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Convexity of the median in the gamma distribution

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Abstract. We show that the median $m(x)$ in the gamma distribution with parameter x is a strictly convex function on the positive half-line.

1. Introduction

The median, $m(x)$, of the gamma distribution with (positive) parameter x is defined implicitly by the formula

$$\int_0^{m(x)} e^{-t}t^{x-1}\,dt = \frac{1}{2}\int_0^{\infty} e^{-t}t^{x-1}\,dt. \tag{1}$$

In a recent paper (see [5]) we showed that $0<m'(x)<1$ for all $x>0$. Consequently, $m(x)-x$ is a decreasing function, which for $x=1,2,...$ yields a positive answer to the Chen–Rubin conjecture, cf. [6]. Other authors have solved this conjecture in its discrete setting (see [2], [1] and [3]).

In [4] convexity of the sequence $m(n+1)$ has been established, and the natural question arises if $m(x)$ is a convex function. The main result of this paper is the following.

Theorem 1.1. *The median $m(x)$ defined in* (1) *satisfies $m''(x)>0$. In particular it is a strictly convex function for $x>0$.*

2. Proofs

The proof is based on some results in [5], which we briefly describe. Convexity of m is studied through the function

$$\varphi(x) \equiv \log\frac{x}{m(x)}, \quad x>0. \tag{2}$$

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This function played a key role in [5], and we recall its crucial properties in the proposition below.

Proposition 2.1. *The function $x\mapsto x\varphi(x)$ is strictly decreasing for $x>0$ and*

$$\lim_{x\to 0_+} x\varphi(x)=\log 2,$$
$$\lim_{x\to\infty} x\varphi(x)=\tfrac{1}{3}.$$

Remark 2.2. Proposition 2.1 is established by showing that $(x\varphi(x))'<0$. It follows that the function $\varphi(x)$ is itself strictly decreasing and $\varphi(x)<-x\varphi'(x)$.

The starting point for proving Theorem 1.1 is the relation [5, (10)]

$$(x\varphi(x))'=-e^{g(x)}(A(x)+B(x)), \tag{3}$$

where

$$\begin{aligned} g(x)&=x(\varphi(x)-1+e^{-\varphi(x)}),\\ A(x)&=\int_0^{x\varphi(x)} e^{-s}e^{x(1-e^{-s/x})}\left(1-\left(1+\frac{s}{x}\right)e^{-s/x}\right)ds,\\ B(x)&=\frac{1}{2}\int_0^\infty te^{-xt}\xi'(t+1)\,dt, \end{aligned}$$

and where ξ, defined in [5, (7)], is a certain positive, increasing and concave function on $[1,\infty)$ satisfying $\xi'(t+1)<\frac{8}{135}$ for $t>0$. To establish these properties of ξ is quite involved, and we refer to [5, Section 5] for details. Let us recall the definition of ξ. The function $f(x)=e^{-x}+x$ has an inverse function $u(t)$ for $x\in]-\infty,0[$ and it has an inverse function $v(t)$ for $x\in]0,\infty[$. Then

$$\xi(t)\equiv u'(t)+v'(t)=\frac{1}{1-e^{-u(t)}}+\frac{1}{1-e^{-v(t)}},\quad t>1.$$

Before proving the theorem we state the following lemmas, whose proofs are given later.

Lemma 2.3. *For the function g we have*

$$\begin{aligned} g(x)&<x\varphi(x),\\ -g'(x)&<-x\varphi'(x)\varphi(x),\\ -g'(x)&<-x\varphi'(x) \end{aligned}$$

for all $x>0$.

Lemma 2.4. *We have for $x>0$,*

$$A(x) < \frac{x\varphi(x)^3}{6},$$
$$-A'(x) < -\tfrac{1}{6}\varphi(x)^3 - \tfrac{1}{2}x\varphi'(x)\varphi(x)^2.$$

Lemma 2.5. *We have for $x>0$,*

$$B(x) < \frac{4}{135x^2},$$
$$-B'(x) < \frac{8}{135x^3}.$$

Proof of Theorem 1.1. From (2) we get

$$m''(x) = -e^{-\varphi(x)}(2\varphi'(x) + x\varphi''(x) - x\varphi'(x)^2),$$

so that $m''(x)>0$ is equivalent to the inequality

$$(x\varphi(x))'' < x\varphi'(x)^2.$$

Differentiation of (3) yields

$$(x\varphi(x))'' = e^{g(x)}(-g'(x))(A(x)+B(x)) + e^{g(x)}(-A'(x)-B'(x)).$$

By using Lemmas 2.4 and 2.5 it follows that

$$\begin{aligned}-A'(x)-B'(x) &< -\frac{1}{2}x\varphi'(x)\varphi(x)^2 + \frac{8}{135x^3} - \frac{1}{6}\varphi(x)^3\\ &= -\frac{1}{2}x\varphi'(x)\varphi(x)^2 + \frac{\varphi(x)^2}{x}\left(\frac{8}{135(x\varphi(x))^2} - \frac{1}{6}x\varphi(x)\right).\end{aligned}$$

Here the expression in the brackets is positive, since $(x\varphi(x))^3 < (\log 2)^3 < \frac{48}{135}$. Therefore, and because $\varphi(x) < -x\varphi'(x)$,

$$\begin{aligned}-A'(x)-B'(x) &< \frac{1}{2}x\varphi'(x)^2\, x\varphi(x) + x\varphi'(x)^2\left(\frac{8}{135(x\varphi(x))^2} - \frac{1}{6}x\varphi(x)\right)\\ &= x\varphi'(x)^2\left(\frac{8}{135(x\varphi(x))^2} + \frac{1}{3}x\varphi(x)\right).\end{aligned}$$

We also have from Lemmas 2.3, 2.4 and 2.5,

$$\begin{aligned}-g'(x)(A(x)+B(x)) &< -x\varphi'(x)\varphi(x)\left(\frac{x\varphi(x)^3}{6} + \frac{4}{135x^2}\right)\\ &< x^2\varphi'(x)^2\varphi(x)^2\left(\frac{x\varphi(x)}{6} + \frac{4}{135(x\varphi(x))^2}\right).\end{aligned}$$

Combining these inequalities, and using Lemma 2.3, yields

$$(x\varphi(x))'' < x\varphi'(x)^2 e^{x\varphi(x)}\left(x\varphi(x)^2\left(\frac{x\varphi(x)}{6}+\frac{4}{135(x\varphi(x))^2}\right)\right.$$
$$\left.+\frac{8}{135(x\varphi(x))^2}+\frac{1}{3}x\varphi(x)\right).$$

Supposing that $x\geq 1$, it follows that

$$\begin{aligned}(x\varphi(x))'' &< x\varphi'(x)^2 e^{x\varphi(x)}\left((x\varphi(x))^2\left(\frac{x\varphi(x)}{6}+\frac{4}{135(x\varphi(x))^2}\right)\right.\\ &\qquad\left.+\frac{8}{135(x\varphi(x))^2}+\frac{1}{3}x\varphi(x)\right)\\ &= x\varphi'(x)^2 e^{x\varphi(x)}\left(\frac{(x\varphi(x))^3}{6}+\frac{4}{135}+\frac{8}{135(x\varphi(x))^2}+\frac{1}{3}x\varphi(x)\right)\\ &= x\varphi'(x)^2 h_1(x\varphi(x)),\end{aligned}$$

where h_1 is given by

$$h_1(t)=e^t\left(\frac{t^3}{6}+\frac{4}{135}+\frac{8}{135t^2}+\frac{t}{3}\right).$$

One can show that h_1 attains its maximum on the interval $\left[\frac{1}{3},\log 2\right]$ at the left end point and that $h_1\left(\frac{1}{3}\right)=\left(\frac{551}{810}\right)\sqrt[3]{e}\approx 0.9494$. Therefore it follows that $(x\varphi(x))''< x\varphi'(x)^2$ for $x\geq 1$.

For $0<x<1$ the estimate $-g'(x)<-x\varphi'(x)$ from Lemma 2.3 is used and in this way we get

$$(x\varphi(x))'' < x\varphi'(x)^2 h_2(x\varphi(x)),$$

where

$$h_2(t)=e^t\left(\frac{t^2}{6}+\frac{4}{135t}+\frac{8}{135t^2}+\frac{t}{3}\right).$$

Since $x<1$ and $x\varphi(x)$ decreases we have $x\varphi(x)>\varphi(1)=-\log\log 2$. One can show that h_2 attains its maximum on the interval $[-\log\log 2,\log 2]$ for $t=-\log\log 2$ and that $h_2(-\log\log 2)\approx 0.9616$. Therefore $(x\varphi(x))''<x\varphi'(x)^2$ for $x<1$. □

Remark 2.6. The function h_2 becomes larger than 1 for t close to $\frac{1}{3}$, so h_2 cannot be used to obtain the inequality $(x\varphi(x))''<x\varphi'(x)^2$ for all $x>0$.

Proof of Lemma 2.3. It is clear that $g(x)<x\varphi(x)$. Differentiation yields

$$\begin{aligned}-g'(x) &= -\varphi(x)+(1-x\varphi'(x))(1-e^{-\varphi(x)})\\ &< -\varphi(x)+(1-x\varphi'(x))\varphi(x) = -x\varphi'(x)\varphi(x),\end{aligned}$$

where we have used that $1-e^{-a}<a$ for $a>0$.

To find an estimate that is more accurate for x near 0 we use

$$\begin{aligned}-g'(x) &= -x\varphi'(x)(1-e^{-\varphi(x)})-\varphi(x)+1-e^{-\varphi(x)}\\ &< -x\varphi'(x)-\varphi(x)+1-e^{-\varphi(x)} < -x\varphi'(x). \quad \square\end{aligned}$$

Proof of Lemma 2.4. Using that $1-e^{-a}<a$ and $1-(1+a)e^{-a}<a^2/2$ for $a>0$, we can estimate $A(x)$ by

$$A(x) < \int_0^{x\varphi(x)} e^{-s}e^{x(s/x)}\left(\frac{s^2}{2x^2}\right)ds = \frac{x\varphi(x)^3}{6}.$$

A computation shows that

$$\begin{aligned}-A'(x) &= -(\varphi(x)+x\varphi'(x))e^{-x\varphi(x)}e^{x(1-e^{-\varphi(x)})}(1-(1+\varphi(x))e^{-\varphi(x)})\\ &\quad -\int_0^{x\varphi(x)} e^{-s}e^{x(1-e^{-s/x})}\left(1-\left(1+\frac{s}{x}\right)e^{-s/x}\right)^2 ds\\ &\quad +\int_0^{x\varphi(x)} e^{-s}e^{x(1-e^{-s/x})}\frac{s^2}{x^3}e^{-s/x}\,ds\\ &< -(\varphi(x)+x\varphi'(x))\frac{1}{2}\varphi(x)^2+\int_0^{x\varphi(x)} s^2e^{-s/x}\,ds\frac{1}{x^3}\\ &< -\tfrac{1}{2}\varphi(x)^3-\tfrac{1}{2}x\varphi'(x)\varphi(x)^2+\tfrac{1}{3}\varphi(x)^3\\ &= -\tfrac{1}{6}\varphi(x)^3-\tfrac{1}{2}x\varphi'(x)\varphi(x)^2. \quad \square\end{aligned}$$

Proof of Lemma 2.5. These estimates follow directly from the inequality

$$\xi'(t+1) < \tfrac{8}{135}. \quad \square$$

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