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Componentwise linear ideals with minimal or maximal Betti numbers

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Abstract. We characterize componentwise linear monomial ideals with minimal Taylor resolution and consider the lower bound for the Betti numbers of componentwise linear ideals.

Introduction

Let $S=K[x_1, ..., x_n]$ denote the polynomial ring in n variables over a field K with each $\deg x_i=1$. Let I be a monomial ideal of S and $G(I)=\{u_1, ..., u_s\}$ its unique minimal system of monomial generators. The Taylor resolution [4, p. 439] provides a graded free resolution of S/I. Fröberg [6, Proposition 1] characterizes the monomial ideals for which the Taylor resolution is minimal. In most cases it is indeed not minimal, however it yields the following upper bound for the Betti numbers of I.

$$\beta_i(I) \leq \binom{s}{i+1} \quad \text{for} \quad i=0, ..., s-1.$$

This upper bound is reached exactly when the Taylor resolution is minimal.

On the other hand, Brun and Römer [3, Corollary 4.1] have shown that

$$\beta_i(I) \geq \binom{p}{i+1} \quad \text{for} \quad i=0, ..., p-1,$$

where p denotes the projective dimension of S/I.

In this note we consider componentwise linear ideals of S, and study the cases when one of the bounds for the Betti numbers described above is obtained. As one of the main results (Theorem 1.5) we have that a componentwise linear monomial ideal has a minimal Taylor resolution if and only if I is Gotzmann with $|G(I)|\leq n$.

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We also prove in Theorem 1.7 the following result: Assume that K is infinite, and let I be a componentwise linear ideal of S and $\mathrm{Gin}(I)$ its generic initial ideal with respect to the reverse lexicographic order. Suppose that $|G(\mathrm{Gin}(I))|=s$, and that $\beta_i(\mathrm{Gin}(I))=\binom{s}{i+1}$ for some $1\le i<s$. Then I is Gotzmann and $\beta_i(I)=\binom{s}{i+1}$ for all i.

In general, if I is a monomial ideal generated by s elements and for some i with $0<i<s-1$, the ith Betti number of I reaches the Taylor bound, i.e. $\beta_i(I)=\binom{s}{i+1}$, then, contrary to the previous result, this does not necessarily imply that the whole Taylor resolution is minimal, as we show by examples. While if $\beta_{s-1}(I)\neq 0$, the Taylor resolution is indeed minimal, as noted by Fröberg [6, Proposition 1].

Concerning the lower bound we have the following result (Theorem 2.2): Let $I\subset\mathfrak{m}^2$ be a componentwise linear ideal with $\operatorname{grade} I=g$ and $\operatorname{proj\,dim} S/I=p$, where $\operatorname{grade} I$ is the length of maximal regular sequences in I. Suppose that $\beta_i(I)=\binom{p}{i+1}$ for some i. Then (a) $i\ge g$, and (b) $\beta_j(I)=\binom{p}{j+1}$ for all $j\ge i$.

1. Componentwise linear ideals with minimal Taylor resolution

A monomial ideal I of S is *lexsegment* if, for a monomial u of S belonging to I and for a monomial v of A with $\deg u=\deg v$ and with $v>_{\mathrm{lex}}u$, one has $v\in I$, where $<_{\mathrm{lex}}$ is the lexicographic order on S induced by the ordering $x_1>...>x_n$ of the variables. A lexsegment ideal I of S is called *universal lexsegment* [2] if, for $m=1,2,...$, the monomial ideal of the polynomial ring $K[x_1,...,x_n,x_{n+1},...,x_{n+m}]$ generated by the monomials belonging to $G(I)$ is lexsegment. In other words, a universal lexsegment ideal of S is a lexsegment ideal I of S which remains being lexsegment in the polynomial ring $K[x_1,...,x_n,x_{n+1},...,x_{n+m}]$ for $m=1,2,....$ It is known [9, Corollary 1.4] that a lexsegment ideal I of S is universal lexsegment if and only if $|G(I)|\le n$, where $|G(I)|$ is the number of monomials belonging to $G(I)$.

For a monomial u of S, let $m(u)$ be the biggest integer j for which x_j divides u. A monomial ideal $I\subset S$ is said to be *stable* if for any monomial $u\in I$ and for any $1\le q<m(u)$ it follows that $(x_q/x_{m(u)})u\in I$. Eliahou and Kervaire [5] proved that if I is a stable monomial ideal of S then

$$\beta_i(I)=\sum_{u\in G(I)}\binom{m(u)-1}{i}\quad\text{for } i=0,1,...,n-1. \tag{1}$$

Lemma 1.1. *The Taylor resolution of a universal lexsegment ideal is minimal.*

Proof. Let I be a universal lexsegment ideal of S and $G(I)=\{u_1,u_2,...,u_s\}$ with $s\le n$, where $\deg u_1\le\deg u_2\le...\le\deg u_s$ and where $u_{i+1}<_{\mathrm{lex}}u_i$ if $\deg u_i=\deg u_{i+1}$. Then [9, Lemma 1.2] says that $m(u_i)=i$. By using (1), the qth total Betti number

$\beta_q(I)$ of I is $\sum_{i=1}^{s}\binom{i-1}{q}=\binom{s}{q+1}$. Thus $\beta_q(I)$ coincides with the rank of the qth free module of the Taylor resolution of I. $\square$

We refer the reader to [7] for fundamental material on componentwise linear ideals and Gotzmann ideals. A homogeneous ideal I of S is *Gotzmann* if $\beta_{ij}(I)=\beta_{ij}(I^{\mathrm{lex}})$ for all i and j, where I^{lex} is the unique lexsegment ideal of S with the same Hilbert function as I. A homogeneous ideal I of S is *componentwise linear* if for all positive integers j the ideal $I_{\langle j\rangle}$ generated by all polynomials of degree j in I has a linear resolution. If I is componentwise linear then $\beta_{ij}(I)=\beta_{ij}(\mathrm{Gin}(I))$ for all i and j, where $\mathrm{Gin}(I)$ is the generic initial ideal of I with respect to the reverse lexicographic order induced by the ordering $x_1>...>x_n$ of the variables; see [1, Theorem 1.1] for the proof of this statement when $\operatorname{char} K=0$, and [8, Lemma 3.3] in the case of a field of arbitrary characteristic. Examples of componentwise linear ideals are the Gotzmann ideals.

We shall need the following three results for the proof of the next theorems.

Lemma 1.2. *A componentwise linear ideal I is Gotzmann if and only if* $\mathrm{Gin}(I)$ *is Gotzmann.*

Proof. Since I is componentwise linear we have $\beta_{ij}(I)=\beta_{ij}(\mathrm{Gin}(I))$ for all i and j. On the other hand, $I^{\mathrm{lex}}=\mathrm{Gin}(I)^{\mathrm{lex}}$. Therefore, $\beta_{ij}(I)=\beta_{ij}(I^{\mathrm{lex}})$ if and only if $\beta_{ij}(\mathrm{Gin}(I))=\beta_{ij}(\mathrm{Gin}(I)^{\mathrm{lex}})$. This proves the assertion. $\square$

Next we have the following result.

Lemma 1.3. *Let I be a stable ideal and $u\in G(I)$ with $m(u)=i\geq 2$. Then there exists $w\in G(I)$ with* $\deg w\leq\deg u$ *and with $m(w)=i-1$. In particular,* $\max\{m(u):u\in G(I)\}\leq|G(I)|$.

Proof. Let $u=vx_i^N$ with $m(v)\leq i-1$ and with $N\geq 1$. Since I is stable, one has $vx_{i-1}^N\in I$. Thus there is $w\in G(I)$ which divides vx_{i-1}^N. Since $u\in G(I)$, it follows that w cannot divide v. Hence $m(w)=i-1$ and $\deg w\leq\deg u$. $\square$

Lemma 1.4. *Let I be a stable ideal with* $\max\{m(u):u\in G(I)\}=|G(I)|$. *Then I is a Gotzmann ideal with $|G(I)|\leq n$.*

Proof. By using Lemma 1.3 we may assume that $G(I)=\{u_1,...,u_s\}$, where $\deg u_1\leq...\leq\deg u_s$ and where $m(u_i)=i$ for $1\leq i\leq s$. Now, [9, Lemma 1.4] guarantees that there exists a universal lexsegment ideal L of S with $G(L)=\{w_1,...,w_s\}$ such that $\deg u_i=\deg w_i$ for $1\leq i\leq s$. Again, by [9, Lemma 1.2], one has $m(w_i)=i$ for $1\leq i\leq s$. Hence the Eliahou–Kervaire formula [5] implies that $\beta_{ij}(I)=\beta_{ij}(L)$ for all i and j. In particular I and L have the same Hilbert function. Thus $L=I^{\mathrm{lex}}$. It follows that I is Gotzmann. Of course, $|G(I)|\leq n$, since $m(u)\leq n$ for all $u\in G(I)$. $\square$

The equivalence of the statements (a) and (b) in the following theorem has been proved for the special case of stable ideals in the paper [10].

Theorem 1.5. *Let I be a componentwise linear monomial ideal of S. Then the following conditions are equivalent*:
(a) *The Taylor resolution of I is minimal*;
(b) $\max\{m(u):u\in G(I)\}=|G(I)|$;
(c) *I is Gotzmann with $|G(I)|\le n$.*

Proof. (a)⇒(b) Let $|G(I)|=s$ and let $J=\mathrm{Gin}(I)$. Then J is strongly stable with $|G(J)|=s$, and the Taylor resolution of J is minimal as well since the Betti numbers do not change. Since $\beta_{s-1}(J)\neq 0$, the Eliahou–Kervaire formula implies that there exists a monomial $u\in G(J)$ with $m(u)=s$.

(b)⇒(c) By using Lemma 1.4, it follows that J is a Gotzmann ideal with $|G(J)|\le n$. Since $J^{\mathrm{lex}}=I^{\mathrm{lex}}$ and J is Gotzmann, we have $\beta_{ij}(I)=\beta_{ij}(J)=\beta_{ij}(J^{\mathrm{lex}})=\beta_{ij}(I^{\mathrm{lex}})$ for all i and j. Thus I is Gotzmann with $|G(I)|\le n$.

(c)⇒(a) Since $|G(I^{\mathrm{lex}})|=|G(I)|\le n$, the lexsegment ideal I^{lex} is a universal lexsegment ideal. Since $\beta_i(I)=\beta_i(I^{\mathrm{lex}})$ for all i, and since by Lemma 1.1 the Taylor resolution of I^{lex} is minimal, it follows that the Taylor resolution of I is minimal, as required. □

Remark 1.6. If some of the Betti numbers of a monomial ideal reach the Taylor bound, then this does not necessarily imply that the whole Taylor resolution is minimal. Given integers $2\le i<s-2$, we consider the ideal I generated by x_1y_1, x_2y_2, ..., $x_{s-1}y_{s-1}$, $y_1\dots y_i$. Then it is easily checked that $\beta_j(I)=\binom{s}{j+1}$ for $j=0$, ..., $i-2$ and $\beta_j(I)<\binom{s}{j+1}$ for $i-2<j\le s-1$.

On the other hand, if I is a monomial ideal with $G(I)=\{u_1,u_2,\dots,u_s\}$, and for some $0\le i\le s$ we have $\beta_i(I)=\binom{s}{i+1}$, then $\beta_j(I)=\binom{s}{j+1}$ for all $j\le i$. Indeed, if $\beta_i(I)=\binom{s}{i+1}$, then from the definition of the Taylor complex it follows that for an arbitrary subset $\{u_{j_1},u_{j_2},\dots,u_{j_i}\}$ of $G(I)$ of cardinality i, one has that $\mathrm{lcm}(u_{j_1},\dots,,u_{j_i})\neq\mathrm{lcm}(u_{j_1},\dots,u_{j_{k-1}},u_{j_{k+1}},\dots,u_{j_i})$ for all $k=1,\dots,i$. It is obvious that similar inequalities hold for any subset of $\{u_{j_1},u_{j_2},\dots,u_{j_i}\}$. This clearly implies that $\beta_j(I)=\binom{s}{j+1}$ for all $j\le i$.

Theorem 1.7. *Let I be a componentwise linear ideal of S with $|G(\mathrm{Gin}(I))|=s$. Suppose that $\beta_i(\mathrm{Gin}(I))=\binom{s}{i+1}$ for some $1\le i<s$. Then I is Gotzmann and $\beta_i(I)=\binom{s}{i+1}$ for all i.*

Proof. Let $q\ge 1$ be the biggest integer for which $m(u)=q$ for some $u\in G(\mathrm{Gin}(I))$. Lemma 1.3 says that for each $j\le q$ there is $u\in G(\mathrm{Gin}(I))$ with $m(u)=j$. Fix i_0 with

$\beta_{i_0}(\mathrm{Gin}(I))=\binom{s}{i_0+1}$. Since

$$\begin{aligned}\beta_{i_0}(\mathrm{Gin}(I)) &= \sum_{u\in G(\mathrm{Gin}(I))}\binom{m(u)-1}{i_0}\\ &= \sum_{\substack{u\in G(\mathrm{Gin}(I))\\ m(u)>i_0}}\binom{m(u)-1}{i_0}\\ &\le \binom{i_0}{i_0}+\binom{i_0+1}{i_0}+\ldots+\binom{q-2}{i_0}+(s-q+1)\binom{q-1}{i_0}\\ &\le \binom{i_0}{i_0}+\binom{i_0+1}{i_0}+\ldots+\binom{s-1}{i_0}\\ &= \binom{s}{i_0+1},\end{aligned}$$

it follows that $q=s$. Hence Lemma 1.4 implies that $\mathrm{Gin}(I)$ is Gotzmann with $|G(\mathrm{Gin}(I))|\le n$. Therefore Theorem 1.5 guarantees that $\beta_i(\mathrm{Gin}(I))=\binom{s}{i+1}$ for all i. Since I is componentwise linear, we have $\beta_i(I)=\beta_i(\mathrm{Gin}(I))$ for all i, and by Lemma 1.2 that I itself is Gotzmann, as desired. $\square$

2. Componentwise linear ideals with minimal Betti numbers

It is not surprising that a stable monomial ideal is rarely a complete intersection. One such example is $I=(x_1,\ldots,x_{s-1},x_s^d)$, where $1\le s\le n$ and $d\ge 1$. If I is contained in the square of the graded maximal ideal $\mathfrak{m}=(x_1,\ldots,x_n)$ of S, as we may always assume, then there exist even less such ideals. Indeed we have the following result.

Lemma 2.1. *Let $I\subset\mathfrak{m}^2$ be a stable ideal. If I is a complete intersection, then $I=(x_1^d)$ for some $d\ge 2$.*

Proof. Let $G(I)=\{u_1,\ldots,u_s\}$. We may assume that $m(u_s)\ge m(u_{s-1})\ge\ldots\ge m(u_1)$. Assume that $s\ge 2$. Then (u_1,u_2) is stable and a complete intersection. It is clear that $u_1=x_1^d$ for some $d\ge 2$. By Lemma 1.3 we have $m(u_2)=2$ and $\deg u_2\ge d$. Since u_1 and u_2 are coprime, it follows that $u_2=x_2^c$ with $c\ge d$. Since I is stable, $x_1x_2^{c-1}\in(u_1,u_2)$, a contradiction. $\square$

Now we can show the following theorem.

Theorem 2.2. *Let $I\subset\mathfrak{m}^2$ be a componentwise linear ideal with* grade $I=g$ *and* proj dim $S/I=p$. *Suppose that $\beta_i(I)=\binom{p}{i+1}$ for some i. Then*

(a) *$i\ge g$;*

(b) *$\beta_j(I)=\binom{p}{j+1}$ for all $j\ge i$.*

Proof. We may replace I by $\mathrm{Gin}(I)$ and hence may assume that I is a stable ideal. As usual we set $m_i(I)=|\{u\in G(I):m(u)=i\}|$. We first show that $m_i(I)>1$ for $i=2,...,g$. Indeed, since I is a stable monomial ideal of grade g, the sequence $x_n, x_{n-1}, ..., x_{n-g+1}$ is a system of homogeneous parameters for the standard graded K-algebra S/I. This is due the fact that for all j, the multiplication map

$$S/(I, x_n, ..., x_{n-j+1}) \xrightarrow{x_{n-j}} S/(I, x_n, ..., x_{n-j+1})$$

has a kernel of finite length, as follows from [4, Proposition 15.24]. Hence S/I modulo the sequence $x_n, x_{n-1}, ..., x_{n-g+1}$ is of dimension 0 and isomorphic to $K[x_1, ..., x_g]/J$, where J is again a stable ideal with $m_i(J)=m_i(I)$ for $i=1,...,g$. We may assume that $g>1$. Then $m_g(I)=m_g(J)=\dim_K(J:x_g)/J$. Since J is stable, $J:x_g=J:(x_1,...,x_g)$, and it follows that $m_i(I)=\dim_K J:(x_1,...,x_g)/J>1$, since S/J is not Gorenstein. In fact, if S/J would be Gorenstein, then, since $\dim S/J=0$ and J is a monomial ideal, it would follow that J is a complete intersection, contradicting Lemma 2.1. If $g>2$, then we consider $K[x_1,...,x_g]/J$ modulo x_g and repeat the argument. This can be done as long as $g>2$.

Now we use the Eliahou–Kervaire formula and get

$$\beta_i(I)=\sum_{u\in G(I)}\binom{m(u)-1}{i}=\sum_{j=i}^{p-1}m_{j+1}(I)\binom{j}{i}\geq\sum_{j=i}^{p-1}\binom{j}{i}=\binom{p}{i+1}$$

with equality if and only if all $m_j(I)=1$ for $j=i+1,...,p$.

For the proof of (a) we may assume that $g\geq 2$. Then $m_g(I)\geq 2$, and we can have the equality $\beta_i(I)=\binom{p}{i+1}$ only if $i+1>g$.

On the other hand, if $\beta_i(I)=\binom{p}{i+1}$ for some $i\geq g$, then $m_j(I)=1$ for all $j=i+1,...,p$. Hence the Eliahou–Kervaire formula implies that $\beta_j(I)=\binom{p}{j+1}$ for all $j\geq i$, as desired. □

In view of the preceding theorem and in view of Lemma 1.1 the following consequence is immediate.

Corollary 2.3. *Let $I\subset\mathfrak{m}^2$ be a componentwise linear ideal whose Taylor resolution is minimal. Then* $\mathrm{grade}\, I=1$. *In particular, any universal lexsegment ideal $I\subset\mathfrak{m}^2$ is of the form x_1J, where J is a monomial ideal.*

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