity of

$$H^{m+1}_K(Z,\mathcal{F})\longrightarrow \varprojlim_{W\supset K} H^{m+1}_c(W,\mathcal{F}),$$

which concludes the proof of the proposition. □

Remark. Keeping the notation as in Proposition 8, $H^i_K(Z,\mathcal{F})$ has finite dimension (resp., vanishes if φ is q-convex on Z) for $j\le\operatorname{prof}_{\partial K}(\mathcal{F})-q$.

End of the proof of Theorem 2. Let $\{Z_\nu\}_\nu$ be a decreasing sequence of q-convex open neighborhoods of K. Let $\varphi_\nu\colon Z_\nu\to[0,\infty)$ be the function displaying the q-convexity of Z_ν such that φ_ν is q-convex on $Z_\nu\backslash S_\nu$, where $S_\nu:=\{x;\varphi_\nu(x)\le 0\}$ contains K. There is no loss in generality to assume that $S_{\nu+1}$ is contained in the interior of S_ν.

Now fix an integer j, $p\le j\le\operatorname{prof}_{\partial K}(\mathcal{F})-q$. Granting the above proposition and the discussion preceding Theorem 6, we derive that

$$H^j(X\backslash S_\nu,\mathcal{F})$$

is separated. Furthermore, by the above remark, the image of

$$H^{j-1}(X,\mathcal{F})\longrightarrow H^{j-1}(X\backslash S_\nu,\mathcal{F})$$

has finite codimension, for all ν. This in turn gives that the projective system

$$\{H^{j-1}(X\setminus S_\nu, \mathcal{F})\}_\nu$$

with the canonical restriction maps satisfies the cML-condition. Finally we conclude applying Theorem 4. The additional case when X is cohomologically p-complete and K admits a base of open q-complete neighborhoods is to be treated similarly (much easier) so we omit the proof. $\square$

Corollary 1. *Let X be a Stein space and K be a compact set admitting a base of q-complete open sets. Then for each coherent analytic sheaf $\mathcal{F}$ on X, the space $H^i(X\setminus K, \mathcal{F})$ vanishes for $1\le i<\mathrm{prof}_{\partial K}(\mathcal{F})-q$ and is separated for $i=\mathrm{prof}_{\partial K}(\mathcal{F})-q$.*

This generalizes a result from [10] where the case X smooth, $\mathcal{F}$ locally free and $q=1$ has been considered.

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