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Rationally convex sets on the unit sphere in $\mathbb{C}^2$

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Abstract. Let X be a rationally convex compact subset of the unit sphere S in $\mathbb{C}^2$, of three-dimensional measure zero. Denote by $R(X)$ the uniform closure on X of the space of functions P/Q, where P and Q are polynomials and $Q\neq 0$ on X. When does $R(X)=C(X)$?

Our work makes use of the kernel function for the $\bar{\partial}_b$ operator on S, introduced by Henkin in [5] and builds on results obtained in Anderson–Izzo–Wermer [3].

We define a real-valued function ε_X on the open unit ball $\operatorname{int} B$, with $\varepsilon_X(z,w)$ tending to 0 as (z,w) tends to X. We give a growth condition on $\varepsilon_X(z,w)$ as (z,w) approaches X, and show that this condition is sufficient for $R(X)=C(X)$ (Theorem 1.1).

In Section 4, we consider a class of sets X which are limits of a family of Levi-flat hypersurfaces in $\operatorname{int} B$.

For each compact set Y in $\mathbb{C}^2$, we denote the rationally convex hull of Y by $\widehat{Y}$. A general reference is Rudin [8] or Aleksandrov [1].

1. Introduction

Let X be a compact subset of the sphere $|z|^2+|w|^2=1$ in $\mathbb{C}^2$. We assume

(1) X is rationally convex, and

(2) $m_3(X)=0$, where m_3 denotes 3-dimensional measure.

For each positive ε we denote by Ω_ε the set of points (z,w) on S whose Euclidean distance from X is less than or equal to ε.

$\widehat{\Omega}_\varepsilon$ denotes the rationally convex hull of Ω_ε. B is the closed unit ball in $\mathbb{C}^2$.

Definition 1.1. For (z,w) in the open unit ball, $\varepsilon(z,w)$ is the smallest number ε such that (z,w) belongs to $\widehat{\Omega}_\varepsilon$.

Theorem 1.1. *Assume that for a.a. z in the unit disk there exists a non-zero number $p>0$ such that*

$$\int_{-\pi}^{\pi} \frac{d\phi}{\varepsilon_X(z, r\exp i\phi)^p}$$

remains bounded as r approaches $\sqrt{1-|z|^2}$. Then $R(X)=C(X)$.

The geometric meaning of the integral in the preceding theorem is as follows: we let, for given z, D_z represent the disk which consists of all (z,w) in $\operatorname{int} B$. D_z has the radius $\sqrt{1-|z|^2}$. Our integral then is taken over the circle of radius r in D_z.

Note. The number p in Theorem 1.1 is allowed to depend on z.

Theorem 1.1 and its proof arise from the following sources: Let μ be a (complex) measure on the ζ-plane $\mathbb{C}$, supported on the compact set K. Let $\hat{\mu}$ be the Cauchy transform of μ. Then for each smooth function f of compact support on $\mathbb{C}$, we have

$$\int_{\mathbb{C}} f\,d\mu = \frac{1}{2\pi i}\int_{\mathbb{C}} (\bar{\delta} f)\wedge\hat{\mu}\wedge d\zeta. \tag{1.1}$$

From this one derives the Hartogs–Rosenthal theorem: if $m_2(K)=0$ then $R(K)=C(K)$.

We let B be the closed unit ball in $\mathbb{C}^2$ and S be its boundary. Henkin, in [5], gave a kernel on S generalizing the Cauchy kernel, and defined for each measure μ on S the corresponding transform K_μ. K_μ is summable on S and smooth on S outside of the support X of μ. Henkin proved, as analogue of (1.1), that

$$\int_S \phi\,d\mu = \frac{1}{4\pi^2}\int_S \bar{\delta}\phi\wedge K_\mu\omega, \tag{1.2}$$

where $\omega=d\zeta_1\wedge d\zeta_2$, and $\phi\in C^1(S)$, provided that μ is orthogonal to all polynomials on $\mathbb{C}^2$.

Let now X be a rationally convex compact subset of S with $m_3(X)=0$ and let μ be a measure supported on X which is orthogonal to $R(X)$. H. P. Lee and the author showed in [7] that K_μ admits holomorphic continuation from $S\setminus X$ to all of $\operatorname{int} B$. Further, J. T. Anderson, A. J. Izzo, and the author showed in [3] that for μ as above, if K_μ lies in the Hardy space $H^1(B)$, then $\mu=0$. It follows from this, that if we can show that for each μ orthogonal to $R(X)$ we have K_μ in $H^1(B)$, then $R(X)=C(X)$.

So we need conditions on X which imply suitable bounds on K_μ. For this purpose one can use the following estimate proved as Lemma 2.3 in [3]:

If $X\subset S$ and μ is a measure on X, then for z in $S\setminus X$, we have

$$|K_\mu|(z)\le\frac{4\|\mu\|}{\operatorname{dist}^4(z,X)}.$$

This result can be used as follows: Fix a point z^0 in $\operatorname{int} B$ and construct a Riemann surface Σ in $\operatorname{int} B$, passing through z^0, with boundary on $S\setminus X$. For μ orthogonal to $R(X)$, K_μ is holomorphic in $\operatorname{int} B$, and so the restriction of K_μ to Σ is holomorphic on Σ. By the maximum principle on Σ, then, $|K_\mu(z^0)|\le|K_\mu(z)|$

for some z in $\operatorname{bd}\Sigma = S\cap\overline{\Sigma}$. Applying this estimate for each z^0 in $\operatorname{int} B$, we can in certain cases show that K_μ lies in $H^1(B)$ for each μ in $R(X)^\perp$, and deduce that $R(X)=C(X)$. In [3] we applied this method, taking X to be certain smooth 2-manifolds contained in S.

To use this method for a general rationally convex compact subset of S, we need some conditions on X. S itself is locally non-pseudoconvex. We require our set X to be "holomorphically flat". We are motivated by an analogy from geometry in $\mathbb{R}^3$. Let Z be the cylinder: $\{(x,y,z):0\le z\le 1 \text{ and } x^2+y^2\le 1\}$ in $\mathbb{R}^3$. The boundary of Z is non-convex. We consider the interval $I=\{(1,0,z):0\le z\le 1\}$ in $\mathbb{R}^3$. I is a flat subset of $\operatorname{bd} Z$. We construct a sequence of planes Π_n in R^3, where Π_n has equation $x=1-1/n$, $n=1,2,\dots$. Each Π_n is flat, and, as n approaches infinity, $\Pi_n\cap Z$ approaches I.

With Z and $\operatorname{bd} Z$, replaced, respectively, by B and S, and I replaced by a rationally convex subset X of S, the role of $\{\Pi_n\}_{n=1}^\infty$ can be played by a sequence of Levi-flat surfaces $\{|F_n|=1\}_{n=1}^\infty$ in $\operatorname{int} B$, converging to X, where F_n is a holomorphic function in $\operatorname{int} B$. For the general compact rationally convex set X contained in S, we do not have Levi-flat hypersurfaces converging to X. It turns out that the sets $\operatorname{bd}\widehat{\Omega}_\varepsilon$ (see the introduction) fulfill the same purpose. The key fact is given in Lemmas 2.5 and 2.6 below, concerning a maximum principle valid on $\operatorname{bd}\widehat{\Omega}_\varepsilon$. Under the hypothesis of Theorem 1.1, we can carry out the argument based on the Henkin transforms of measures in $R(X)^\perp$ which we sketched earlier.

2. Properties of $\widehat{\Omega}_\varepsilon$

Lemma 2.1. *Let U be a relatively open neighborhood of X in B. There exists $a>0$ such that for $0<\varepsilon<a$, $\widehat{\Omega}_\varepsilon\subset U$.*

Proof. Suppose no such a exists. Then there is a sequence $\{\varepsilon_n\}_{n=1}^\infty$ tending to 0 and for each n there is a point $p_n\in\widehat{\Omega}_{\varepsilon_n}$ with $p_n\notin U$. Let p be an accumulation point of $\{p_n\}_{n=1}^\infty$. Then $p\notin U$.

Choose a polynomial Q with $Q(p)=0$. Then for all n, $Q-Q(p_n)$ vanishes at p_n, Hence there exists y_n in Ω_{ε_n} with $(Q-Q(p_n))(y_n)=0$. Let y be an accumulation point of $\{y_n\}_{n=1}^\infty$. Then y belongs to X. Letting $n\to\infty$, we see that $Q(y)=0$. So Q has a zero in X. Thus p lies in $\widehat{X}$. This contradicts the rational convexity of X. So the claim holds. □

Lemma 2.2. *Fix $\varepsilon>0$. Then $\operatorname{int}\widehat{\Omega}_\varepsilon$ is non-empty, and is a pseudoconvex open set.*

Proof. Ω_ε has non-empty relative interior on S. It follows that the family of analytic disks with boundary in Ω_ε is contained in $\widehat{\Omega}_\varepsilon$ and has non-empty interior in $\mathbb{C}^2$. So $\operatorname{int}\widehat{\Omega}_\varepsilon$ is non-empty. Every function in $R(\Omega_\varepsilon)$, restricted to $\operatorname{int}\widehat{\Omega}_\varepsilon$, is analytic there.

Suppose that $\operatorname{int}\widehat{\Omega}_\varepsilon$ fails to be pseudoconvex. Then there exists an open ball β which has non-trivial intersection with both $\widehat{\Omega}_\varepsilon$ and with its complement such that every function in $R(\Omega_\varepsilon)$ has an analytic continuation from $\operatorname{int}\widehat{\Omega}_\varepsilon\cap\beta$ to all of β. Fix a point y in β. Then the map: $G\to G(y)$ is a homomorphism of the algebra $R(\Omega_\varepsilon)\to\mathbb{C}$. It follows that y is in $\widehat{\Omega}_\varepsilon$. Then β is contained in $\widehat{\Omega}_\varepsilon$, contrary to the choice of β. So $\operatorname{int}\Omega_\varepsilon$ is pseudoconvex, and we are done. □

Lemma 2.3. *Fix a point y in $\operatorname{int}B\cap\operatorname{bd}\widehat{\Omega}_\varepsilon$ and choose an open bidisk D^2, centered at y and contained in $\operatorname{int}B$. Then $D^2\cap\operatorname{int}\widehat{\Omega}_\varepsilon$ is pseudoconvex.*

Proof. The assertion follows from Lemma 2.2. □

In [9] Słodkowski showed that the complement of the polynomially convex hull of a compact subset Y of $\mathbb{C}^2$ is locally pseudoconvex, away from Y. In [2, Chapter 23], a proof is given of Słodkowski's theorem . The argument given there applies equally well when "polynomially convex hull" is replaced by "rationally convex hull". This argument yields the following result.

Lemma 2.4. *Let Y be a compact set in $\mathbb{C}^2$ and $\widehat{Y}$ its rationally convex hull. Fix y in $\widehat{Y}\setminus Y$ and let D^2 be an open bidisk, centered at y and missing Y. Then $D^2\setminus\widehat{Y}$ is pseudoconvex.*

We give a sketch of the proof of Lemma 2.4 in the Appendix A.

We need some terminology. Let T be a compact space and A an algebra of continuous functions on T. We say that the *local maximum modulus principle* (LMMP) holds for A on T if for each point p in T, and each neighborhood N of p, $|f(p)|\leq\max_{\operatorname{bd}N}|f|$ for every f in A.

Fix a bidisk D^2 as in Lemma 2.3. We can write D^2 as the union of the following three sets:

D^2 intersected with the complement of $\widehat{\Omega}_\varepsilon$ in B,
D^2 intersected with the boundary of $\widehat{\Omega}_\varepsilon$, and
D^2 intersected with $\operatorname{int}\widehat{\Omega}_\varepsilon$.
These three sets are disjoint.

Claim 2.1. *$D^2\setminus\widehat{\Omega}_\varepsilon$ is pseudoconvex.*

Proof. It follows from Lemma 2.4 that $D^2\setminus\widehat{\Omega}_\varepsilon$ is pseudoconvex. □

By Lemma 2.2, we have the following result.

Claim 2.2. *$D^2\cap\operatorname{int}\widehat{\Omega}_\varepsilon$ is pseudoconvex.*

We now use the following result, given as Theorem 1 in [11].

Let X be a relatively closed subset of the open bidisk $D^2=\{(z,w):|z|<1$ and $|w|<1\}$ in $\mathbb{C}^2$. Assume further that X is a singularity set of an analytic function Φ in the following sense: Φ is analytic on $D^2\setminus X$ and if Γ is any bidisk $\{(z,w): |z-z_0|<R$ and $|w-w_0|<R\}$ with $(z_0,w_0)\in D^2\setminus X$ and $\Gamma\cap X$ is non-empty, then Φ cannot be analytically continued to all of Γ. Then the algebra of polynomials in z and w satisfies LMMP on X.

Claim 2.3. *Put $\Xi=D^2\cap\operatorname{bd}\widehat{\Omega}_\varepsilon$ Then the algebra of polynomials in z and w, restricted to Ξ, satisfies LMMP on Ξ.*

Proof. By the preceding two claims, $D^2\setminus\Xi$ is pseudoconvex. From Theorem 1 in [11], stated above, we get our claim. □

We now write Y_ε for $\operatorname{bd}\widehat{\Omega}_\varepsilon$. Put $B_R=\{(z,w):|z|^2+|w|^2\le R^2\}$.

Lemma 2.5. *Let Φ be a function defined and continuous on Y_ε. Assume that Φ has a holomorphic extension to a neighborhood of Y_ε in $\operatorname{int} B$, and further assume that for each $R<1$, Φ is uniformly approximable by polynomials on $Y_\varepsilon\cap B_R$. Then for each point p in Y_ε, $|\Phi(p)|\le\max_{\operatorname{bd}\Omega_\varepsilon}|\Phi|$.*

Proof. Fix R, $0<R<1$. B_R is the closed ball of radius R and center at the origin, and let S_R be its boundary. Let A_R be the uniform closure on $Y_\varepsilon\cap B_R$ of polynomials in z and w, restricted to Y_ε.

Claim 2.4. *Every peak point of the algebra A_R lies on $\operatorname{bd}\widehat{\Omega}_\varepsilon\cap S_R$.*

Fix a peak point y of A_R and assume that $y\notin S_R$. Choose an open bidisk D^2 centered at y, and contained in $\operatorname{int} B_R$. We have

$$\Xi=D^2\cap\operatorname{bd}\widehat{\Omega}_\varepsilon.$$

By the preceding, the algebra of polynomials, restricted to Ξ, satisfies LMMP on Ξ. Let U be a neighborhood of y on Ξ. By choice of y, there exists f in A_R with $f(y)=1$ and $|f(y)|\le 1-\varepsilon$ on $\operatorname{bd} U$, with $\varepsilon>0$.

Choose a polynomial P such that $|f-P|<\varepsilon/4$ on $Y_\varepsilon\cap B_R$. Then $|f-P|<\varepsilon/4$ on $\operatorname{bd} U$ and $|(f-P)(y)|<\varepsilon/4$. Hence $|P|\le 1-\varepsilon/4$ on $\operatorname{bd} U$ and $|P(y)|>1-\varepsilon/4$. This violated the LMMP on the algebra of polynomials in z and w, restricted to Ξ. Thus y is in S_R, and so the claim is proved.

It follows from the claim that the Silov boundary of A_R is contained in $\operatorname{bd}\widehat{\Omega}_\varepsilon\cap S_R$.

Choose a sequence of numbers $\{R_j\}_{j=1}^\infty$ tending to 1 from below. Let p and Φ be as in the statement of Lemma 2.5. Fix j. Since the Silov boundary of A_R is contained in $\mathrm{bd}\,\widehat{\Omega}_\varepsilon \cap S_{R_j}$, there exists a point p_j in that set such that $|\Phi(p)| \le |\Phi(p_j)|$. Let p_∞ denote an accumulation point of the sequence $\{p_j\}_{j=1}^\infty$. Then p_∞ lies in the closure of $\mathrm{bd}\,\widehat{\Omega}_\varepsilon$ as well as in S. So p_∞ lies in Ω_ε.

Suppose p_∞ lies in the interior of Ω_ε, relative to S. Then there exists a non-empty neighborhood U of p_∞ in B with U contained in $\widehat{\Omega}_\varepsilon$. There exist points p_j in $\mathrm{bd}\,\widehat{\Omega}_\varepsilon$ belonging to U. Such points then do not lie in the boundary of $\widehat{\Omega}_\varepsilon$. This is a contradiction. So p_∞ belongs to the boundary of Ω_ε, relative to S.

By the continuity of Φ on Y_ε and by the fact that $|\Phi(p)| \le |\Phi(p_j)|$ for each j, $|\Phi(p)| \le |\Phi(p_\infty)| \le \max_{\mathrm{bd}\,\Omega_\varepsilon} |\Phi|$. Lemma 2.5 is proved. $\square$

Lemma 2.6. *Let ϕ be a function holomorphic on* $\mathrm{int}\,B$ *and continuous on $B \setminus X$. Fix a point (z_0, w_0) in* $\mathrm{int}\,B$ *and let $\varepsilon_0 = \varepsilon(z_0, w_0)$. Then there exists a point $(z', w') \in S$ with* $\mathrm{dist}((z', w'), X) \ge \varepsilon_0$ *such that $|\phi(z_0, w_0)| \le |\phi(z', w')|$.*

Proof. Fix $\varepsilon < \varepsilon_0$. Put $V = \mathrm{int}\,B \setminus \widehat{\Omega}_\varepsilon$. Then (z_0, w_0) is in V. By the maximum principle, there exists $(z, w) \in \mathrm{bd}\,V$ such that $|\phi(z_0, w_0)| \le |\phi(z, w)|$.

Case 1. $(z, w) \in S$. Since $(z, w) \notin \Omega_\varepsilon$, we have $\mathrm{dist}((z, w), X) \ge \varepsilon$.

Case 2. $(z, w) \in \mathrm{int}\,B$. Then $(z, w) \notin \mathrm{int}\,\widehat{\Omega}_\varepsilon$, and $(z, w) \in \widehat{\Omega}_\varepsilon$. So $(z, w) \in \mathrm{bd}\,\widehat{\Omega}_\varepsilon = Y_\varepsilon$. Then by Lemma 2.4 there is a point $(a, b) \in \mathrm{bd}\,\Omega_\varepsilon$ so that $|\phi(z, w)| \le |\phi(a, b)|$. Thus $\mathrm{dist}((a, b), X) = \varepsilon$.

In either case, we let ε approach ε_0 from below. By passing to a convergent subsequence of the relevant sequence, we get a point $(z', w') \in S$ such that $\mathrm{dist}((z', w'), X) \ge \varepsilon_0$ and $|\phi(z_0, w_0)| \le |\phi(z', w')|$. We are done. $\square$

3. Preliminary material

We now proceed to list material from the work of Henkin in [5] and results from [3] and [7] which we shall use in the proof of Theorem 1.1.

Let X be a compact set on S and let μ be a finite complex measure on X which is orthogonal to polynomials.

Henkin defines the kernel $H(\zeta, z) = A/B$, where $A = \bar{\zeta}_1 \bar{z}_2 - \bar{\zeta}_2 \bar{z}_1$, $B = |1 - (\zeta, z)|^2$ and $(\cdot, \cdot)$ denotes the Hermitean inner product on $\mathbb{C}^2$. Define the function K_μ for ζ in S by

$$K_\mu(\zeta) = \int_S H(\zeta, z)\, d\mu(z).$$

Then K_μ is summable over S, and K_μ is smooth on $S \setminus \mathrm{supp}\,\mu$.

Let now X be a set satisfying (1) and (2) and let μ be a measure on X orthogonal to $R(X)$, and hence also orthogonal to polynomials.

It is shown in [7] that under these conditions, K_μ has a holomorphic extension from $S\setminus X$ to the open ball $\operatorname{int} B$. We denote the extension again by K_μ. Let Δ denote the closed unit disk in the z-plane.

For z in Δ we consider the set D_z of all (z,w) with $|w|<\sqrt{1-|z|^2}$. For each z, the restriction of K_μ to that disk is analytic on the disk.

Note. If the boundary of D_z lies in $S\setminus X$, then K_μ is continuous up to the boundary, while if the boundary of D_z meets X, then K_μ will in general become singular there.

Since K_μ is summable over S, K_μ is summable over the boundary of D_z with respect to 1-dimensional measure for almost all z.

The following is proved in [3, Lemma 2.6]. Assume that for a.a. z there exists a number $p>0$, depending on z, such that the restriction of K_μ to D_z lies in $H^p(D_z)$. Then K_μ lies in $H^1(B)$.

Further, it is shown in [3, Lemma 2.2] that: $K_\mu\in H^1(B)$ implies that $\mu=0$.

We now proceed to the proof of Theorem 1.1.

Proof of Theorem 1.1. Let X be as in Theorem 1.1. If we can show that for each measure μ on X with μ orthogonal to $R(X)$, $\mu=0$, then by the Hahn–Banach theorem we can deduce that $R(X)=C(X)$.

By the preliminary material given above, it suffices to show that for a.a. z there exists $p>0$ such that the restriction of K_μ to D_z lies in H^p. Write ε for ε_X.

By the hypothesis of Theorem 1.1, for a.a. z in Δ, there exists $p>0$ such that

$$(*)\qquad \int_{-\pi}^{\pi} \frac{d\phi}{\varepsilon(z, r\exp i\phi)^p} = O(1), \quad \text{as } r\to\sqrt{1-|z|^2}.$$

Fix a z for which $(*)$ holds and fix $r<\sqrt{1-|z|^2}$. Choose w with $|w|=r$. The point (z,w) is in $\operatorname{int} B$. Put $\varepsilon_0=\varepsilon(z,w)$. Since K_μ is holomorphic on $\operatorname{int} B$ and continuous on $B\setminus X$, Lemma 2.5 gives that there exists a point (z',w') in S such that $|K_\mu(z,w)|\le|K_\mu(z',w')|$ and $\operatorname{dist}((z',w'),X)\ge\varepsilon_0$.

We now need the following result from [3, Lemma 2.3]. If (a,b) is a point of S, and μ a measure on X, then

$$|K_\mu(a,b)| \le \frac{4\|\mu\|}{(\operatorname{dist}^4((a,b),X))}.$$

It follows that $|K_\mu(z',w')|\le 4\|\mu\|/\varepsilon_0^4$. Hence

$$|K_\mu(z,w)|\le \frac{4\|\mu\|}{\varepsilon_0^4}=\frac{4\|\mu\|}{\varepsilon(z,w)^4}.$$

We set $p'=p/4$ and raise this inequality to the p'-th power. We get

$$|K_\mu(z,w)|^{p'}\le \frac{(4\|\mu\|)^{p'}}{\varepsilon(z,w)^p}=\frac{\varkappa}{\varepsilon(z,w)^p}.$$

So for $-\pi\le\phi\le\pi$,

$$|K_\mu(z,r\exp i\phi)|^{p'}\le \frac{\varkappa}{\varepsilon(z,r\exp i\phi)^p}$$

for all $r<\sqrt{1-|z|^2}$. It follows that

$$(**)\qquad \int_{-\pi}^{\pi}|K_\mu(z,r\exp i\phi)|^{p'}\,d\phi\le\varkappa\int_{-\pi}^{\pi}\frac{d\phi}{\varepsilon(z,r\exp i\phi)^p}.$$

By hypothesis, there exists a constant M, depending only on z, such that the right-hand side in $(**)$ is bounded by M for all $r<\sqrt{1-|z|^2}$. Thus K_μ belongs to $H^{p'}(D_z)$.

Since this holds for a.a. z in the unit disk, by the results from [3] stated in the preliminary material, $\mu=0$ and since this holds for each measure μ on X which is orthogonal to $R(X)$, $R(X)=C(X)$. Theorem 1.1 is proved. □

4. A class of sets

In this section, we shall study the class of sets X on S described as follows:

Let E be an open disk in $|z|<1$ and put W equal to the set of (z,w) in $\operatorname{int} B$ with z in E, and choose a compact set X on S which lies over E. Let F be a function holomorphic on W and continuous on $\overline{W}$, such that

$$(4.1)\qquad |F|<1\quad\text{on } W,$$

and

$$(4.2)\qquad |F|=1\quad\text{on } X.$$

Lemma 4.1. *X is rationally convex.*

Note. Standard arguments give that every function analytic on W and continuous on the closure of W is uniformly approximable on the closure of W by polynomials.

Proof. Fix (z_0, w_0) in W. Put $\lambda=|F(z_0, w_0)|$. Then $0\le\lambda<1$. Fix $\eta>0$ and choose a polynomial Q on $\mathbb{C}^2$ such that $|Q-F|<\eta$ on $\overline{W}$. Put $Q_1=Q-Q(z_0, w_0)$. Then Q_1 vanishes at (z_0, w_0). Fix (z, w) in X. We have

$$Q_1(z,w) = (F-F(z_0,w_0))(z,w)+(Q-Q(z_0,w_0))(z,w)-(F-F(z_0,w_0))(z,w).$$

Then

$$|Q_1(z,w)| \ge |F(z,w)-F(z_0,w_0)|-2\eta \ge 1-\lambda-2\eta.$$

For η sufficiently small, the right-hand side $\neq 0$. So Q_1 does not vanish on X. Hence (z_0, w_0) does not lie in $\widehat{X}$.

It is easy to see, directly, that the remaining points of $\mathbb{C}^2\setminus X$ do not lie in $\widehat{X}$. So X is rationally convex, as asserted. □

Lemma 4.2. *Assume* (4.1) *and* (4.2) *and also assume that F is in $C^1(\overline{W})$. There exists a constant M such that the following holds*: *for every (z,w) in W, if we put $\lambda=|F(z,w)|$ and if (a,b) is any point on $S\cap\{(z,w):|F(z,w)|=\lambda\}$, then*

$$1-|F(z,w)| \le M\,\mathrm{dist}((a,b),X). \tag{4.3}$$

Proof. We now write $\phi=|F|$, and introduce the real coordinates in $\mathbb{C}^2$ so that we may regard ϕ as a function on $\mathbb{R}^4$. We put $\nabla\phi$ equal to the gradient of ϕ.

Fix (c,d) in X and put $f(t)=\phi((a,b)+t(c-a,d-b))$, $0<t<1$. Then

$$f'(t) = \nabla\phi((a,b)+t(c-a,d-b))\cdot((c,d)-(a,b))$$

and $\phi(c,d)-\phi(a,b)=f(1)-f(0)=f'(T)$ for some T with $0<T<1$. Let $\lambda=|F(a,b)|$. So $1-\lambda=\nabla\phi\cdot((c,d)-(a,b))$. Then $1-\lambda\le|\nabla\phi|\,|(c,d)-(a,b)|$. This holds for all (c,d) in X. Hence $1-|F(a,b)|=1-\lambda\le M\,\mathrm{dist}((a,b),X)$, where $M=\max|\nabla\phi|$ over $\overline{W}$. So

$$1-|F(z,w)| = 1-|F(a,b)| \le M\,\mathrm{dist}((a,b),X).$$

Thus (4.3) holds. □

Lemma 4.3. *Let μ be a measure on X orthogonal to $R(X)$. Then there exist constants C and M' such that*

$$|K_\mu(z,w)| \le \max\left(\frac{C}{(1-|F(z,w)|)^4}, M'\right), \tag{4.4}$$

if $(z,w)\in W$.

Proof. Fix (z,w) in W. Put $T=\{(u,v)\in B\}$ with $u\in\operatorname{bd}E$. Put $\lambda=|F(z,w)|$. Since K_μ is holomorphic in $\operatorname{int}B$, and $\{(z,w):|F(z,w)|=\lambda\}$ is a Levi-flat hypersurface, $|K_\mu(z,w)|\le|K_\mu(a,b)|$ for some (a,b) lying either on the set $\{(z,w):|F(z,w)|=\lambda\}\cap S$ or on the set $\{(z,w):|F(z,w)|=\lambda\}\cap T$.

Case 1. $(a,b)\in\{(z,w):|F(z,w)|=\lambda\}\cap S$. By Lemma 2.3 in [3], $|K_\mu(a,b)|\le k/\operatorname{dist}^4((a,b),X)$, for some constant k. By (4.3), $\operatorname{dist}((a,b),X)\ge(1-|F(z,w)|)/M$. It follows that

$$|K_\mu(z,w)|\le|K_\mu(a,b)|\le\frac{k'}{(1-|F(z,w)|)^4},$$

for some constant k'. So (4.4) holds.

Case 2. $(a,b)\in\{(z,w):|F(z,w)|=\lambda\}\cap T$. T is a compact subset of B, disjoint from X. So there exists a constant M' with $|K_\mu(a,b)|\le M'$, and so $|K_\mu(z,w)|\le|K_\mu(a,b)|\le M'$.

So in either case, (4.4) holds. □

Theorem 4.1. *Let E, W, F and X be as before. Let μ be a measure on X orthogonal to $R(X)$. Fix $p>0$ and put $p'=p/4$. Then there exist constants C_p and C'_p such that*

(4.5)
$$\int_{-\pi}^{\pi}|K_\mu(z,r\exp i\phi)|^{p'}\,d\phi\le C_p+C'_p\int_{-\pi}^{\pi}\frac{d\phi}{(1-|F(z,r\exp i\phi)|)^p},\quad 0<r<\sqrt{1-|z|^2}.$$

Proof. Fix z in E and fix r. Put $w=r\exp i\phi$. By (4.4), we have

$$|K_\mu|^{p'}\le\max\biggl(\frac{C^{p'}}{(1-|F(z,w)|)^p},(M')^{p'}\biggr),$$

and so

$$|K_\mu|^{p'}\le\frac{C^{p'}}{(1-|F(z,w)|)^p}+(M')^{p'}.$$

Integrating this inequality with respect to $d\phi$, we get (4.5). □

Claim 4.1. *Assume that, with $w=r\exp i\phi$,*

$$\int_{-\pi}^{\pi}\frac{d\phi}{(1-|F(z,w)|)^p}=O(1),\quad \textit{as } r\to\sqrt{1-|z|^2}, \tag{4.6}$$

for every z in E. Then $m_3(X)=0$.

Proof. Fix z in E. Let Λ_z be the set of all w with $(z,w)\in X$. Assume that $m_1(\Lambda_z)>0$. Fix $w\in\Lambda_z$, $w=r\exp i\phi$, and let $r\to\sqrt{1-|z|^2}$. Then

$$\int_{-\pi}^{\pi}\liminf_{r\to\sqrt{1-|z|^2}}\left(\frac{1}{(1-|F(z,w)|)^p}\right)d\phi\le\liminf_{r\to\sqrt{1-|z|^2}}\int_{-\pi}^{\pi}\frac{d\phi}{(1-|F(z,w)|)^p}. \tag{4.7}$$

Since (z,w) is in X and $|F(z,w)|=1$ on X, the left-hand side in (4.7) is infinite, and so the right-hand side is infinite as well. This contradicts (4.6). So $m_1(\Lambda_z)$ is zero. Since this holds for every z in E, $m_3(X)=0$. We are done. □

The preceding allows us to give conditions for the equality of $R(X)=C(X)$ in certain cases. As earlier we fix a disk E in the z-plane and define the region W as the set of all points $(z,w)\in\operatorname{int}B$ such that $z\in E$. We consider a function F, holomorphic in W and C^1 in $\overline{W}$, such that $|F|<1$ on W.

We now choose a set X on S lying over E, such that $|F|=1$ on X.

Theorem 4.2. *Assume that for all z in E, for some $p>0$, we have*

$$\int_{-\pi}^{\pi}\frac{d\phi}{(1-|F(z,r\exp i\phi)|)^p}=O(1),\quad \text{as } r\to\sqrt{1-|z|^2}. \tag{4.8}$$

Then $R(X)=C(X)$.

Proof. Making use of (4.5), the proof follows the argument in the proof of Theorem 1.1. □

5. A special case

Let E and W be as in Section 2 and assume that $\overline{E}$ is disjoint from 0.

In this section we verify condition (4.8) for the case that F is the function $z/\sqrt{1-w^2}$. We take $X=\{(z,\sqrt{1-|z|^2}):z\in E\}$. Then (4.1) and (4.2) hold.

For $(z,w)\in W$ we shall write $w=r\exp i\phi$. For fixed $z\in E$, an elementary calculation gives

$$1-|F|\ge k2r^2(1-\cos 2\phi), \tag{5.1}$$

where k is a constant independent of r.

For $w\in D_z$, $w=r\exp i\phi$, $r\ge\frac{1}{2}\sqrt{1-|z|^2}$, we have

$$\frac{1}{1-|F(z,w)|}\le C\frac{1}{1-\cos 2\phi},$$

where C is a constant.

It follows that for some positive constant c',

$$\int_{-\pi}^{\pi} \frac{d\phi}{(1-|F(z,w)|)^{1/4}} \le c' \int_{-\pi}^{\pi} \frac{d\phi}{(1-\cos 2\phi)^{1/4}}.$$

The right-hand side is finite, so we have (4.8) holding in this case, with $p=\frac{1}{4}$.

Note. The approximation result $R(X)=C(X)$ in this case can be obtained directly from the fact that $w=\sqrt{1-|z|^2}$ on X.

Appendix A

In this appendix we sketch a proof of Lemma 2.4.

Definition A.1. A *Euclidean Hartogs figure* in $\mathbb{C}^2$ is a pair of sets (P,H), chosen as follows. Let $P=\{z=(z_1,z_2)\in\mathbb{C}^2 : |z_1|\le 1 \text{ and } |z_2|\le 1\}$ and choose numbers q_1 and q_2, $0<q_1,q_2<1$. Let $H=\{z=(z_1,z_2)\in P : |z_1|\ge q_1 \text{ or } |z_2|\le q_2\}$.

Definition A.2. Let (P,H) be a Euclidean Hartogs figure in $\mathbb{C}^2$ and let Φ be a biholomorphism: $P\to\mathbb{C}^2$. Set $\widetilde{P}=\Phi(P)$ and $\widetilde{H}=\Phi(H)$. Then the pair $(\widetilde{P},\widetilde{H})$ is a *general Hartogs figure* in $\mathbb{C}^2$.

Lemma A.1. *Let W be a bounded domain in $\mathbb{C}^2$ that is not pseudoconvex. Then there exists a general Hartogs figure $(\widetilde{P},\widetilde{H})$ in $\mathbb{C}^2$ such that $\widetilde{H}\subset W$ and $\widetilde{P}$ is not contained in W.*

Proof. Lemma A.1 is the same as [2, Lemma 23.2], and a proof of it is given there. □

Proof of Lemma 2.4. We are given a compact set Y in $\mathbb{C}^2$. Let $\widehat{Y}$ be its rationally convex hull. Fix a point $y\in\widehat{Y}\setminus Y$ and a ball β in $\mathbb{C}^2$, centered at y, with β disjoint from Y.

Let W be a connected component of $\beta\setminus\widehat{Y}$. Arguing by contradiction, we assume that W is not pseudoconvex. By Lemma A.1, we then have a general Hartogs figure $(\widetilde{P},\widetilde{H})$ in $\mathbb{C}^2$ such that $\widetilde{H}\subset W$ and $\widetilde{P}$ is not contained in W. Here "ball" is replaced by "bidisk", but the argument is the same.

The argument in the proof of [2, Theorem 23.1] applies to the present case of a rationally convex hull $\widehat{Y}$ even though it is concerned with the polynomially convex hull $\widehat{Y}_P$. The proof uses the following result.

Claim A.1. *Let K be a compact subset of $\widehat{Y}_P\setminus Y$. Then the LMMP holds for the algebra of functions on K which are restrictions to K of functions holomorphic in a C^2-neighborhood of K.*

This claim holds also for the rationally convex hull $\widehat{Y}$. This follows from a general result on commutative Banach algebras due to Graham Allan, and given as Theorem 9.3 in [10].

Using this claim, the argument in the proof of Theorem 23.1 of [2] arrives at a contradiction. Hence W is pseudoconvex, and Lemma 2.4 is proved. □

A related result on rational approximation is given by Kytmanov in [6]. The result concerns sets X on the unit sphere which are peak sets for functions in the ball algebra.

Note. The author is grateful to John Anderson for help with this paper. In particular, John Anderson pointed out to me the following result, which follows from work of T. Duchamp and E. L. Stout [4, Section 5].

Theorem A.1. *With B and S as above, let F be a function holomorphic on* $\operatorname{int} B$ *and continuous on B. Let X be a compact subset of S such that*

(i) $|F|<1$ *on B outside of X, and*

(ii) $|F|=1$ *on X.*

Then $R(X)=C(X)$.

This result gives a treatment, based on the theory of uniform algebras, related to the class of sets in Section 4 above.

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