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Extreme Jensen measures

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Abstract. Let Ω be an open subset of $\mathbf{R}^d$, $d\geq 2$, and let $x\in\Omega$. A *Jensen measure* for x on Ω is a Borel probability measure μ, supported on a compact subset of Ω, such that $\int u\,d\mu\leq u(x)$ for every superharmonic function u on Ω. Denote by $J_x(\Omega)$ the family of Jensen measures for x on Ω. We present two characterizations of $\operatorname{ext}(J_x(\Omega))$, the set of extreme elements of $J_x(\Omega)$. The first is in terms of finely harmonic measures, and the second as limits of harmonic measures on decreasing sequences of domains.

This allows us to relax the local boundedness condition in a previous result of B. Cole and T. Ransford, Jensen measures and harmonic measures, *J. Reine Angew. Math.* **541** (2001), 29–53.

As an application, we give an improvement of a result by Khabibullin on the question of whether, given a complex sequence $\{\alpha_n\}_{n=1}^{\infty}$ and a continuous function $M\colon \mathbf{C}\to\mathbf{R}^+$, there exists an entire function $f\not\equiv 0$ satisfying $f(\alpha_n)=0$ for all n, and $|f(z)|\leq M(z)$ for all $z\in\mathbf{C}$.

1. Introduction

Let Ω be an open subset of $\mathbf{R}^d$, $d\geq 2$. We denote by $\mathrm{SH}(\Omega)$ the set of superharmonic functions on Ω (see e.g. [AG, Definition 3.1.2]), and by $\mathrm{SH}^+(\Omega)$ the set of positive superharmonic functions on Ω.

Given $x\in\Omega$, a *Jensen measure* for x with respect to Ω is a probability measure μ, supported on a compact subset of Ω, such that

$$\int u\,d\mu \leq u(x), \quad u\in\mathrm{SH}(\Omega).$$

Let us denote by $J_x(\Omega)$ the set composed of these measures.

Jensen measures have proved to be a very useful tool in many domains such as (pluri)potential theory, complex analysis, analytic multifunctions theory and uniform algebra theory. For a survey of this, see [Ra]. The aim of this paper is to study Jensen measures and get more information about them.

As a few examples of Jensen measures, there is the Dirac measure ε_x at x, the normalized Lebesgue measure on a closed ball B centered at x and contained in Ω, and the normalized surface measure σ on the sphere ∂B with B as above. More generally than the last example, there is the set of harmonic measures for x on domains D with $x\in D\Subset\Omega$ (i.e. $\overline{D}$ is a compact subset of Ω), which we denote by $H_x(\Omega)$.

The space of continuous real-valued functions on Ω, denoted by $C(\Omega)$, is a Fréchet space with the usual uniform convergence on compact subsets. The dual space $C(\Omega)^*$ may be identified with the space of finite signed Borel measures on Ω with compact support. In particular, $J_x(\Omega)\subset C(\Omega)^*$ for each $x\in\Omega$. The following proposition gives some properties of $J_x(\Omega)$ as a subset of $C(\Omega)^*$.

Proposition 1.1. ([CR2, Proposition 2.3]) *Let Ω be an open subset of $\mathbf{R}^d$, $d\geq 2$, and let $x\in\Omega$.*

(a) *The set $J_x(\Omega)$ is convex and weak*-closed in $C(\Omega)^*$.*

(b) *For each compact $K\subset\Omega$, the set*

$$J_x(\Omega,K):=\{\mu\in J_x(\Omega):\operatorname{supp}\mu\subset K\}$$

is convex, weak-compact and metrizable.*

Denote by $\operatorname{ext}(J_x(\Omega))$ the set of extreme elements of $J_x(\Omega)$. As a consequence of Choquet's theory (see e.g. [Ph]) and the above proposition, each Jensen measure can be expressed as an 'average' of extreme Jensen measures, and moreover $\operatorname{ext}(J_x(\Omega))$ is the minimal set with this property. This vague statement is formalized in [CR2, Proposition 6.1]. So, extreme Jensen measures can tell us a lot about $J_x(\Omega)$, and our work is based on this perspective.

In [CR2, Theorem 1.5], it is shown that

$$H_x(\Omega)\cup\{\varepsilon_x\}\subset\operatorname{ext}(J_x(\Omega)),$$

and that this inclusion is strict. This result and the minimal property of $\operatorname{ext}(J_x(\Omega))$ mentioned above dash our hopes to express Jensen measures as averages of harmonic measures.

What we shall do in this paper is to give a complete characterization of $\operatorname{ext}(J_x(\Omega))$ in terms of *finely harmonic measures* (defined below) and also in terms of sequences of harmonic measures. This is the content of the following theorems, which are proved in Sections 3–7. Before stating them, let us give some explanatory comments.

On $\mathbf{R}^d$, the so-called *fine topology* related to the set $\mathrm{SH}(\mathbf{R}^d)$ can be defined. It is the coarsest topology on $\mathbf{R}^d$ which makes every superharmonic function on $\mathbf{R}^d$ continuous in the extended sense of functions taking values in $[-\infty,+\infty]$. It is strictly finer than the Euclidean topology. As is the case for relatively compact domains of Ω containing x, it is possible to define the harmonic measure on a *fine domain* V (open and connected with respect to the fine topology) for x, provided that $x\in V\Subset\Omega$. We denote this measure by $\varepsilon_x^{\Omega\setminus V}$, and by $\mathrm{FH}_x(\Omega)$, the set containing all of them. A detailed treatment is given in Section 2. Finally, by a *regular domain* we mean a domain which is regular with respect to the classical Dirichlet problem (see e.g. [AG, Chapter 6]).

Theorem 1.2. *Let Ω be an open subset of $\mathbf{R}^d$, $d\geq 2$. Then, for each $x\in\Omega$,*

$$\mathrm{ext}(J_x(\Omega)) = \mathrm{FH}_x(\Omega)\cup\{\varepsilon_x\}.$$

Theorem 1.3. *Let Ω be an open subset of $\mathbf{R}^d$, $d\geq 2$, and $x\in\Omega$.*

(i) *For each decreasing sequence $\{D_n\}_{n=1}^\infty$ of domains such that $x\in D_n\Subset\Omega$, the sequence $\{\omega_n\}_{n=1}^\infty$ converges to ε_x or to a measure in $\mathrm{FH}_x(\Omega)$ in the weak*-topology of $C(\mathbf{R}^d)^*$, where ω_n is the harmonic measure on D_n for x.*

(ii) *Let V be a fine domain in Ω such that $x\in V\Subset\Omega$. Then, there exists a decreasing sequence of regular domains $\{D_n\}_{n=1}^\infty$ such that $V\subset D_n\Subset\Omega$ and $(\bigcap_{n=1}^\infty D_n)\setminus V$ is polar.*

(iii) *If $\{D_n\}_{n=1}^\infty$ and V are as described in* (ii), *then $\omega_n\to\varepsilon_x^{\Omega\setminus V}$ in the weak*-topology of $C(\mathbf{R}^d)^*$, where ω_n is the harmonic measure on D_n for x.*

These two theorems allow us to give an improved version of the main result of [CR2], and moreover, to prove it in a shorter and more transparent way. This is the content of the following theorem which is proved in Section 8. Let us denote by $H_x^r(\Omega)$ the set of harmonic measures of $H_x(\Omega)$ defined on regular domains.

Theorem 1.4. *Let Ω be an open subset of $\mathbf{R}^d$, $d\geq 2$, and $x\in\Omega$. Let $\varphi\colon\Omega\to[-\infty,+\infty]$ be a universally measurable function which satisfies the following property*:

For each open set D with $x\in D\Subset\Omega$, there exists a subharmonic function s on D such that $s(x)>-\infty$ and $\varphi\geq s$ on D.

Then, for each $\mu\in J_x(\Omega)$,

$$\int\varphi\,d\mu\leq\sup\biggl\{\int\varphi\,d\omega:\omega\in H_x^r(\Omega)\cup\{\varepsilon_x\}\biggr\}.$$

Note that the condition on φ implies that the integrals above exist. To fully understand this result and its implications, one should have a look at [CR1] and [CR2].

Finally, in Section 9, we present an application of our work to entire functions based on Theorem 1.4 and a theorem of Khabibullin [Kh, Section 2, p. 1069] stated in the same section. Let us explain the context within which this is set.

Given an entire function $f\not\equiv 0$, it is well known that there is a relation between the rate of growth of f and the rate of convergence toward infinity of its sequence of zeros. In particular, if $M\colon \mathbf{C}\to(0,\infty)$ is a continuous function and $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence of non-zero complex numbers converging to infinity, one might ask which conditions M and $\{\alpha_n\}_{n=1}^{\infty}$ must satisfy to assure the existence of f.

Denote by G_D the Green function of a bounded domain D and by H_{φ}^{D} the generalized solution of the Dirichlet problem on D with boundary data φ. If such an f exists, then we can show that

$$\sum_n G_D(\alpha_n,0)\le H_{\log M}^{D}(0)-\log|f(0)|$$

for all bounded domains D which contain 0, where n runs over the integers such that $\alpha_n\in D$. The proof of this statement is elementary and a full justification appears in Section 9. Surprisingly, this necessary condition turns out to be almost sufficient, and this is the content of the next result.

Theorem 1.5. *Let $M\colon \mathbf{C}\to(0,\infty)$ be a continuous function. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence of non-zero complex numbers. Suppose that there exists a constant $c\in\mathbf{R}$ such that*

$$\sum_n G_D(\alpha_n,0)\le H_{\log M}^{D}(0)+c \tag{1}$$

for each bounded domain D which contains 0, where n runs over the integers such that $\alpha_n\in D$. Then, for each $\delta>0$, there exists an entire function $f\not\equiv 0$, whose zero set includes $\{\alpha_n\}_{n=1}^{\infty}$ (counting multiplicities), such that

$$|f(z)|\le \max_{|\zeta-z|\le\delta} M(\zeta),\quad z\in\mathbf{C}.$$

2. Definitions and preliminaries

In this section, we introduce the definitions and the results one should know to feel comfortable with the sequel. If there is no contraindication, Ω will denote an open subset of $\mathbf{R}^d$, $d\ge 2$, and x a point of Ω.

As in measure theory, potential theory has its own negligible sets, called *polar sets* (see e.g. [AG, Chapter 5]). We will say that a proposition $P(y)$, concerning a point y in a set A, holds *quasi-everywhere* (*q.e.*) or holds for *quasi-every* $y\in A$, if it is true for all $y\in A$ apart from a polar set.

For a positive superharmonic function $u\in\mathrm{SH}^+(\Omega)$ and $A\subset\Omega$, define

$$R_u^A := \inf\{v : v\in\mathrm{SH}^+(\Omega) \text{ and } v\ge u \text{ on } A\}.$$

This function is called the *réduite* of u on A (in Ω). Its lower semicontinuous regularization $\widehat{R}_u^A$ is called the *balayage* of u on A (in Ω) (see e.g. [AG, Section 5.3]).

Recall that $J_x(\Omega)$ denotes the set of Jensen measures. Most of the literature on the subject defines Jensen measures in terms of subharmonic functions. Nevertheless, we decided to proceed in a different way and define them in terms of superharmonic functions since most of the cited literature on potential theory which appears in this paper is built on superharmonic functions. We thought it would be more natural in this way.

Let us define an auxiliary class of measures for $J_x(\Omega)$. Write $\mu \in \mathrm{SJ}_x(\Omega)$ if μ is a Radon measure on Ω such that

$$\int u\,d\mu \leq u(x), \quad u \in \mathrm{SH}^+(\Omega).$$

The next properties of $\mathrm{SJ}_x(\Omega)$ are more or less evident. Recall that $\operatorname{ext}(A)$ denotes the set of extreme elements of the convex set A.

Basic properties of $\mathrm{SJ}_x(\Omega)$

(i) If $\mu \in \mathrm{SJ}_x(\Omega)$, then the support of μ only meets the connected component of Ω which contains x.

(ii) Suppose that Ω is Greenian (i.e. that Ω possesses a Green function; see e.g. [AG, Chapter 4]). If $P \subset \Omega$ is polar and $\mu \in \mathrm{SJ}_x(\Omega)$, then P is μ-measurable and $\mu(P \setminus \{x\}) = 0$.

(iii) If $\mu \in \operatorname{ext}(\mathrm{SJ}_x(\Omega))$ and $\mu \neq \varepsilon_x$, then $\mu(\{x\}) = 0$.

Note that Properties (ii) and (iii) imply that $\mu(P) = 0$ if $P \subset \Omega$ is polar and $\mu \in \operatorname{ext}(\mathrm{SJ}_x(\Omega)) \setminus \{\varepsilon_x\}$.

Here are some classical definitions.

Definition 2.1. Let Ω be a Greenian open subset of $\mathbf{R}^d$, $d \geq 2$.

(a) ([AG, Chapter 1]) For $y \in \mathbf{R}^d$, we define $U_y\colon \mathbf{R}^d \to (-\infty, +\infty]$ by

$$U_y(z) := \begin{cases} -\log\|z-y\|, & d=2, \\ \|z-y\|^{2-N}, & d \geq 3, \end{cases} \qquad z \in \mathbf{R}^d,$$

where it is understood that $U_y(y) = +\infty$. This function is usually called the *fundamental harmonic function* with pole y. It is harmonic on $\mathbf{R}^d \setminus \{y\}$ and superharmonic on $\mathbf{R}^d$.

(b) ([AG, Chapter 4] and [CC, Section 7.1]) Let $G_\Omega(\cdot\,,\cdot)$ denote the Green function of Ω. If μ is a Borel measure, then the function

$$G_\Omega\mu(z) := \int_\Omega G_\Omega(z,y)\,d\mu(y), \quad z \in \Omega$$

is the (*Green*) *potential* for μ on Ω if it is superharmonic. In the affirmative, we say that μ is an *admissible measure.*

(c) ([CC, Section 7.1]) If μ is admissible and $A\subset\Omega$, then there exists a unique Borel measure μ^A such that

$$\int u\,d\mu^A = \int \widehat{R}_u^A\,d\mu, \quad u\in \mathrm{SH}^+(\Omega).$$

The measure μ^A turns out to be admissible, and is called the *balayage* of μ on A.

As we mentioned in the introduction, $\mathbf{R}^d$ can be equipped with the fine topology relative to $\mathrm{SH}(\mathbf{R}^d)$. To avoid confusion, we will use the terms *fine* and *finely* when referring to the fine topology. (For more details, see e.g. [AG, Chapter 7].)

Definition 2.2. ([Fu, Section 3]) Let Ω be a Greenian open subset of $\mathbf{R}^d$, $d\geq 2$, and $A\subset\mathbf{R}^d$.

(a) Given $y\in\mathbf{R}^d$, we shall say that A is *thin* at y if y is not a fine limit point of A (that is, y is not a limit point of A with respect to the fine topology).

(b) When $A\subset\Omega$, we denote by $b(A)$ and $i(A)$ respectively the subset of Ω consisting of the points for which A is non-thin, and the set of finely isolated points of A.

(c) The sets $\tilde{A}$ and $\partial_f A$ denote respectively the fine closure and the fine boundary of A.

(d) We say that $A\subset\Omega$ is a *base* if $A=b(A)$. It is easy to see that base sets of Ω are relatively finely closed in Ω.

The notion of base set is intimately related to the regularity of an open set for the Dirichlet problem, as we can see in the next result.

Proposition 2.3. ([AG, Section 7.5] and [CC, Section 7.1]) *Let Ω be a Greenian open subset of $\mathbf{R}^d$, $d\geq 2$. Let $U\Subset\Omega$ open. Then, the following statements are equivalent*:

(i) *U is regular for the Dirichlet problem*;

(ii) *$\Omega\setminus U$ is a base*;

(iii) *$i(\Omega\setminus U)=\varnothing$.*

Note. To avoid misunderstanding, we will only consider the Dirichlet problem on relatively compact subsets of Ω, since there are different definitions for general open sets.

This relation between regular open sets and base sets allows us to give a wider sense for the former notion. We will say that a finely open subset $U\Subset\Omega$ is *regular* if $\Omega\backslash U$ is a base (see [Fu, p. 34]). Note that, like in the Euclidean case, the set $i(\Omega\backslash U)$ is polar if U is finely open [CC, Corollary 7.2.1].

Let us give a last result on the regular sets.

Proposition 2.4. ([Fu, p. 150]) *Let Ω be a Greenian open subset of $\mathbf{R}^d$, $d\geq 2$. Let $U\subset\Omega$ be finely open and set $V:=U\cup i(\Omega\backslash U)$. Then, U is finely connected (i.e. connected with respect to the fine topology) if and only if V is finely connected. Also, V is regular.*

We now present a series of results which will be needed in the following sections. Most of them come from the literature.

Proposition 2.5. *Let Ω be a Greenian open subset of $\mathbf{R}^d$, $d\geq 2$. Then, the fine Borel subsets of Ω are μ-measurable for each $\mu\in\mathrm{SJ}_x(\Omega)$. In particular, the fine Borel functions $f\colon\Omega\to[-\infty,+\infty]$ are μ-measurable for each $\mu\in\mathrm{SJ}_x(\Omega)$.*

Proof. Let F be a relatively finely closed subset of Ω and $\mu\in\mathrm{SJ}_x(\Omega)$. We can write $F=b(F)\cup i(F)$. The set $i(F)$ is polar, so it is μ-measurable (property (ii) of $\mathrm{SJ}_x(\Omega)$). Also, the set $b(F)$ is of type G_δ. This is not obvious, but we can find a proof of this in [CC, Corollary 7.2.1]. This implies that F is μ-measurable. Since it is true for all closed sets, we get the desired result. $\square$

Here is a result which links the balayage of an admissible measure with the notion of base set.

Proposition 2.6. ([Fu, Section 4.7]) *Let Ω be a Greenian open subset of $\mathbf{R}^d$, $d\geq 2$. Let μ be an admissible measure on Ω and $A\subset\Omega$. Then,*

(a) *$\mu^A=\mu^{b(A)}$ is carried by $b(A)$;*

(b) *$\mu^A=\mu$ if and only if μ is carried by $b(A)$.*

The next result is a motivation for the definition of finely harmonic measure.

Proposition 2.7. ([CC, Theorems 7.1.1 and 7.1.2]) *Let Ω be a Greenian open subset of $\mathbf{R}^d$, $d\geq 2$. Let U be an open subset of Ω (in the Euclidean topology) such that $U\Subset\Omega$. Then,*

$$\varepsilon_y^{\Omega\backslash U}=\begin{cases}\varepsilon_y & \textit{for } y\in b(\Omega\backslash U),\\ \omega & \textit{for } y\in U,\end{cases}$$

where ω denotes the harmonic measure on U for y.

We are now ready to define the finely harmonic measure.

Definition 2.8. ([Fu, p. 37]) Let Ω be a Greenian open subset of $\mathbf{R}^d$, $d\geq 2$. Let U be a finely open subset of Ω such that $U\Subset\Omega$, and let $x\in U$. The *finely harmonic measure* on U for x is the measure $\varepsilon_x^{\Omega\setminus U}$. We denote by $\mathrm{FH}_x(\Omega)$ the set of finely harmonic measures $\varepsilon_x^{\Omega\setminus V}$ such that $V\Subset\Omega$ is a fine domain containing x.

By the last proposition, we can see that the finely harmonic measure on a relatively compact open subset of Ω is nothing other than the classical harmonic measure.

Proposition 2.9. *Let Ω be a Greenian open subset of $\mathbf{R}^d$, $d\geq 2$, and $x\in\Omega$. Let U be a finely open subset of Ω with $x\in U$. Then,*

(i) *$\varepsilon_x^{\Omega\setminus U}$ is carried by $\partial_f U\cap\Omega$ and does not charge polar sets*;

(ii) *if U is finely connected, then the measures $\varepsilon_y^{\Omega\setminus U}$, $y\in U$, all have the same null sets.*

Proof. (i) See [Fu, p. 37].

(ii) Follows from Propositions 2.4, 2.6 and [Fu, p. 150]. □

We define an auxiliary class of measures for $\mathrm{FH}_x(\Omega)$. Like the set $\mathrm{SJ}_x(\Omega)$, it will play an important rôle later.

Definition 2.10. Let Ω be a Greenian open subset of $\mathbf{R}^d$, $d\geq 2$. For each $x\in\Omega$, we denote by $\mathrm{SFH}_x(\Omega)$ the set which contains the measures of the form ε_x^A, where $A\subset\Omega$.

For a finely open subset U of a Greenian open subset Ω, we adopt the same definitions as [Fu, Section 8] for *finely hyperharmonic*, *finely hypoharmonic* and *finely harmonic functions*. Note that all these functions are ν-measurable, $\nu\in\mathrm{FH}_x(\Omega)$, as a consequence of Proposition 2.5.

Proposition 2.11. ([Fu, Corollary of p. 86]) *Let Ω be a Greenian open subset of $\mathbf{R}^d$, $d\geq 2$. Let $u\in\mathrm{SH}^+(\Omega)$ and $A\subset\Omega$. Then, the function $\widehat{R}_u^A$ is finely harmonic on $\{x\in\Omega:\widehat{R}_u^A(x)<+\infty\}\setminus b(A)$.*

By a *semibounded potential*, we mean a potential for which the lower semicontinuous regularization of $\inf\{\widehat{R}_p^{A_\lambda}:\lambda>0\}$ equals 0, where $A_\lambda=\{x\in\Omega:p(x)>\lambda\}$ (see e.g. [Fu, Section 2]).

Proposition 2.12. ([Fu, Theorems 9.1 and 12.6]) *Let Ω be a Greenian subset of $\mathbf{R}^d$, $d\geq 2$. Let f be a finely hyperharmonic function on a finely open subset $U\subset\Omega$.*

(i) *If*

$$\operatorname*{fine-lim\,inf}_{\substack{y\to z\\ y\in U}} f(x)\geq 0$$

for quasi-every $z\in\partial_f U\cap\Omega$, *and if moreover there exists a semibounded potential* p *on* Ω *such that* $f\geq -p$ *in* U, *then* $f\geq 0$.

(ii) *If* U *is finely connected and* f *attains a global minimum on* U, *then* f *is constant.*

These results are respectively known as the *fine boundary minimum principle* and the *fine global minimum principle.* Note that if $U\Subset\Omega$ and f is lower bounded in U, then the semibounded potential always exists.

Proposition 2.13. *Let* Ω *be a Greenian open subset of* $\mathbf{R}^d$, $d\geq 2$, *and* $x\in\Omega$. *Let* F *be a relatively finely closed subset of* $\Omega\setminus\{x\}$. *Denote by* U *the finely connected component of* $\Omega\setminus F$ *which contains* x. *If* $U\Subset\Omega$, *then*

(i) $\varepsilon_x^F=\varepsilon_x^{\Omega\setminus U}$;

(ii) $\varepsilon_x^{\Omega\setminus U}(\Omega)=1$.

Proof. (i) Since

$$\int u\,d\varepsilon_x^F=\widehat{R}_x^F(x) \text{ and } \int u\,d\varepsilon_x^{\Omega\setminus U}=\widehat{R}_u^{\Omega\setminus U}(x),\quad u\in \mathrm{SH}^+(\Omega),$$

it suffices to show that

$$\widehat{R}_u^F(x)=\widehat{R}_u^{\Omega\setminus U}(x),\quad u\in \mathrm{SH}^+(\Omega)\cap C(\Omega).$$

Take $u\in\mathrm{SH}^+(\Omega)\cap C(\Omega)$. By Proposition 2.11, the function $\widehat{R}_u^F-\widehat{R}_u^{\Omega\setminus U}$ is finely harmonic in U. Note also that it is bounded on $\overline{U}$. Moreover,

$$\liminf_{\substack{y\to z\\ y\in U}}\big(\widehat{R}_u^F-\widehat{R}_u^{\Omega\setminus U}\big)(y)\geq \liminf_{\substack{y\to z\\ y\in U}}\widehat{R}_u^F(y)-u(z)\geq \widehat{R}_u^F(z)-u(z)=0$$

for quasi-every $z\in\partial_f U\cap\Omega$. By the fine boundary minimum principle, it follows that $\widehat{R}_u^F(x)\geq\widehat{R}_u^{\Omega\setminus U}(x)$. The reverse inequality can be shown in the same way.

(ii) It is needed to verify that $\widehat{R}_1^{\Omega\setminus U}(x)=1$ since

$$\varepsilon_x^{\Omega\setminus U}(\Omega)=\int 1\,d\varepsilon_x^{\Omega\setminus U}=\widehat{R}_1^{\Omega\setminus U}(x).$$

By Proposition 2.11, the function $\widehat{R}_1^{\Omega\setminus U}-1$ is finely harmonic on U. It is also bounded on $\overline{U}$ and

$$\liminf_{\substack{y\to z\\ y\in U}}\,(\widehat{R}_1^{\Omega\setminus U}(y)-1)=0$$

for quasi-every $z\in\partial_f U\cap\Omega$. The fine boundary minimum principle then implies the desired equality. □

Proposition 2.14. *Let Ω_1 and Ω_2 be Greenian open subsets of $\mathbf{R}^d$, $d\ge 2$, and $x\in\Omega_1\cap\Omega_2$. Let U be a finely open set such that $x\in U\Subset\Omega_1\cap\Omega_2$. If ${}^{\Omega_1}\varepsilon_x^{\Omega_1\setminus U}$ and ${}^{\Omega_2}\varepsilon_x^{\Omega_2\setminus U}$ denote the finely harmonic measures for U at x with respect to Ω_1 and Ω_2, respectively, then*

$$ {}^{\Omega_1}\varepsilon_x^{\Omega_1\setminus U} = {}^{\Omega_2}\varepsilon_x^{\Omega_2\setminus U}. $$

Proof. Set $\Omega'=\Omega_1\cap\Omega_2$. By applying a reasoning similar to the one in Proposition 2.13, we can show that ${}^{\Omega_1}\varepsilon_x^{\Omega_1\setminus U} = {}^{\Omega'}\varepsilon_x^{\Omega'\setminus U} = {}^{\Omega_2}\varepsilon_x^{\Omega_2\setminus U}$. $\square$

Proposition 2.15. ([Fu, Theorem 14.1]) *Let Ω be a Greenian open subset of $\mathbf{R}^d$, $d\ge 2$. Let U be a finely open subset of Ω and $f\colon b(\partial_f U)\to(-\infty,+\infty)$ be a finely continuous function. If $|f|\le p$ on $b(\partial_f U)$ for some finite and semibounded potential p on Ω, then the function*

$$ u(y) := \int f\, d\varepsilon_y^{\Omega\setminus U}, \quad y\in\widetilde{U}, $$

is a finely continuous extension of f from $b(\partial_f U)$ to $\widetilde{U}$ which is also finely harmonic in U, and such that $|u|\le p$ on $\widetilde{U}$.

The function u is unique and is called the *proper fine Dirichlet solution* on U with boundary data f. Note that if $U\Subset\Omega$ and f is bounded on $b(\partial_f U)$, then the solution always exists and is bounded.

Definition 2.16. ([Fu, p. 173]) Let Ω be a Greenian open subset of $\mathbf{R}^d$, $d\ge 2$. Let $f:\partial_f U\to[-\infty,+\infty]$ be a function on the fine boundary of an open subset $U\Subset\Omega$.

(i) A *fine superfunction* u for f (with respect to U) is a finely hyperharmonic function defined on U and bounded below there, such that

$$ \operatorname*{fine-lim\,inf}_{\substack{y\to z\\ y\in U}} u(y) \ge f(z), \quad z\in\partial_f U. $$

We denote by $\overline{H}_f^U$ the infimum of the fine superfunctions for f.

(ii) The *fine subfunctions* and $\underline{H}_f^U U$ are defined by 'inverting' the preceding definitions.

(iii) In the case where $\overline{H}_f^U$ and $\underline{H}_f^U$ coincide and are finite valued on U, we denote their common value by H_f^U and we say that f is *finely resolutive.*

Proposition 2.17. ([Fu, Theorem 14.6]) *Let Ω be a Greenian open subset of $\mathbf{R}^d$, $d\ge 2$. Let $f\colon\partial_f U\to[-\infty,+\infty]$ be a function on the fine boundary of an open subset $U\Subset\Omega$. If f is $\varepsilon_x^{\Omega\setminus U}$-measurable for each $x\in U$ and if the following* *integral exists, then*

$$\underline{H}_f^U(x) = \overline{H}_f^U(x) = \int f \, d\varepsilon_x^{\Omega \setminus U}.$$

Moreover, f is finely resolutive if and only if it is integrable with respect to $\varepsilon_x^{\Omega\setminus U}$, $x \in U$. In the affirmative case, the function H_f^U is finely harmonic and

$$H_f^U(x) = \int f \, d\varepsilon_x^{\Omega \setminus U}, \quad x \in U.$$

This result is usually called the *generalized fine Dirichlet problem* for U with boundary data f. Note that this coincides with the classical case when U is open in the Euclidean topology.

Proposition 2.18. (Generalized fine gluing principle) *Let Ω be a Greenian open subset of $\mathbf{R}^d$, $d \geq 2$. Let u and v be finely hyperharmonic functions on the finely open subset U and V of Ω, respectively. Suppose also that $V \subset U$. If*

$$\operatorname*{fine-lim\,inf}_{\substack{y \to x \\ y \in V}} v(y) \geq u(x)$$

for quasi-every $x \in \partial_f V \cap U$ and if v is lower bounded in some deleted fine neighbourhood of each point of the exceptional set where v does not satisfy the fine lower limit condition, then the function

$$w = \begin{cases} \min\{u, v\} & \text{on } V, \\ u & \text{on } U \setminus V, \end{cases}$$

is finely hyperharmonic in U.

Proof. This follows from [Fu, Theorem 9.14 and Lemma 10.1]. $\square$

3. First steps toward Theorem 1.2

The aim of the next four sections is to prove Theorem 1.2. To achieve this, we shall proceed in the way described below.

Steps toward Theorem 1.2

Let Ω be a Greenian open subset of $\mathbf{R}^d$, $d \geq 2$, and $x \in \Omega$. We shall proceed by showing the following relations:

$$\text{(2)}\qquad \mathrm{SFH}_x(\Omega) \subset \mathrm{SJ}_x(\Omega),$$

$$\text{(3)}\qquad \mathrm{FH}_x(\Omega) \subset J_x(\Omega),$$

$$\text{(4)}\qquad \mathrm{SFH}_x(\Omega) \subset \mathrm{ext}(\mathrm{SJ}_x(\Omega)),$$

$$\text{(5)}\qquad \mathrm{FH}_x(\Omega) \subset \mathrm{ext}(J_x(\Omega)),$$

$$\text{(6)}\qquad \mathrm{SFH}_x(\Omega) = \mathrm{ext}(\mathrm{SJ}_x(\Omega)),$$

$$\text{(7)}\qquad \mathrm{FH}_x(\Omega) \cup \{\varepsilon_x\} = \mathrm{SFH}_x(\Omega) \cap J_x(\Omega),$$

$$\text{(8)}\qquad \mathrm{ext}(\mathrm{SJ}_x(\Omega)) \cap J_x(\Omega) = \mathrm{ext}(J_x(\Omega)).$$

The first four equations are the content of this section. In the next section, we prove (6) with the aid of a paper of Mokobodzki [Mo]. Equation (7) is shown in Section 5. Most of the tools from fine harmonicity are used there. Finally, the last equation is proved in Section 6. It is mostly based on an approximation result from Gardiner [Gd, Lemma 6.2].

Let us now prove the first two equations.

Proof of (2). Take $\varepsilon_x^A \in \mathrm{SFH}_x(\Omega)$, where $A \subset \Omega$. If $u \in \mathrm{SH}^+(\Omega)$, then

$$\int u \, d\varepsilon_x^A = \widehat{R}_u^A(x) \le u(x).$$

This shows that $\varepsilon_x^A \in \mathrm{SJ}_x(\Omega)$. □

Proof of (3). Take $\varepsilon_x^{\Omega \setminus V} \in \mathrm{FH}_x(\Omega)$, where $V \Subset \Omega$ is a fine domain containing x. The support of $\varepsilon_x^{\Omega \setminus V}$ is relatively compact in Ω since this measure is carried by $\partial_f V$. Moreover, Proposition 2.13 implies that $\varepsilon_x^{\Omega \setminus V}$ is a probability measure.

It remains to prove that $\int u \, d\varepsilon_x^{\Omega \setminus V} \le u(x)$, $u \in \mathrm{SH}(\Omega)$. Take $u \in \mathrm{SH}(\Omega)$. Since $V \Subset \Omega$ and u is lower semicontinuous, there exists a sequence of continuous functions $\{f_n\}_{n=1}^\infty$ on $\mathbf{R}^d$, increasing pointwise to u on $\overline{V}$ (see e.g. [AG, Lemma 3.2.1]). Consider

$$h_n(y) = \int f_n \, d\varepsilon_y^{\Omega \setminus V}, \quad y \in \widetilde{V},$$

the proper fine Dirichlet solution on V with boundary data $f_n|_{b(\partial_f V)}$. The function $u - h_n$ is then finely hyperharmonic and lower bounded on V (see the remark after Proposition 2.15). Moreover,

$$\operatorname*{fine-lim\,inf}_{\substack{y \to z \\ y \in V}} (u - h_n)(y) \ge u(z) - h_n(z), \quad z \in \partial_f V.$$

The right-hand side of this expression equals $u(z)-f_n(z)$ for quasi-every $z\in\partial_f V$, and $u\geq f_n$ on $\partial_f V$. By the fine boundary minimum principle, it follows that $u\geq h_n$ on V. Hence,

$$\int f_n\,d\varepsilon_x^{\Omega\setminus V}=h_n(x)\leq u(x).$$

Letting $n\to\infty$, we get the desired result. □

The last part of this proposition is inspired by [CR2, Proposition 3.2], where the same result is shown for $H_x(\Omega)$ instead of $\mathrm{FH}_x(\Omega)$. Before proceeding to (4) and (5), let us give a preliminary lemma.

Lemma 3.1. *Let Ω be a Greenian open subset of $\mathbf{R}^d$, $d\geq 2$, and $x\in\Omega$. If $\mu\in\mathrm{SJ}_x(\Omega)$ is carried by a base B of Ω, then*

$$\int u\,d\mu\leq\int u\,d\varepsilon_x^B,\quad u\in\mathrm{SH}^+(\Omega).$$

Proof. Let B a base of Ω and $\mu\in\mathrm{SJ}_x(\Omega)$ carried by B. Proposition 2.6(b) implies that $\mu=\mu^B$. Hence,

$$\int u\,d\mu=\int u\,d\mu^B=\int \widehat{R}_u^B\,d\mu\leq\widehat{R}_u^B(x)=\int u\,d\varepsilon_x^B,\quad u\in\mathrm{SH}^+(\Omega).\quad\square$$

Proof of (4). Let $\varepsilon_x^A\in\mathrm{SFH}_x(\Omega)$, where $A\subset\Omega$. Let $\mu_1,\mu_2\in\mathrm{SJ}_x(\Omega)$ and $\alpha\in(0,1)$ be such that

$$\varepsilon_x^A=\alpha\mu_1+(1-\alpha)\mu_2.$$

This relation implies that $\operatorname{supp}\mu_j\subset\operatorname{supp}\varepsilon_x^A$, $j=1,2$. So, by Proposition 2.6(a), we see that μ_1 and μ_2 are carried by $b(A)$. From Lemma 3.1 we get that

$$\int u\,d\mu_j\leq\int u\,d\varepsilon_x^{b(A)}=\int u\,d\varepsilon_x^A,\quad u\in\mathrm{SH}^+(\Omega),\ j=1,2.$$

This implies, by considering the equality $\varepsilon_x^A=\alpha\mu_1+(1-\alpha)\mu_2$, that

$$\int u\,d\mu_1=\int u\,d\mu_2=\int u\,d\varepsilon_x^A,\quad u\in\mathrm{SH}^+(\Omega).$$

We then conclude that $\mu_1=\mu_2=\varepsilon_x^A$. □

Proof of (5). This is similar to the proof of (4). □

4. Mokobodzki's paper

The context in which Mokobodzki's paper [Mo] takes place is much more general than the present one. It is not obvious at first sight that it can be applied to ours, but a closer look reveals that it is indeed suitable for our purpose. Applied in the right way, one can see that the main result implies (6). Let us describe the abstract setting of this paper.

Recall that $C(\Omega)$ is the set of continuous real-valued functions defined on Ω. Denote by $C^+(\Omega)$ the set of nonnegative functions of $C(\Omega)$.

Definition 4.1. Let Ω be an open subset of $\mathbf{R}^d$, $d\geq 2$. A subset C of $C^+(\Omega)$ is called a *convex cone* if it satisfies the following properties:

(i) $f,g\in C\Rightarrow f+g\in C$;

(ii) $f\in C$ and $\lambda\geq 0\Rightarrow \lambda f\in C$.

On $C(\Omega)$ we can define a new order relation with respect to a convex cone C. For $f,g\in C(\Omega)$, we will say that f is *specifically smaller than* g (with respect to C), and write $f\preceq g$, if there exists $h\in C$ such that $f+h=g$. This is the *specific order* on $C(\Omega)$ relative to C.

Definition 4.2. Let Ω be an open subset of $\mathbf{R}^d$, $d\geq 2$. Let C be a convex cone in $C^+(\Omega)$.

(i) C is said to be *finitely stable from below* if the minimum of each finite family of functions in C is also a function in C.

(ii) C is said to be *linearly separating* if, for each pair of points $x,y\in\Omega$, $x\neq y$, there exist two functions $f,g\in C$ such that

$$f(x)g(y)\neq f(y)g(x).$$

(iii) C is said to be *adapted* if, for each $f\in C$, there exists $g\in C$ such that, given $\varepsilon>0$, the set $\{x\in\Omega:f(x)>\varepsilon g(x)\}$ is relatively compact in Ω.

(iv) C is said to satisfy property (P) if, given $f,g_1,g_2\in C$ with $f\preceq g_1+g_2$, there exist $f_1,f_2\in C$ such that $f=f_1+f_2$, $f_1\preceq g_1$ and $f_2\preceq g_2$.

(v) We say that a Radon measure μ is C-*integrable* if

$$\int f\,d\mu<+\infty,\quad f\in C.$$

We now forget about the previous definition of balayage and define two other ones: *the balayage of a measure on a set*, and *the balayage of a measure with respect to a convex cone.*

Definition 4.3. Let Ω be an open subset of $\mathbf{R}^d$, $d\geq 2$. Let $C\subset C^+(\Omega)$ be a convex cone and μ be a C-integrable measure.

A C-integrable measure ν is said to be a *balayage of μ with respect to C* if

$$\int f\,d\nu \leq \int f\,d\mu, \quad f\in C.$$

The set of measures which are balayage of μ with respect to C is denoted by A_μ.

For the second definition, we need to work a little more.

Definition 4.4. Let Ω be an open subset of $\mathbf{R}^d$, $d\geq 2$. Let $C\subset C^+(\Omega)$ be a convex cone and $K\subset\Omega$ a compact set. For $f\in C$, we define

$$S_f^K = \inf\{g\in C : g\geq f \text{ on } K\}.$$

Proposition 4.5. ([Mo, Corollary 5]) *Let Ω be an open subset of $\mathbf{R}^d$, $d\geq 2$. Let $C\subset C^+(\Omega)$ be a convex cone satisfying properties* (i)–(iv) *of Definition* 4.2*, and let μ be a C-integrable measure.*

(i) *If K is a compact subset of Ω, then there exists a unique measure μ^K which is C-integrable, carried by K, and such that*

$$\int f\,d\mu^K = \int S_f^K\,d\mu, \quad f\in C.$$

(ii) *If A is a Borel subset of Ω, then there exists a unique measure μ^A which is C-integrable, and such that*

$$\int f\,d\mu^A = \sup\biggl\{\int f\,d\mu^K : K \textit{ is compact and } K\subset A\biggr\}, \quad f\in C.$$

Here is the second definition of the balayage.

Definition 4.6. Let Ω be an open subset of $\mathbf{R}^d$, $d\geq 2$. Let $C\subset C^+(\Omega)$ be a convex cone satisfying properties (i)–(iv) of Definition 4.2, and μ be a C-integrable measure. Let A be a Borel subset of Ω. The measure μ^A defined in the last proposition is called the *balayage of μ on A.*

As explained in the introduction of [Mo], the aim of the paper was to relate these two definitions of the balayage. The main result is given in [Mo, Propositions 8 and 9]. We now present a combination of them both.

Proposition 4.7. ([Mo, Propositions 8 and 9]) *Let Ω be an open subset of $\mathbf{R}^d$, $d\geq 2$. Let $C\subset C^+(\Omega)$ be a convex cone satisfying properties* (i)–(iv) *of Definition* 4.2*, and μ be a C-integrable measure. Then* $\operatorname{ext}(A_\mu)$ *coincides with the set of measures of the form μ^B, where B is a Borel subset of Ω.*

For our purpose, consider the convex cone of continuous real potentials on Ω, for Ω Greenian. Denote this cone by P^c.

Proposition 4.8. *Let Ω be a Greenian open subset of $\mathbf{R}^d$, $d\geq 2$. Then P^c satisfies properties* (i)–(iv) *of Definition* 4.2.

Proof. (i) P^c is clearly finitely stable from below.

(ii) The fact that P^c is linearly separating is a consequence of [CC, Proposition 2.3.2].

(iii) The fact that P^c is adapted is a consequence of [CC, Proposition 2.2.4].

(iv) As on $C(\Omega)$, we can consider the specific order on $\mathrm{SH}(\Omega)$ relative to P^c. Let $p,q_1,q_2\in P^c$ be such that $p\preceq q_1+q_2$. By [Fu, p. 16], there exist $p_1,p_2\in \mathrm{SH}^+(\Omega)$ such that $p=p_1+p_2$, $p_1\preceq q_1$ and $p_2\preceq q_2$. Both p_1 and p_2 are clearly potentials. Moreover, since p_1 and p_2 are lower semicontinuous and their sum is continuous, they are continuous. Hence, P^c satisfies property (P). $\square$

Given $x\in\Omega$, one can easily verify that the set A_{ε_x} corresponds to $\mathrm{SJ}_x(\Omega)$. So, what we would like to get is a relation between the usual notion of balayage introduced in Section 2 and the second one introduced in the present section. Since there is a possibility of confusion, let us fix the following notation. Denote by ε_x^A the usual notion of the balayage of ε_x on A, and by δ_x^A the one encountered in Definition 4.6.

Here is the relation between ε_x^A and δ_x^A.

Proposition 4.9. *Let Ω be a Greenian open subset of $\mathbf{R}^d$, $d\geq 2$. If B is a Borel subset of Ω, then*

$$\delta_x^B=\begin{cases} \varepsilon_x, & \text{if } x\in B,\\ \varepsilon_x^B, & \text{if } x\notin B.\end{cases}$$

In particular, if B is a base, then $\delta_x^B=\varepsilon_x^B$, $x\in\Omega$.

Before continuing with the proof of this proposition, let us state a preliminary lemma.

Lemma 4.10. *Let Ω be a Greenian subset of $\mathbf{R}^d$, $d\geq 2$. If $p\in P^c$ and K is a compact subset of Ω, then $\widehat{R}_p^K=\widehat{S}_p^K$.*

Proof. Clearly, $\widehat{R}_p^K\leq\widehat{S}_p^K$. Let us show the opposite inequality. The set $\{q\in P^c : q\geq p \text{ on } K\}$ is a saturated family on $\Omega\setminus K$ (see e.g. [AG, p. 79]). Hence, the function S_p^K is harmonic and equal to $\widehat{S}_p^K$ on $\Omega\setminus K$. Let $u\in\mathrm{SH}^+(\Omega)$ be such that $u \geq p$ on K. By applying the domination principle [AG, Theorem 5.1.11] with u and $\widehat{S}_p^K$, we see that $u \geq \widehat{S}_p^K$ on Ω. The conclusion follows by taking the infimum over $u \in \mathrm{SH}^+(\Omega)$ such that $u \geq p$ on K. $\square$

Proof of Proposition 4.9. First, suppose that $x \notin B$. If K is a compact subset of B, then

$$\int p \, d\delta_x^K = S_p^K(x) = \widehat{S}_p^K(x) = \widehat{R}_p^K(x), \quad p \in P^c.$$

This and the fact that $A \mapsto \widehat{R}_p^A$, $A \subset \Omega$ is a capacity (see e.g. [CC, p. 123]) imply that

$$\int p \, d\delta_x^B = \sup\bigl\{\widehat{R}_p^K(x) : K \text{ is compact and } K \subset B\bigr\} = \widehat{R}_p^B(x), \quad p \in P^c.$$

Hence, $\int p \, d\delta_x^B = \widehat{R}_p^B(x)$, $p \in P^c$, and this shows that $\varepsilon_x^B = \delta_x^B$.

Now let us treat the case where $x \in B$. If K is a compact subset of B and if $x \in K$, then

$$\int p \, d\delta_x^K = S_p^K(x) = p(x), \quad p \in P^c.$$

So,

$$\int p \, d\delta_x^B = \sup\biggl\{\int p \, d\delta_x^K : K \text{ is compact and } K \subset B\biggr\} = p(x) = \int p \, d\varepsilon_x, \quad p \in P^c.$$

It follows that $\delta_x^B = \varepsilon_x$.

Finally, the last remark follows from the fact that $\varepsilon_x^B = \varepsilon_x$ if B is a base and $x \in B$. $\square$

Proof of (6). By (4), we already know that $\mathrm{SFH}_x(\Omega) \subset \mathrm{ext}(\mathrm{SJ}_x(\Omega))$. Let therefore $\mu \in \mathrm{ext}(\mathrm{SJ}_x(\Omega))$. By Proposition 4.7, there exists a Borel set B such that $\mu = \delta_x^B$. If $x \in B$, then $\delta_x^B = \varepsilon_x$. On the other hand, if $x \notin B$, then $\delta_x^B = \varepsilon_x^B$. In both cases, $\mu \in \mathrm{SFH}_x(\Omega)$. $\square$

5. Relation between $\mathbf{SFH}_{\boldsymbol{x}}(\boldsymbol{\Omega})$ and $\mathbf{FH}_{\boldsymbol{x}}(\boldsymbol{\Omega})$

This section is devoted to proving (7). The main result on which it depends is Proposition 5.2. First, let us give a preliminary lemma.

Lemma 5.1. *Let Ω be a Greenian open subset of $\mathbf{R}^d$, $d \geq 2$. Let F be a finely closed subset of Ω such that $F \Subset \Omega$.*

(i) $\widehat{R}_1^F\equiv 1$ *on each bounded finely connected component* V *of* $\mathbf{R}^d\setminus F$ *entirely contained in* Ω.

(ii) *If each bounded finely connected component of* $\mathbf{R}^d\setminus F$ *is entirely contained in* Ω, *then* $\widehat{R}_1^F<1$ *on* $U\cap\Omega$, *where* U *is the unbounded component of* $\mathbf{R}^d\setminus F$.

Proof. (i) Let V be a bounded finely connected component of $\mathbf{R}^d\setminus F$ entirely contained in Ω. The function $\widehat{R}_1^F-1$ is finely harmonic on V, lower bounded on $\overline{V}$ and

$$\liminf_{\substack{y\to z\\ y\in V}}(\widehat{R}_1^F(y)-1)=0$$

for quasi-every $z\in\partial_f V$. By the fine boundary minimum principle, it then follows that $\widehat{R}_1^F\equiv 1$ on V.

(ii) For a contradiction, suppose that the hypothesis of (ii) is satisfied and $\widehat{R}_1^F(x_0)=1$ for a certain $x_0\in U\cap\Omega$. By the global fine minimum principal, we see that $\widehat{R}_1^F\equiv 1$ on the fine component of $U\cap\Omega$ which contains x_0. This and (i) would imply that $\widehat{R}_1^F\equiv 1$ on the component of Ω which contains x_0. This is impossible since $\widehat{R}_1^F$ is a potential (see e.g. [AG, p. 134]). $\square$

Proposition 5.2. *Let* Ω *be a Greenian open subset of* $\mathbf{R}^d$, $d\geq 2$. *Let* $\mu\in J_x(\Omega)$ *be a Jensen measure carried by a finely closed set* F *such that* $F\Subset\Omega$. *If* V *is either a bounded finely connected component of* $\mathbf{R}^d\setminus F$ *such that* $V\setminus\Omega\neq\varnothing$, *or the unbounded finely connected component of* $\mathbf{R}^d\setminus F$, *then* $x\notin V\cap\Omega$.

Proof. For a contradiction, first suppose that there exists a bounded finely connected component V of $\mathbf{R}^d\setminus F$ such that $V\setminus\Omega\neq\varnothing$ and $x\in V\cap\Omega$. Let $x_0\in V\setminus\Omega$. Suppose temporarily that Ω is bounded. Under this hypothesis, the set $\Omega':=\Omega\cup V$ is Greenian. The function U_{x_0} is finely harmonic on $\Omega'\setminus\{x_0\}$ and finely hyperharmonic on Ω'. Since U_{x_0} is bounded on $b(\partial_f V)$ and $V\Subset\Omega'$, the proper fine Dirichlet solution on $b(\partial_f V)$ with boundary data $U_{x_0}|_{b(\partial_f V)}$ exists and is bounded. Let us denote it by f. The function $U_{x_0}-f$ is finely hyperharmonic and lower bounded on V and

$$\operatorname*{fine-lim\,inf}_{\substack{y\to z\\ y\in V}}(U_{x_0}-f)(y)=0$$

for quasi-every $z\in\partial_f V$. By the fine boundary minimum principle, it follows that $U_{x_0}\geq f$. Moreover, by applying the global fine minimum principal, we see that $U_{x_0}>f$. Let us consider now the function

$$u(y)=\begin{cases}(f-U_{x_0})(y), & \text{if } y\in V\cap\Omega,\\ 0, & \text{if } y\in\Omega\setminus V.\end{cases}$$

The generalized fine gluing principle implies first that u is finely hyperharmonic on Ω and [Fu, Theorem 9.8] then implies that it is superharmonic. Hence, we come to the contradiction

$$\int u\,d\mu = 0 \quad \text{and} \quad u(x) < 0.$$

We deduce the same result for Ω unbounded.

Suppose now that V is the unbounded component of $\mathbf{R}^d \backslash F$, that $x \in V \cap \Omega$, and that each bounded finely connected component of $\mathbf{R}^d \backslash F$ is entirely contained in Ω. Part (ii) of the last lemma implies then $\widehat{R}_1^F(x) < 1$. This is impossible since it would imply that

$$1 = \int 1\,d\mu = \int \widehat{R}_1^F\,d\mu \leq \widehat{R}_1^F(x) < 1,$$

where the second equality follows from the fact that μ is carried by F and does not charge the polar set not containing x (property (ii) of $\mathrm{SJ}_x(\Omega)$).

Finally, suppose that V is the unbounded component of $\mathbf{R}^d \backslash F$, that $x \in V \cap \Omega$, and that there exists a bounded finely connected component W of $\mathbf{R}^d \backslash F$ such that $W \backslash \Omega \neq \varnothing$. Let $x_0 \in W \backslash \Omega$ and let B be a closed ball with center x_0 which is entirely contained in W° (the Euclidean interior of W).

Denote by F' and V' the inverse of F and V, respectively, with respect to the sphere ∂B (see e.g. [AG, Section 1.6]). Let f be the proper fine Dirichlet solution on $b(\partial_f V')$ with boundary data $U_{x_0}|_{b(\partial_f V')}$. As before, $U_{x_0} - f > 0$ on V'. By the generalized fine gluing principle, the function

$$u(y) = \begin{cases} (f - U_{x_0})(y), & \text{if } y \in V', \\ 0, & \text{if } y \in B^\circ \backslash V', \end{cases}$$

is finely hyperharmonic on B°. By [Fu, Theorem 9.8], it follows that u is superharmonic on B°. Let us denote by v the Kelvin transform of u with respect to ∂B (see again [AG, Section 1.6]), and extend v to all of $\mathbf{R}^d \backslash \{x_0\}$ by defining $v = 0$ on $\overline{B} \backslash \{x_0\}$. Then, $v \equiv 0$ on F, and $v(x) < 0$. This is all we need for a contradiction. $\square$

Proof of (7). We already know that $\mathrm{FH}_x(\Omega) \cup \{\varepsilon_x\} \subset \mathrm{SFH}_x(\Omega) \cap J_x(\Omega)$. Let $\mu \in \mathrm{SFH}_x(\Omega) \cap J_x(\Omega)$, and write $\mu = \varepsilon_x^B$, where B is a base (by part (a) of Proposition 2.6). If $x \in B$, then $\varepsilon_x^B = \varepsilon_x$. On the other hand, suppose that $x \notin B$. Since ε_x^B is carried by B, and $\mu \in J_x(\Omega)$, we deduce that B is relatively compact in Ω. By Proposition 5.2, it follows that x is contained in a bounded finely connected component V of $\mathbf{R}^d \backslash B$ entirely contained in Ω. Part (i) of Proposition 2.13 implies that $\varepsilon_x^B = \varepsilon_x^{\Omega \backslash V}$. In both cases, $\mu \in \mathrm{FH}_x(\Omega) \cup \{\varepsilon_x\}$. $\square$

6. Proof of Theorem 1.2

The purpose of this section is to prove (8). At the same time, it will conclude the proof of Theorem 1.2 for Greenian open sets.

Notation. Given a compact subset K of Ω, we denote by $\omega_1, ..., \omega_m$ the bounded components of $\mathbf{R}^d \backslash K$ which are not contained in Ω and by ω_{m+1} the unbounded component of $\mathbf{R}^d \backslash K$.

The next proposition is a modification of a lemma from [Gd]. Let us first state this lemma.

Lemma 6.1. ([Gd, Lemma 6.2]) *Let Ω be an open subset of $\mathbf{R}^d$, $d \geq 2$, and K be a compact subset of Ω. Let $y_k \in \omega_k$, $k=1, ..., m$. If $u \in \mathrm{SH}(\Omega)$, then there exists $v \in \mathrm{SH}(\mathbf{R}^d)$ and $c \geq 0$ such that*

$$u(x) = v(x) - c \sum_{k=1}^{m} U_{y_k}(x)$$

on an open set which contains K.

Proposition 6.2. *Let Ω be an open subset of $\mathbf{R}^d$, $d \geq 2$. Let K be a compact subset of Ω. Suppose that $\omega_i \backslash \Omega$ is non-polar for $i=1, ..., m$, if $d \geq 3$, and for $i=1, ..., m+1$, if $d=2$. If $u \in \mathrm{SH}(\Omega)$, then there exists a lower bounded function $v \in \mathrm{SH}(\Omega)$ such that $v=u$ on K.*

Proof. Let W_i be a regular open subset of ω_i such that $\omega_i \backslash \Omega \subset W_i$ with ∂W_i compact, $i=1, ..., m+1$. Set $U_i := W_i \cap \Omega$ and $U := \bigcup_{i=1}^{m+1} U_i$. Define a harmonic function h on U as follows. On U_i, the function h is the proper Dirichlet solution on U_i with boundary data 0 on ∂W_i, and 1 on $\partial \Omega \cap \omega_i$, $i=1..., m$ and $(\partial \Omega \cap \omega_{m+1}) \cup \{\infty\}$. The hypothesis of the statement and the regularity of the sets W_i imply that $h > 0$ and $h \to 0$ on $\bigcup_{i=1}^{m+1} \partial W_i$.

Let V_i be a regular open subset of Ω such that $\partial W_i \subset V_i$ and $\overline{V}_i \Subset \Omega \cap \omega_i$. Set $V = \bigcup_{i=1}^{m+1} V_i$ and

$$w(x) = \begin{cases} H_u^V(x), & \text{if } x \in V, \\ u(x), & \text{if } x \in \Omega \backslash V. \end{cases}$$

By [AG, Theorem 6.6.4] and the classical gluing principal, it follows that $w \in \mathrm{SH}(\Omega) \cap H(V)$. If

$$M > \sup\{w(x) : x \in \partial U\} \quad \text{and} \quad c > \sup\left\{ \frac{M - w(x)}{h(x)} : x \in U \cap \partial V \right\}$$

then the function

$$v(x) := \begin{cases} w(x) & \text{on } \Omega \setminus U, \\ \min\{w(x), M - ch(x)\} & \text{on } U \cap V, \\ M - ch(x) & \text{on } U \setminus V, \end{cases}$$

is superharmonic and lower bounded on Ω, and $v = u$ on K. $\square$

Remark. The converse of the last proposition is false. Consider for example $\Omega := B(0,1) \setminus \{0\}$ and $K := S\big(0, \frac{1}{2}\big)$. Of course, the set $\omega_1 \setminus \Omega$ is polar. On the other hand, if $u \in \mathrm{SH}(\Omega)$, then Lemma 6.1 implies the existence of a function $v \in \mathrm{SH}(\mathbf{R}^d)$ and a constant $c \geq 0$ such that $u = v - cU_0$ on K. Set

$$w = v - cU_0\big(\big(\tfrac{1}{2}, 0, ..., 0\big)\big).$$

The function w is lower bounded and superharmonic on Ω. Moreover, $u = w$ on $S\big(0, \frac{1}{2}\big)$.

Proposition 6.3. *Let Ω be a Greenian open subset of $\mathbf{R}^d$, $d \geq 2$. Then, for each $x \in \Omega$, the following statements are equivalent*:

(a) *The set $\{\mu \in \mathrm{SJ}_x(\Omega) : \operatorname{supp} \mu \Subset \Omega$ and $\mu(\Omega) = 1\} = J_x(\Omega)$;*

(b) *For each compact K contained in the same component of Ω as x, the sets $\omega_i \setminus \Omega$ are non-polar for $i = 1, ..., m$, if $d \geq 3$, and for $i = 1, ..., m+1$, if $d = 2$;*

(c) *For each compact K contained in the same component of Ω as x and each function $u \in \mathrm{SH}(\Omega)$, there exists a lower bounded function $v \in \mathrm{SH}(\Omega)$ such that $v = u$ on K.*

Note that if Ω is regular, then the previous statements are necessarily fulfilled.

Proof. (b)$\Rightarrow$(c) This is a direct consequence of Proposition 6.2.

(c)$\Rightarrow$(a) Let $\mu \in \{\mu \in \mathrm{SJ}_x(\Omega) : \operatorname{supp} \mu \Subset \Omega$ and $\mu(\Omega) = 1\}$ and set $K = \operatorname{supp} \mu \cup \{x\}$. The set K is necessarily contained in the same component of Ω as x since $\mu \in \mathrm{SJ}_x(\Omega)$. Let $u \in \mathrm{SH}(\Omega)$. If (c) is fulfilled, then there exists a lower bounded superharmonic function on Ω such that $u = v$ on K. Choose a number $\alpha > 0$ such that $v + \alpha \geq 0$. Then

$$\int u \, d\mu = \int v \, d\mu = \int (v + \alpha) \, d\mu - \alpha \leq v(x) = u(x).$$

This shows that $\mu \in J_x(\Omega)$. The reverse inclusion is obvious.

(a)$\Rightarrow$(b) For a contradiction, suppose there exists a compact set K contained in the same component of Ω as x, and such that $\omega_i \setminus \Omega$ is polar for a certain $1 \leq i \leq m$.

Modify K in such a way that $x\in\omega_i\cap\Omega$. The measure ε_x^K does not belong to $J_x(\Omega)$, by Proposition 5.2. The function $\widehat{R}_1^K$ is harmonic on $\Omega\setminus K$. By the removable singularities principle, it possesses an extension to ω_i. The minimum principle then implies that $\widehat{R}_1^K(x)=1$, that is $\varepsilon_x^K(\Omega)=1$, which brings us to a contradiction.

Suppose, again for a contradiction, that $d=2$ and $\omega_{m+1}\setminus\Omega$ is polar. Modify K in such a way that $x\in\omega_{m+1}\cap\Omega$. The measure ε_x^K does not belong to $J_x(\Omega)$, by Proposition 5.2. On the other hand, [AG, Theorem 5.2.6] implies that $\widehat{R}_1^K(x)=1$, which brings us again to a contradiction. $\square$

Proof of (8)*, and at the same time of Theorem* 1.2 *for Greenian open sets.* We already know that $\operatorname{ext}(\mathrm{SJ}_x(\Omega))\cap J_x(\Omega)\subset\operatorname{ext}(J_x(\Omega))$. Let us prove the reverse inclusion. Suppose first that Ω and x satisfy Proposition 6.3(b). This assures us that $\{\mu\in\mathrm{SJ}_x(\Omega):\operatorname{supp}\mu\Subset\Omega \text{ and } \mu(\Omega)=1\}=J_x(\Omega)$. Let $\mu\in\operatorname{ext}(J_x(\Omega))$ and write

$$\mu=\alpha\mu_1+(1-\alpha)\mu_2,$$

where $\mu_1,\mu_2\in\mathrm{SJ}_x(\Omega)$ and $0<\alpha<1$. This implies that $\operatorname{supp}\mu_i\Subset\Omega$ and $\mu_i(\Omega)=1$, $i=1,2$. By the last proposition, it follows that $\mu=\mu_1=\mu_2$. This proves (8) and Theorem 1.2 for Ω and x as mentioned above.

We next remove the restriction on Ω and x. Let $\mu\in\operatorname{ext}(J_x(\Omega))$, and set $K=\operatorname{supp}\mu\cup\{x\}$. There exists a regular open set Ω' with $K\subset\Omega'\subset\Omega$, and such that each bounded component of $\mathbf{R}^d\setminus\Omega'$ meets $\mathbf{R}^d\setminus\Omega$. By [CR2, Proposition 2.1], we get $\mu\in J_x(\Omega')$, and it is not hard to see that $\mu\in\operatorname{ext}(J_x(\Omega'))$. From this and the first part, we deduce that $\mu=\varepsilon_x$ or $\mu=\delta_x^{\Omega'\setminus V}$, where V is a fine domain of Ω' which contains x and is such that $V\Subset\Omega'$, and $\delta_x^{\Omega'\setminus V}$ represents the finely harmonic measure on V for x with respect to Ω'. In the former case, the conclusion follows trivially. In the latter case, we get from Proposition 2.14 that $\delta_x^{\Omega'\setminus V}=\varepsilon_x^{\Omega\setminus V}$, and this concludes the proof. $\square$

7. From Greenian to general open sets

Up to now, we successfully proved Theorem 1.2 for Greenian open sets. This seems quite satisfactory since we only defined finely harmonic measures with respect to these sets and, in fact Definition 2.8 does not make sense anymore for general open sets. In this section, we give a wider version of Definition 2.8 and prove Theorem 1.2 for general open sets. Next, we prove Theorem 1.3.

Definition 7.1. Let Ω be an open subset of $\mathbf{R}^d$, $d\geq2$, and $x\in\Omega$. Let $U\Subset\Omega$ be a finely open set which contains x. Then, the *finely harmonic measure* on U for x

(with respect to Ω), denoted $\varepsilon_x^{\Omega\setminus U}$, is defined by

$$\varepsilon_x^{\Omega\setminus U} := {}^{\Omega'}\varepsilon_x^{\Omega'\setminus U},$$

where Ω' is a Greenian open subset of Ω such that $U\Subset\Omega'$. This makes sense because of Proposition 2.14. Denote by $\mathrm{FH}_x(\Omega)$ the set of finely harmonic measures $\varepsilon_x^{\Omega\setminus V}$ such that $V\Subset\Omega$ is a fine domain containing x.

Note that $\mathrm{FH}_x(\Omega)=\bigcup \mathrm{FH}_x(\Omega)_x(\Omega')$, where Ω' runs over the Greenian open subsets of Ω which contain x. Since $\mathrm{FH}_x(\Omega)_x(\Omega')\subset J_x(\Omega')\subset J_x(\Omega)$ for these sets, it follows that $\mathrm{FH}_x(\Omega)\subset J_x(\Omega)$.

Proof of Theorem 1.2 *in its general form.* Let $\varepsilon_x^{\Omega\setminus V}\in\mathrm{FH}_x(\Omega)$, where V is a fine domain of Ω with $x\in V\Subset\Omega$. Let Ω' be a Greenian open set with $V\Subset\Omega'\subset\Omega$ and such that each bounded component of $\mathbf{R}^d\setminus\Omega'$ meets $\mathbf{R}^d\setminus\Omega$. By applying the weak version of Theorem 1.2, it follows that $\varepsilon_x^{\Omega\setminus V}\in\mathrm{FH}_x(\Omega)_x(\Omega')=\mathrm{ext}(J_x(\Omega'))$. Let $\mu_1,\mu_2\in J_x(\Omega)$ and $0<\alpha<1$ be such that

$$\varepsilon_x^{\Omega\setminus V}=\alpha\mu_1+(1-\alpha)\mu_2.$$

By [CR2, Proposition 2.1(ii)], it follows that $\mu_1,\mu_2\in J_x(\Omega')$. Hence, $\varepsilon_x^{\Omega\setminus V}=\mu_1=\mu_2$, and this shows that $\varepsilon_x^{\Omega\setminus V}\in\mathrm{ext}(J_x(\Omega))$.

On the other hand, let $\mu\in\mathrm{ext}(J_x(\Omega))$ and let Ω' be a Greenian open subset of Ω with $\operatorname{supp}\mu\cup\{x\}\Subset\Omega'$ and such that each bounded component of $\mathbf{R}^d\setminus\Omega'$ meets $\mathbf{R}^d\setminus\Omega$. By [CR2, Proposition 2.1(i) and (ii)], it follows that $\mu\in\mathrm{ext}(J_x(\Omega'))$. Hence, applying the weak version of Theorem 1.2, we conclude that $\mu\in\mathrm{FH}_x(\Omega)$. $\square$

Proof of Theorem 1.3. (i) Let $\{D_n\}_{n=1}^\infty$ be a decreasing sequence of domains such that $x\in D_n\Subset\Omega$. Let ω_n be the harmonic measure on D_n for x. By [CC, Proposition 7.2.4], the sequence $\{\omega_n\}_{n=1}^\infty$ converges to $\varepsilon_x^{\Omega\setminus D}$ in the weak*-topology of $C(\mathbf{R}^d)^*$, where $D:=\bigcap_{n=1}^\infty D_n$. Proposition 2.6(a) implies that $\varepsilon_x^{\Omega\setminus D}=\varepsilon_x^{b(\Omega\setminus D)}$. If $x\in b(\Omega\setminus D)$, then $\varepsilon_x^{b(\Omega\setminus D)}=\varepsilon_x$. On the other hand, suppose that $x\notin b(\Omega\setminus D)$. Since $\Omega\setminus b(\Omega\setminus D)\Subset\Omega$, we can apply Proposition 2.13(i) to find that $\varepsilon_x^{b(\Omega\setminus D)}=\varepsilon_x^{\Omega\setminus V}$, where V is the fine component of $\Omega\setminus b(\Omega\setminus D)$ which contains x.

(ii) Let us first suppose that V is open with respect to the Euclidean topology. Let Ω' be a regular domain with $V\subset\Omega'\Subset\Omega$ and such that $\overline{\Omega}'\setminus V$ has non-empty interior. Applying [CR2, Lemma 4.3] to $\overline{\Omega}'\setminus V$, we find an increasing sequence $\{L_n\}_{n=1}^\infty$ of compact bases contained in $\overline{\Omega}'\setminus V$ such that $(\overline{\Omega}'\setminus V)\setminus(\bigcup_{n=1}^\infty L_n)$ is polar. Define D_n to be the connected component of $\Omega'\setminus L_n$ which contains V. The sequence $\{D_n\}_{n=1}^\infty$ has the desired properties.

Let us remove the restriction on V. By [AG, Theorem 7.3.11], there exists a sequence $\{F_n\}_{n=1}^\infty$ of closed sets contained in $\Omega\setminus V$ such that $(\Omega\setminus V)\setminus(\bigcup_{n=1}^\infty F_n)$ is polar. Let Ω' be a regular open set such that $V\subset\Omega'\Subset\Omega$. Set $V_n=\Omega'\setminus F_n$. Each V_n is then a relatively compact open subset of Ω, and $(\bigcap_{n=1}^\infty V_n)\setminus V$ is polar. By applying the first part to V_n, we find a sequence $\{U_{n,m}\}_{n,m=1}^\infty$ of regular open sets with $V\subset U_{n,m}\Subset\Omega$ and such that $(\bigcap_{n,m=1}^\infty U_{n,m})\setminus V$ is polar. We then construct inductively the desired sequence.

(iii) Let $\varepsilon_x^{\Omega\setminus V}\in\mathrm{FH}_x(\Omega)$, where V is a fine domain such that $x\in V\Subset\Omega$. Let $\{D_n\}_{n=1}^\infty$ be a sequence as described in the statement of (ii), and set $D':=\bigcap_{n=1}^\infty D_n$ and $P:=D'\setminus V$. By [CC, Proposition 7.2.4(a)], we get

$$\varepsilon_x^{\Omega\setminus D_n}\to\varepsilon_x^{\Omega\setminus D'}.$$

Since P is polar, it follows that

$$\varepsilon_x^{\Omega\setminus D'}=\varepsilon_x^{P\cup(\Omega\setminus D')}=\varepsilon_x^{\Omega\setminus V}.$$

We then get the desired conclusion. □

8. Proof of Theorem 1.4

In this section, we prove Theorem 1.4. The proof proceeds via a number of preliminary results. Given a compact set $K\subset\Omega$, recall that $J_x(\Omega,K):=\{\mu\in J_x(\Omega):\operatorname{supp}\mu\subset K\}$.

Lemma 8.1. *Let Ω be an open subset of $\mathbf{R}^d$, $d\geq2$, and $x\in\Omega$. Let $\mu\in J_x(\Omega)$. Then, there exists a probability measure σ on $\operatorname{ext}(J_x(\Omega))\cap J_x(\Omega,\operatorname{supp}\mu)$ such that*

$$\int\varphi\,d\mu=\int_{\operatorname{ext}(J_x(\Omega))}\left(\int\varphi\,d\nu\right)d\sigma(\nu) \tag{9}$$

for every function $\varphi\colon\Omega\to(-\infty,+\infty]$ which is universally measurable and locally bounded below.

Proof. Let $\mu\in J_x(\Omega)$. By [CR2, Proposition 6.1] and its proof, there exists a probability measure σ on $\operatorname{ext}(J_x(\Omega))\cap J_x(\Omega,\operatorname{supp}\mu)$ such that (9) holds for every continuous function on Ω.

If $\varphi\colon\Omega\to(-\infty,+\infty]$ is lower semicontinuous, then there exists an increasing sequence $\{\varphi_n\}_{n=1}^\infty$ in $C(\mathbf{R}^d)$ such that $\varphi_n\to\varphi$ on $\operatorname{supp}\mu$. Since (9) holds for each φ_n, it then holds for φ. The same is also true for upper semicontinuous functions.

Suppose that $\varphi\colon \Omega\to(-\infty,+\infty]$ is a Borel function which is locally bounded below. Suppose temporarily that φ is bounded from above. By Vitali–Carathéodory's theorem [Ru, Theorem 2.24], given $\varepsilon>0$, we can find two functions u and v which are upper semicontinuous and lower semicontinuous, respectively, such that $u\le\varphi\le v$, and

$$\int (v-u)\,d\mu<\varepsilon. \tag{10}$$

This implies that

$$\int u\,d\mu\le\int\varphi\,d\mu\le\int v\,d\mu,$$

and

$$\int\biggl(\int u\,d\nu\biggr)\,d\sigma(\nu)\le\int\biggl(\int \varphi\,d\nu\biggr)\,d\sigma(\nu)\le\int\biggl(\int v\,d\nu\biggr)\,d\sigma(\nu).$$

From (10) and (9) applied to u and v, it follows that

$$\biggl|\int\biggl(\int \varphi\,d\nu\biggr)\,d\sigma(\nu)-\int\varphi\,d\mu\biggr|\le\varepsilon.$$

By the arbitrary nature of ε, the function φ satisfies (9). If φ is not bounded above, apply (9) to $\min\{n,\varphi\}$ and take the limit.

Finally, suppose that $\varphi\colon \Omega\to(-\infty,+\infty]$ is universally measurable and locally bounded below. There exist two Borel functions ψ_1 and ψ_2 which are locally bounded below, such that $\psi_2\le\varphi\le\psi_2$, and $\psi_1=\psi_2$ outside a set E with $\mu(E)=0$. By a reasoning similar to above, we get the desired result. □

Proposition 8.2. *Let Ω be an open subset of $\mathbf{R}^d$, $d\ge2$, and $x\in\Omega$. Let $\mu\in J_x(\Omega)$. Then, for each function $\varphi\colon \Omega\to[-\infty,+\infty]$ which is universally measurable and such that the integrals $\int\varphi\,d\mu$ and $\int\varphi\,d\nu$, $\nu\in\operatorname{ext}(J_x(\Omega))$, exist,*

$$\int\varphi\,d\mu\le\sup\biggl\{\int\varphi\,d\nu:\nu\in\operatorname{ext}(J_x(\Omega))\biggr\}.$$

Proof. If $\int\varphi\,d\mu=-\infty$, then the result is trivial. Suppose this is not the case. Let σ be as in Lemma 8.1, and set $\varphi_n:=\min\{\varphi,n\}$. Then,

$$\begin{aligned}\int\varphi_n\,d\mu&=\int\biggl(\int\varphi_n\,d\nu\biggr)\,d\sigma(\nu)\\&\le\sup\biggl\{\int\varphi_n\,d\nu:\nu\in\operatorname{ext}(J_x(\Omega))\biggr\}\le\sup\biggl\{\int\varphi\,d\nu:\nu\in\operatorname{ext}(J_x(\Omega))\biggr\}.\end{aligned}$$

We get the desired result by letting $n\to\infty$. □

Lemma 8.3. *Let Ω be an open subset of $\mathbf{R}^d$, $d\geq 2$. Let $\varphi\colon \Omega\to[0,+\infty]$ be a universally measurable function. Given $x\in\Omega$, let V and D be a finely open and an open set, respectively, such that $x\in V\subset D\Subset\Omega$. Set $\psi:=\varphi\chi$, where χ is the characteristic function of $\partial_f V\cap\partial D$. Then,*

$$\int \psi\, d\varepsilon_x^{\Omega\setminus V} \leq \int \varphi\, d\omega,$$

where ω is the harmonic measure on D for x.

Proof. Let $u\in\mathrm{SH}(D)$ be a superfunction for φ on D. By the minimum principal and the fact that $\varphi\geq 0$, we get $u\geq 0$ on D. The fact that u is a finely hyperharmonic function on V satisfying

$$\operatorname*{fine-\liminf}_{\substack{y\to z\\ y\in V}} u(y)\geq \psi(z), \quad z\in\partial_f V,$$

implies that it is a fine superfunction for ψ on V. So,

$$\overline{H}{}_\psi^V(x)\leq \overline{H}{}_\varphi^D(x).$$

The conclusion then follows from 2.17. $\square$

Lemma 8.4. *Let Ω be an open subset of $\mathbf{R}^d$, $d\geq 2$, and $x\in\Omega$. Let V and D be a finely open and an open set, respectively, such that $x\in V\subset D\Subset\Omega$. Then, for each subharmonic function s defined on a neighbourhood of $\overline{D}$,*

$$\int s\, d\varepsilon_x^{\Omega\setminus V} \leq \int s\, d\omega,$$

where ω is the harmonic measure on D for x.

Proof. Let $u\in\mathrm{SH}(D)$ be lower bounded and satisfy

$$\liminf_{\substack{y\to z\\ y\in D}} u(y)\geq s(z), \quad z\in\partial D.$$

Applying the minimum principle to $u-s$, it follows that $u\geq s$ on D. The function u is then a fine superfunction for s with respect to V. By Proposition 2.17, we get

$$\overline{H}{}_s^V(x)\leq H_s^D(x).$$

Since the integral $\int s\, d\varepsilon_x^{\Omega\setminus V}$ exists, we get

$$\int s\, d\varepsilon_x^{\Omega\setminus V} \leq \int s\, d\omega. \quad \square$$

Recall that $H_x^r(\Omega)$ is the set of harmonic measures on relatively compact regular domains in Ω which contain x.

Theorem 8.5. *Let Ω be an open subset of $\mathbf{R}^d$, $d\geq 2$, and $x\in\Omega$. Let $\varphi\colon \Omega\to[-\infty,+\infty]$ be a universally measurable function which satisfies the following property*:

For each open set D with $x\in D\Subset\Omega$, there exists a subharmonic function s on D such that $s(x)>-\infty$ and $\varphi\geq s$ on D.

Then, for each $\nu\in\operatorname{ext}(J_x(\Omega))$,

$$\int\varphi\,d\nu\leq\sup\biggl\{\int\varphi\,d\omega:\omega\in H_x^r(\Omega)\cup\{\varepsilon_x\}\biggr\}.$$

Note that the condition on φ implies that the integrals above exist.

Proof. The conclusion is trivial if $\nu=\varepsilon_x$, or if $\int\varphi\,d\nu=-\infty$. Suppose then that $\nu=\varepsilon_x^{\Omega\setminus V}$, where V is a fine domain with $x\in V\Subset\Omega$, and $\int\varphi\,d\nu>-\infty$. Let Ω' be an open subset of Ω such that $\overline{V}\subset\Omega'\Subset\Omega$. Let s be a subharmonic function on Ω' which satisfies the property of the statement and suppose also that $s\leq 0$. Set $s_m:=\max\{s,-m\}$. Consider $\{D_n\}_{n=1}^\infty$ and $\{\omega_n\}_{n=1}^\infty$ as in Theorem 1.3 and suppose also that $\overline{D}_n\Subset\Omega'$. Denote by χ_n the characteristic function of $\partial_f V\cap D_n$, and set $\psi_n:=(\varphi-s)\chi_n$. By applying the last two lemmas, we get

$$\int\psi_n\,d\nu\leq\int(\varphi-s)\,d\omega_n\quad\text{and}\quad\int s_m\,d\nu\leq\int s_m\,d\omega_n.$$

Put together these two relations to obtain

$$\int(\psi_n+s_m)\,d\nu\leq\int(\varphi+s_m-s)\,d\omega_n.$$

Since $\varphi\chi_n\leq\psi_n+s_m$ and $\int(\varphi+s_m-s)\,d\omega_n\to\int\varphi\,d\omega_n$, as $m\to\infty$, we get

$$\int\varphi\chi_n\,d\nu\leq\int\varphi\,d\omega_n.$$

Now, the conclusion follows by taking the lower limit on each side. $\square$

Proof of Theorem 1.4. Let φ and μ satisfy the hypothesis of the statement. If $\int\varphi\,d\mu=-\infty$, then the conclusion is trivial. Suppose this is not the case. Set $\varphi_n:=\min\{n,\varphi\}$. By Proposition 8.2 and Theorem 8.5, we get

$$\int\varphi_n\,d\mu\leq\sup\biggl\{\int\varphi_n\,d\omega:\omega\in H_x^r(\Omega)\cup\{\varepsilon_x\}\biggr\}\leq\sup\biggl\{\int\varphi\,d\omega:\omega\in H_x^r(\Omega)\cup\{\varepsilon_x\}\biggr\}.$$

The conclusion follows by letting $n\to+\infty$. $\square$

9. Application to entire functions

In this final section, we give an application to entire functions stated in Theorem 1.5. Denote by $M\colon \mathbf{C}\to(0,\infty)$ a continuous function and by $\{\alpha_n\}_{n=1}^{\infty}$ a sequence of non-zero complex numbers converging to infinity. Let us first explain why (1) is a necessary condition for the existence of an entire function $f\not\equiv 0$, whose zero set includes $\{\alpha_n\}_{n=1}^{\infty}$, and is such that $|f(z)|\le M(z)$, $z\in\mathbf{C}$.

Suppose that such a function exists, and let us see the resulting implications. By replacing f by $Cf(z)/z^k$ for appropriate constants C and k, we can suppose that $f(0)\neq 0$. Let D be a bounded domain which contains 0. By [AG, Theorems 4.3.7 and 4.4.1], we get the following representation for $\log|f|$:

$$\log|f(z)| = h(z) - \sum_{n=1}^{\infty} G_D(\alpha_n, z), \quad z\in D,$$

where h is the greatest harmonic minorant of $\log|f|$ on D, and where n runs over the integers such that $\alpha_n\in D$. By [AG, Theorem 6.4.10] and [CC, Theorem 7.2.4], the function h can be written as

$$h(z) = \int_{\partial D} \log|f(\zeta)|\, d\omega_z(\zeta), \quad z\in D,$$

where ω_z is the harmonic measure for z on D. It then follows that

$$\log|f(z)| = \int_{\partial D} \log|f(\zeta)|\, d\omega_z(\zeta) - \sum_{n=1}^{\infty} G_D(\alpha_n, z), \quad z\in D.$$

By setting $z=0$ in the above equation and considering that $f\le M$ on D, we get (1).

Before continuing with Theorem 1.5, let us state a theorem of Khabibullin [Kh, Section 2, p. 1069 and Remark 2, p. 1072] on which the previous relies.

Theorem 9.1. (Khabibullin) *Let $M\colon \mathbf{C}\to(0,\infty)$ be a continuous function. If there exists an entire function g such that $g(0)\neq 0$ and*

$$\sup_{\mu\in J_0(\Omega)} \int (\log|g| - \log M)\, d\mu < +\infty$$

then, for each $\delta>0$, there exists an entire function $f\not\equiv 0$, whose zero set includes that of g, and which satisfies

$$|f(z)| \le \max_{|\zeta-z|\le\delta} M(\zeta), \quad z\in\mathbf{C}.$$

Proof of Theorem 1.5. By Weierstrass's theorem (see e.g. [Ru, Theorem 15.9]), there exists an entire function g whose zero set is exactly $\{\alpha_n\}_{n=1}^{\infty}$ (counting multiplicities). Let D be a bounded domain which contains 0. By a reasoning similar to the one preceding the statement of the present theorem, we get the formula

$$\sum_n G_D(\alpha_n, 0) = H^D_{\log|g|}(0) - \log|g(0)|,$$

where n runs over the integers such that $\alpha_n \in D$. If (1) holds, then

$$H^D_{\log|g|}(0) - H^D_{\log M}(0) \leq -c - \log|g(0)|,$$

and this is true for all bounded domains D which contain 0. This implies that

$$\sup_{\omega \in H_0(\mathbf{C})} \int (\log|g| - \log M)\, d\omega < +\infty.$$

The conclusion then follows from Theorem 1.4 and Khabibullin's theorem. $\square$

Remark. In condition (1), it is possible to replace the family of bounded domains which contain 0 by a smaller one. Consider for instance a family $\mathcal{D}$ of bounded domains containing 0, which satisfies the following property:

For each bounded domain D containing 0*, there exists an increasing sequence* $\{D_n\}_{n=1}^{\infty} \subset \mathcal{D}$ *such that* $D = \bigcup_{n=1}^{\infty} D_n$.

Then, it is easy to see that it is enough for (1) to hold for the domains in $\mathcal{D}$. In particular, one could consider the family of bounded domains containing 0 with smooth boundary.

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