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Weighted integral formulas on manifolds

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Abstract. We present a method of finding weighted Koppelman formulas for (p,q)-forms on n-dimensional complex manifolds X which admit a vector bundle of rank n over $X\times X$, such that the diagonal of $X\times X$ has a defining section. We apply the method to $\mathbb{P}^n$ and find weighted Koppelman formulas for (p,q)-forms with values in a line bundle over $\mathbb{P}^n$. As an application, we look at the cohomology groups of (p,q)-forms over $\mathbb{P}^n$ with values in various line bundles, and find explicit solutions to the $\bar\partial$-equation in some of the trivial groups. We also look at cohomology groups of $(0,q)$-forms over $\mathbb{P}^n\times\mathbb{P}^m$ with values in various line bundles. Finally, we apply our method to developing weighted Koppelman formulas on Stein manifolds.

1. Introduction

The Cauchy integral formula provides a decomposition of a holomorphic function in one complex variable in simple rational functions, and is a cornerstone in function theory in one complex variable. The kernel is holomorphic and works for any domain. In several complex variables it is harder to find appropriate representations. The simplest multivariable analog, the Bochner–Martinelli kernel, is not as useful since the kernel is not holomorphic. The Cauchy–Fantappiè–Leray formula is a generalization which gives a holomorphic kernel in domains which admit a holomorphic support function. Henkin and Ramirez, in [16] and [22], obtained holomorphic kernels in strictly pseudoconvex domains G by finding such support functions. Henkin also found solutions to the $\bar\partial$-equation in such domains. This was done by means of a Koppelman formula, which represents a (p,q)-form ϕ defined in some domain D as a sum of integrals

$$\phi(z)=\int_{\partial D}K\wedge\phi+\int_D K\wedge\bar\partial\phi+\bar\partial_z\int_D K\wedge\phi+\int_D P\wedge\phi,$$

by means of the current K and the smooth form P. If ϕ is a closed form and the first and fourth terms of the right hand-side of Koppelman's formula vanish, we get a solution of the $\bar{\partial}$-problem for ϕ. Henkin's result paved the way for the Henkin–Skoda theorem (see [17] and [23]), which provided improved L^1-estimates on ∂G for solutions of the $\bar{\partial}$-equation by weighting the integral formulas.

Andersson and Berndtsson [9] found a flexible method of generating weighted formulas for representing holomorphic functions and solutions of the $\bar{\partial}$-equation. It was further developed by Berndtsson [6] to find solutions to division and interpolation problems. If V is a regular analytic subvariety of some domain D in $\mathbb{C}^n$ and h is holomorphic in V, then Berndtsson found a kernel K such that

$$H(z)=\int_V h(\zeta)K(\zeta,z)$$

is a holomorphic function which extends h to D. If $f=(f_1,...,f_m)$ are holomorphic functions without common zeros, he also found a solution to the division problem $\phi=f\cdot p$ for a given holomorphic function ϕ. Passare [19] used weighted integral formulas to solve a similar division problem, where the f_i's do have common zeros, but the zero sets have a complete intersection. He also proved the duality theorem for complete intersections (also proved independently by Dickenstein and Sessa [13]). Since then weighted integral formulas have been used by a number of authors to obtain qualitative estimates of solutions of the $\bar{\partial}$-equation and of division and interpolation problems, for example sharp approximation by polynomials [24], estimates of solutions to the Bézout equation [5], and explicit versions of the fundamental principle [10]. More examples and references can be found in the book [4]. More recently, Andersson [3] introduced a method generalizing [9] and [6] which is even more flexible and also easier to handle. It allows for some recently found representations with residue currents, for applications to division and interpolation problems, and also allows for f to be a matrix of functions.

There have been several attempts to obtain integral formulas on manifolds. Berndtsson [8] gave a method of obtaining integral kernels on n-dimensional manifolds X which admit a vector bundle of rank n over $X\times X$ such that the diagonal has a defining section, but did not consider weighted formulas. Formulas on Stein manifolds were first treated in Henkin and Leiterer [18], where formulas for $(0,q)$-forms are found, then in Demailly and Laurent–Thiébaut [11], where the leading term in a kernel for (p,q)-forms is found, in Andersson [1], which is a generalization of [9] following Henkin and Leiterer, and finally in Berndtsson [8], where the method described therein is applied to Stein manifolds. Formulas on $\mathbb{P}^n$ have been considered in [20], where they were constructed by using known formulas in $\mathbb{C}^{n+1}$, and in [7], where they were constructed directly on $\mathbb{P}^n$. There is also an example at the end of Berndtsson [8], where the method of that article is applied to $\mathbb{P}^n$.

In this article, we begin in Section 2 by developing a method for generating weighted integral formulas on $\mathbb{C}^n$, following [2]. Section 3 describes a similar method which can be used on n-dimensional manifolds X which admit a vector bundle of rank n over $X\times X$ such that the diagonal has a defining section. It has similar results as the method described in [8], but with the added benefit of yielding weighted formulas. The method of Section 3 is applied to the complex projective space $\mathbb{P}^n$ in Section 4, where we find a Koppelman formula for differential forms with values in a line bundle over $\mathbb{P}^n$. In the $\mathbb{P}^n$ case we get formulas which coincide with Berndtsson's formulas in [7] in the case $p=0$, but they are not the same in the general (p,q)-case.

As an application, in Section 5 we look at the cohomology groups of (p,q)-forms over $\mathbb{P}^n$ with values in various line bundles, and find which of them are trivial (though we do not find all the trivial groups). Berndtsson's formulas in [7] give the same result. The trivial cohomology groups of the line bundles over $\mathbb{P}^n$ are, of course, known before, but our method gives explicit solutions of the $\bar\partial$-equations. In Section 6 we look instead at cohomology groups of $(0,q)$-forms over $\mathbb{P}^n\times\mathbb{P}^m$ with values in various line bundles. Finally, in Section 7 we apply the method of Section 3 to finding weighted integral formulas on Stein manifolds, following [18] but also developing weighted formulas.

2. Weighted Koppelman formulas in $\mathbb{C}^n$

As a model for obtaining representations on manifolds, we present the $\mathbb{C}^n$ case in some detail. The material in this section follows the last section of [2]. The article [2] is mostly concerned with representation of holomorphic functions, but in the last section a method of constructing weighted Koppelman formulas in $\mathbb{C}^n$ is indicated. We expand this material and give proofs in more detail. We begin with some motivation from the one-dimensional case:

One way of obtaining a representation formula for a holomorphic function would be to solve the equation

$$\bar\partial u=[z],$$

where $[z]$ is the Dirac measure at z considered as a $(1,1)$-current, since then one would get an integral formula by Stokes' theorem. Less obviously, note that the kernel of Cauchy's integral formula in $\mathbb{C}$ also satisfies the equation

$$\delta_{\zeta-z}u=1,$$

where $\delta_{\zeta-z}$ denotes contraction with the vector field $2\pi i(\zeta-z)\partial/\partial\zeta$. These two can be combined into the equation

$$\nabla_{\zeta-z}u=1-[z], \tag{1}$$

where $\nabla_{\zeta-z}:=\delta_{\zeta-z}-\bar{\partial}$. To find representation formulas for holomorphic functions in $\mathbb{C}^n$, we look for solutions to (1) in $\mathbb{C}^n$, where $\delta_{\zeta-z}$ is contraction with

$$2\pi i\sum_{j=1}^{n}(\zeta_j-z_j)\frac{\partial}{\partial\zeta_j}.$$

Since the right-hand side of (1) contains one form of bidegree $(0,0)$ and one of bidegree (n,n), we must in fact have $u=u_{1,0}+u_{2,1}+...+u_{n,n-1}$, where $u_{k,k-1}$ has bidegree $(k,k-1)$. We can then write (1) as the system of equations

$$\delta_{\zeta-z}u_{1,0}=1,\quad \delta_{\zeta-z}u_{1,2}-\bar{\partial}u_{1,0}=0,\quad ...,\quad \bar{\partial}u_{n,n-1}=[z].$$

In this case, $u_{n,n-1}$ will satisfy $\bar{\partial}u_{n,n-1}=[z]$ and will give a kernel for a representation formula. The advantage of this approach is that it easily allows for weighted integral formulas, as we will see.

To get Koppelman formulas for (p,q)-forms, we need to consider z as a variable and not a constant. If we find $u_{n,n-1}$ such that $\bar{\partial}u_{n,n-1}=[\Delta]$, where $\Delta=\{(\zeta,z):\zeta=z\}$ is the diagonal of $\mathbb{C}^n_\zeta\times\mathbb{C}^n_z$ and $[\Delta]$ is the current of integration over Δ, then $u_{n,n-1}$ will be the kernel that we seek. In fact, if we let ϕ be a (p,q)-form, and ψ an $(n-p,n-q)$ test form, we have

$$\int_{\mathbb{C}^n_z}\left(\int_{\mathbb{C}^n_\zeta}\phi(\zeta)\wedge[\Delta]\right)\wedge\psi(z)=\int_{\mathbb{C}^n_z\times\mathbb{C}^n_\zeta}\phi(\zeta)\wedge\psi(z)\wedge[\Delta]=\int_{\mathbb{C}^n_z}\phi(z)\wedge\psi(z)$$

so that $\int_{\mathbb{C}^n_\zeta}\phi(\zeta)\wedge[\Delta]=\phi(z)$ in the current sense.

In more detail: Let Ω be a domain in $\mathbb{C}^n$ and let $\eta(\zeta,z)=2\pi i(z-\zeta)$, where $(\zeta,z)\in\Omega\times\Omega$. Note that η vanishes to the first order on the diagonal. Consider the subbundle $E^*=\mathrm{Span}\{d\eta_1,...,d\eta_n\}$ of the cotangent bundle $T^*_{1,0}$ over $\Omega\times\Omega$. Let E be its dual bundle, and let δ_η be an operation on E^*, defined as contraction with the section

(2) $$\sum_{j=1}^{n}\eta_j e_j,$$

where $\{e_j\}_{j=1}^n$ is the dual basis to $\{d\eta_j\}_{j=1}^n$. Note that δ_η anticommutes with $\bar{\partial}$.

Consider the bundle $\Lambda(T^*(\Omega\times\Omega)\oplus E^*)$ over $\Omega\times\Omega$. An example of an element of the fiber of this bundle at (ζ,z) is $d\zeta_1\wedge d\bar{z}_2\wedge d\eta_3$. We define

(3) $$\mathcal{L}^m=\bigoplus_{p=0}^{n}C^\infty(\Omega\times\Omega,\Lambda^pE^*\wedge\Lambda^{p+m}T^*_{0,1}(\Omega\times\Omega)).$$

Note that $\mathcal{L}^m$ is a subset of the space of sections of $\Lambda(T^*(\Omega\times\Omega)\oplus E^*)$. Let $\mathcal{L}^m_{\mathrm{curr}}$ be the corresponding space of currents. If $f\in\mathcal{L}^m$ and $g\in\mathcal{L}^k$, then $f\wedge g\in\mathcal{L}^{m+k}$.

We define the operator

$$\nabla=\nabla_\eta=\delta_\eta-\bar\partial,$$

which maps $\mathcal{L}^m$ to $\mathcal{L}^{m+1}$. We see that ∇ obeys Leibniz' rule, that is,

$$\nabla(f\wedge g)=\nabla f\wedge g+(-1)^m f\wedge\nabla g, \tag{4}$$

if $f\in\mathcal{L}^m$. Note that $\nabla^2=0$, which means that

$$\ldots\xrightarrow{\nabla}\mathcal{L}^m\xrightarrow{\nabla}\mathcal{L}^{m+1}\xrightarrow{\nabla}\ldots$$

is a complex. We also have the following useful property: If $f(\zeta,z)$ is a form of bidegree $(2n,2n-1)$ and $D\subset\Omega\times\Omega$, then

$$\int_{\partial D} f=-\int_D\nabla f. \tag{5}$$

This follows from Stokes' theorem and the fact that $\int_D\delta_\eta f=0$. The operator ∇ is defined also for currents, since $\bar\partial$ is defined for currents, and δ_η just amounts to multiplying with a smooth function, which is also defined for a current.

As in the beginning of this section, we want to find a solution to the equation

$$\nabla_\eta u=1-[\Delta]. \tag{6}$$

with $u\in\mathcal{L}^{-1}_{\mathrm{curr}}$ (since the left-hand side lies in $\mathcal{L}^0_{\mathrm{curr}}$), so as before, we have $u=u_{1,0}+u_{2,1}+\ldots+u_{n,n-1}$, where $u_{k,k-1}$ has degree k in E^* and degree $k-1$ in $T^*_{0,1}$.

Proposition 2.1. *Let*

$$b(\zeta,z)=\frac{1}{2\pi i}\frac{\partial|\eta|^2}{|\eta|^2}$$

and

$$u_{\mathrm{BM}}=\frac{b}{\nabla_\eta b}=\frac{b}{1-\bar\partial b}=b+b\wedge\bar\partial b+\ldots+b\wedge(\bar\partial b)^{n-1}, \tag{7}$$

where we get the right-hand side by expanding the fraction in a geometric series. Then u solves (6)*.*

The crucial step in the proof is showing that $\bar\partial(b\wedge(\bar\partial b)^{n-1})=[\Delta]$, which is common knowledge, as $b\wedge(\bar\partial b)^{n-1}$ is actually the well-known Bochner–Martinelli kernel.

A form u which satisfies $\nabla_\eta u=1$ outside Δ is a good candidate for solving (6). The following proposition gives us a criterion for when such a u in fact is a solution.

Proposition 2.2. *Suppose that* $u \in \mathcal{L}^{-1}(\Omega\times\Omega\setminus\Delta)$ *solves* $\nabla_\eta u = 1$, *and that* $|u_k| \lesssim |\eta|^{-(2k-1)}$. *We then have* $\nabla_\eta u = 1 - [\Delta]$.

Proof. Let u_{BM} be the form defined by (7), and let u be a form satisfying the conditions in the proposition. We know that $\nabla(u\wedge u_{\mathrm{BM}}) = u_{\mathrm{BM}} - u$ pointwise outside Δ, in light of (4). We want to show that this also holds in the current sense, i.e.

$$\int_{\mathbb{C}^n} \nabla(u\wedge u_{\mathrm{BM}})\wedge\phi = \int_{\mathbb{C}^n} (u_{\mathrm{BM}} - u)\wedge\phi, \tag{8}$$

where ϕ is a test form in $\Omega\times\Omega$. Using firstly that $u\wedge u_{\mathrm{BM}}$ is locally integrable (since $u\wedge u_{\mathrm{BM}} = \mathcal{O}(|\eta|^{-(2n-2)})$ near Δ), and secondly (5), we get that

$$\begin{aligned}\int_{\mathbb{C}^n} \nabla(u\wedge u_{\mathrm{BM}})\wedge\phi &= -\lim_{\varepsilon\to0}\int_{|\eta|>\varepsilon} (u\wedge u_{\mathrm{BM}})\wedge\nabla\phi \\ &= \lim_{\varepsilon\to0}\biggl(\int_{|\eta|=\varepsilon} u\wedge u_{\mathrm{BM}}\wedge\phi + \int_{|\eta|>\varepsilon}\nabla(u\wedge u_{\mathrm{BM}})\wedge\phi\biggr).\end{aligned} \tag{9}$$

The boundary integral in (9) will converge to 0 when $\varepsilon\to0$, as $u\wedge u_{\mathrm{BM}} = \mathcal{O}(|\eta|^{-2n+2})$ and $\mathrm{Vol}(\{\eta:|\eta|=\varepsilon\}\cap\mathrm{supp}(\phi)) = \mathcal{O}(\varepsilon^{2n-1})$. As for the last integral in (9), we get

$$\lim_{\varepsilon\to0}\int_{|\eta|>\varepsilon}\nabla(u\wedge u_{\mathrm{BM}})\wedge\phi = \lim_{\varepsilon\to0}\int_{|\eta|>\varepsilon}(u_{\mathrm{BM}} - u)\wedge\phi = \int_{\mathbb{C}^n}(u_{\mathrm{BM}} - u)\wedge\phi,$$

since $u_{\mathrm{BM}} - u$ is locally integrable. Thus $\nabla(u\wedge u_{\mathrm{BM}}) = u_{\mathrm{BM}} - u$ as currents. It follows that $\nabla u = \nabla u_{\mathrm{BM}}$ since $\nabla^2 = 0$, and since u_{BM} satisfies (6), u must also do so. □

Example 1. If s is a smooth $(1,0)$-form in $\Omega\times\Omega$ such that $|s| \lesssim |\eta|$ and $|\delta_\eta s| \gtrsim |\eta|^2$, we can set $u = s/\nabla s$. By Proposition 2.2, u will satisfy (6), and

$$u_{n,n-1} = \frac{s\wedge(\bar\partial s)^{n-1}}{(\delta_\eta s)^n}$$

is the classical Cauchy–Fantappiè–Leray kernel.

We now introduce weights, which will allow us to get more flexible integral formulas.

Definition 1. A form $g \in \mathcal{L}^0(\Omega\times\Omega)$ is a *weight* if $g_{0,0}(z,z) = 1$ and $\nabla_\eta g = 0$.

The form $1 + \nabla Q$ is an example of a weight, if $Q \in \mathcal{L}^{-1}$. In fact, we have considerable flexibility when choosing weights: if Q is a $(1,0)$-form, $g = 1 + \nabla Q$, and

$G(\lambda)$ is a holomorphic function such that $G(0)=1$, then it is easy to see that

$$G(g)=\sum_{k=0}^{n}\frac{1}{k!}G^{(k)}(\delta_\eta Q)(-\bar\partial Q)^k$$

is also a weight. We can now prove the following representation formula.

Theorem 2.3. (Koppelman's formula) *Assume that* $D\Subset\Omega$, $\phi\in\mathcal{E}_{p,q}(\overline{D})$, *and that the current* K *and the smooth form* P *solve the equation*

$$\bar\partial K=[\Delta]-P. \tag{10}$$

We then have

$$\begin{aligned}\phi(z)=&\int_{\partial D_\zeta}K(\zeta,z)\wedge\phi(\zeta)+\int_{D_\zeta}K(\zeta,z)\wedge\bar\partial\phi(\zeta)\\&+\bar\partial_z\int_{D_\zeta}K(\zeta,z)\wedge\phi(\zeta)+\int_{D_\zeta}P(\zeta,z)\wedge\phi(\zeta).\end{aligned}\tag{11}$$

Proof. First assume that ϕ has compact support in D, so that the first integral in (11) vanishes. Take a test form $\psi(z)$ of bidegree $(n-p,n-q)$ in Ω. Then we have

$$\begin{aligned}&\int_{\mathbb{C}^n_z}\left(\int_{\mathbb{C}^n_\zeta}K\wedge\bar\partial\phi+\bar\partial_z\int_{\mathbb{C}^n_\zeta}K\wedge\phi+\int_{\mathbb{C}^n_\zeta}P\wedge\phi\right)\wedge\psi\\&\quad=\int_{\mathbb{C}^n_z\times\mathbb{C}^n_\zeta}K\wedge d\phi\wedge\psi+(-1)^{p+q}\int_{\mathbb{C}^n_z\times\mathbb{C}^n_\zeta}K\wedge\phi\wedge d\psi+\int_{\mathbb{C}^n_z\times\mathbb{C}^n_\zeta}P\wedge\phi\wedge\psi\\&\quad=\int_{\mathbb{C}^n_z\times\mathbb{C}^n_\zeta}K\wedge d(\phi\wedge\psi)+\int_{\mathbb{C}^n_z\times\mathbb{C}^n_\zeta}P\wedge\phi\wedge\psi\\&\quad=\int_{\mathbb{C}^n_z\times\mathbb{C}^n_\zeta}dK\wedge\phi\wedge\psi+\int_{\mathbb{C}^n_z\times\mathbb{C}^n_\zeta}P\wedge\phi\wedge\psi\\&\quad=\int_{\mathbb{C}^n_z}\phi\wedge\psi,\end{aligned}$$

where we use Stokes' theorem repeatedly. If ϕ does not have compact support in D, we can prove the general case, e.g., by replacing ϕ with $\chi_k\phi$, where $\chi_k\to\chi_D$, and let $k\to\infty$. $\square$

It is easy to obtain K and P which solve (10): If we take g to be a weight and u to be a solution of (6), then we can solve the equation

$$\nabla_\eta v=g-[\Delta]$$

by choosing $v=u\wedge g$. This means that $K=(u\wedge g)_{n,n-1}$ and $P=g_{n,n}$ will solve (10).

Example 2. Let

$$g(\zeta,z)=1-\nabla\frac{1}{2\pi i}\frac{\bar{\zeta}\cdot d\eta}{1+|\zeta|^2}=\frac{1+\bar{\zeta}\cdot z}{1+|\zeta|^2}-\bar{\partial}\frac{i}{2\pi}\frac{\bar{\zeta}\cdot d\eta}{1+|\zeta|^2},$$

then g is a weight for all (ζ,z). Take a (p,q)-form $\phi(\zeta)$ which grows polynomially as $|\zeta|\to\infty$. If we let $K=(u\wedge g^k)_{n,n-1}$ and $P=(g^k)_{n,n}$, then

$$\phi(z)=\int_{|\zeta|=R}K\wedge\phi+\int_{|\zeta|\le R}K\wedge\bar{\partial}\phi+\bar{\partial}_z\int_{|\zeta|\le R}K\wedge\phi+\int_{|\zeta|\le R}P\wedge\phi.$$

If k is large enough, then the weight will compensate for the growth of ϕ, so that the boundary integral will go to zero when $R\to\infty$. We get the representation

$$\phi(z)=\int_{\mathbb{C}^n}K\wedge\bar{\partial}\phi+\bar{\partial}_z\int_{\mathbb{C}^n}K\wedge\phi+\int_{\mathbb{C}^n}P\wedge\phi.$$

Note that if ϕ in (11) is a closed form and the first and fourth terms of the right-hand side of Koppelman's formula vanish, we get a solution of the $\bar{\partial}$-problem for ϕ. Note also that the proof of Koppelman's formula works equally well over $X\times X$, where X is any complex manifold, provided that we can find K and P such that (10) holds. The purpose of the next section is to find such K and P for a special type of manifold.

3. A method for finding weighted Koppelman formulas on manifolds

We will now describe a method which can be used to find integral formulas on manifolds in certain cases, and which is modelled on the one in the previous section. The method is similar to one presented in [8], see Remark 2 at the end of this section for a comparison.

Let X be a complex manifold of dimension n, and let $E\to X_\zeta\times X_z$ be a vector bundle of rank n such that we can find a holomorphic section η of E that defines the diagonal $\Delta=\{(\zeta,z):\zeta=z\}$ of $X\times X$. In other words, η must vanish to the first order on Δ and be non-zero elsewhere. Let $\{e_j\}_{j=1}^n$ be a local frame for E, and $\{e_j^*\}_{j=1}^n$ the dual local frame for E^*. Contraction with η is an operation on E^* which we denote by δ_η; if $\eta=\sum_{j=1}^n\eta_je_j$ then

$$\delta_\eta\biggl(\sum_{j=1}^n\sigma_je_j^*\biggr)=\sum_{j=1}^n\eta_j\sigma_j.$$

Set

$$\nabla_\eta=\delta_\eta-\bar{\partial}.$$

Choose a Hermitian metric h for the vector bundle E, let D_E be the Chern connection on E, and D_{E^*} the induced connection on E^*. Consider $G_E = C^\infty(X\times X, \Lambda[T^*(X\times X)\oplus E\oplus E^*])$. If A lies in $C^\infty(X\times X, T^*(X\times X)\otimes E\otimes E^*)$, then we define $\tilde{A}$ as the corresponding element in G_E, arranged with the differential form first, then the section of E and finally the section of E^*. For example, if $A=dz_1\otimes e_1\otimes e_1^*$, then $\tilde{A}=dz_1\wedge e_1\wedge e_1^*$.

To define a derivation D on G_E, we first let $Df=\widetilde{D_E f}$ for a section f of E, and $Dg=\widetilde{D_{E^*} g}$ for a section g of E^*. We then extend the definition by

$$D(\xi_1\wedge\xi_2) = D\xi_1\wedge\xi_2 + (-1)^{\deg\xi_1}\xi_1\wedge D\xi_2,$$

where $D\xi_i=d\xi_i$ if ξ_i happens to be a differential form, and $\deg\xi_1$ is the total degree of ξ_1. For example, $\deg(\alpha\wedge e_1\wedge e_1^*)=\deg\alpha+2$, where $\deg\alpha$ is the degree of α as a differential form. We let

$$\mathcal{L}^m = \bigoplus_{p=0}^{n} C^\infty(X\times X, \Lambda^p E^*\wedge\Lambda^{p+m}T^*_{0,1}(X\times X));$$

note that $\mathcal{L}^m$ is a subspace of G_E. The operator ∇ will act in a natural way as $\nabla\colon \mathcal{L}^m\to\mathcal{L}^{m+1}$. Notice also the analogy with the construction (3) in $\mathbb{C}^n$. As before, if $f\in\mathcal{L}^m$ and $g\in\mathcal{L}^k$, then $f\wedge g\in\mathcal{L}^{m+k}$. We also see that ∇ obeys Leibniz' rule, and that $\nabla^2=0$. Let $\mathrm{End}(E)$ denote the bundle of endomorphisms of E.

Proposition 3.1. *If v is a differential form taking values in* $\mathrm{End}(E)$*, and $D_{\mathrm{End}(E)}$ is the induced Chern connection on* $\mathrm{End}(E)$*, then*

$$\widetilde{D_{\mathrm{End}(E)} v} = D\tilde{v}. \tag{12}$$

Proof. Suppose that $v=f\otimes g$, where f is a section of E and g a section of E^*. We prove first that

$$D_{\mathrm{End}(E)} v = D_E f\otimes g + f\otimes D_{E^*} g. \tag{13}$$

In fact, if s takes values in E, we have

$$\begin{aligned}(D_{\mathrm{End}(E)} v).s &= D_E((g.s)f) - (g.(D_E s))f = d(g.s)f + (g.s)D_E f - (g.(D_E s))f\\ &= (g.s)D_E f + (D_{E^*} g.s)f = (D_E f\otimes g + f\otimes D_{E^*} g).s,\end{aligned}$$

which proves (13). We have

$$\widetilde{D_{\mathrm{End}(E)} v} = \widetilde{D_E f\otimes g} + \widetilde{f\otimes D_{E^*} g} = Df\wedge g - f\wedge Dg = D\tilde{v}$$

which proves (12). If $v=\alpha\otimes f\otimes g$, where α is a differential form, we would have $D_{\mathrm{End}(E)} v = d\alpha\otimes f\otimes g + (-1)^{\deg\alpha}\alpha\otimes D_{\mathrm{End}(E)}(f\otimes g)$, so the result follows by an application of $\sim$. Since any differential form taking values in $\mathrm{End}(E)$ is a sum of such elements, the result follows by linearity. □

Definition 2. For a form $f(\zeta,z)$ on $X\times X$, we define

$$\int_E f(\zeta,z)\wedge e_1\wedge e_1^*\wedge\ldots\wedge e_n\wedge e_n^* = f(\zeta,z).$$

Note that if I is the identity on E, then $\tilde{I}=e\wedge e^*=e_1\wedge e_1^*+\ldots+e_n\wedge e_n^*$. It follows that $\tilde{I}_n=e_1\wedge e_1^*\wedge\ldots\wedge e_n\wedge e_n^*$, with the notation $a_n=a^n/n!$, so the definition above is independent of the choice of frame.

Proposition 3.2. *If* $F\in G_E$ *then*

$$d\int_E F=\int_E DF.$$

Proof. If $F=f\wedge\tilde{I}_n$ we have $d\int_E F=df$ and

$$\int_E DF=\int_E [df\wedge\tilde{I}_n\pm f\wedge D(\tilde{I}_n)].$$

It is obvious that $D_{\mathrm{End}(E)}I=0$, and by Proposition 3.1 it follows that $D\tilde{I}=0$, so we are finished. □

We will now construct integral formulas on $X\times X$. As a first step, we find a section σ of E^* such that $\delta_\eta\sigma=1$ outside Δ. For reasons that will become apparent, we choose σ to have minimal pointwise norm with respect to the metric h, which means that $\sigma=\sum_{j,k=1}^n h_{jk}\bar{\eta}_k e_j^*/|\eta|^2$. Close to Δ, it is obvious that $|\sigma|\lesssim 1/|\eta|$, and a calculation shows that we also have $|\bar{\partial}\sigma|\lesssim 1/|\eta|^2$. Next, we construct a section u with the property that $\nabla u=1-R$, where R has support on Δ. We set

$$u=\frac{\sigma}{\nabla_\eta\sigma}=\sum_{k=0}^{\infty}\sigma\wedge(\bar{\partial}\sigma)^k. \tag{14}$$

Note that $u\in\mathcal{L}^{-1}$. By $u_{k,k-1}$ we will mean the term in u with degree k in E^* and degree $k-1$ in $T^*_{0,1}(X\times X)$. It is easily checked that $\nabla u=1$ outside Δ.

We will need the following lemma.

Lemma 3.3. *If* Θ *is the Chern curvature tensor of* E, *then*

$$\nabla_\eta\left(\frac{D\eta}{2\pi i}+\frac{i\widetilde{\Theta}}{2\pi}\right)=0.$$

Proof. The lemma will follow from the more general statement that if v takes values in $\mathrm{End}(E)$, then $\delta_\eta \tilde{v} = -v.\eta$. In fact, let $v = f \otimes g$, where f is a section of E and g a section of E^*; then we have $\delta_\eta(f \wedge g) = -f \wedge \eta . g = -(f \otimes g).\eta$. Now, note that $\bar\partial\widetilde{\Theta} = 0$ since D is the Chern connection. We have

$$\nabla_\eta\left(\frac{D\eta}{2\pi i} + \frac{i\widetilde{\Theta}}{2\pi}\right) = -\frac{1}{2\pi i}[\bar\partial D\eta + \delta_\eta \widetilde{\Theta}] = -\frac{1}{2\pi i}[\Theta\eta - \Theta\eta] = 0.$$

In the calculations we use that η is holomorphic and that $\bar\partial\theta = \Theta$, where θ is the connection matrix of D_E with respect to the frame e. □

The following theorem yields a Koppelman formula by Theorem 2.3.

Theorem 3.4. *Let $E \to X \times X$ be a vector bundle with a section η which defines the diagonal Δ of $X \times X$. We have*

$$\bar\partial K = [\Delta] - P,$$

where

$$K = \int_E u \wedge \left(\frac{D\eta}{2\pi i} + \frac{i\widetilde{\Theta}}{2\pi}\right)_n \quad \text{and} \quad P = \int_E \left(\frac{D\eta}{2\pi i} + \frac{i\widetilde{\Theta}}{2\pi}\right)_n, \tag{15}$$

and u is defined by (14).

Note that since $D\eta$ contains no e_i's, we have

$$P = \int_E \left(\frac{i\widetilde{\Theta}}{2\pi}\right)_n = \det \frac{i\Theta}{2\pi} = c_n(E),$$

i.e. the nth Chern class of E.

Proof. We claim that

$$\frac{1}{(2\pi i)^n} \int_E R \wedge (D\eta)_n = [\Delta], \tag{16}$$

where R is defined by $\nabla u = 1 - R$. If this were true, we would have, by Lemma 3.3 and Proposition 3.2,

$$\begin{aligned} \bar\partial \int_E u \wedge \left(\frac{D\eta}{2\pi i} + \frac{i\widetilde{\Theta}}{2\pi}\right)_n &= \int_E \bar\partial \left[u \wedge \left(\frac{D\eta}{2\pi i} + \frac{i\widetilde{\Theta}}{2\pi}\right)_n\right] = -\int_E \nabla \left[u \wedge \left(\frac{D\eta}{2\pi i} + \frac{i\widetilde{\Theta}}{2\pi}\right)_n\right] \\ &= -\int_E \left(\frac{D\eta}{2\pi i} + \frac{i\widetilde{\Theta}}{2\pi}\right)_n + \frac{1}{(2\pi i)^n} \int_E R \wedge (D\eta)_n = [\Delta] - P. \end{aligned}$$

We want to use Proposition 2.2 to prove the claim (16), so we need to express the left-hand side of (16) in local coordinates. Since η defines Δ, we can choose

$\eta_1, ..., \eta_n$ together with some functions $\tau_1, ..., \tau_n$ to form a coordinate system locally in a neighborhood of Δ. We have

$$\frac{1}{(2\pi i)^n}\int_E R\wedge(D\eta)_n = \bar\partial\frac{1}{(2\pi i)^n}\int_E \sigma\wedge(\bar\partial\sigma)^{n-1}\wedge(D\eta)_n,$$

and

$$\int_E \sigma\wedge(\bar\partial\sigma)^{n-1}\wedge(D\eta)_n = s\wedge(\bar\partial s)^{n-1} + A,$$

where $s=\sum_{j=1}^n \sigma_j\, d\eta_j$ and A contains only terms which lack some $d\eta_j$, i.e., every term in A will contain at least one η_j. Note that both s and A are now forms in $\mathbb{C}^n$. Recall that we have $|\sigma|\lesssim 1/|\eta|$ and $|\bar\partial\sigma|\lesssim 1/|\eta|^2$ close to Δ (this is why we chose σ to have minimal norm). Thus, by Theorem 2.2 we know that

$$\bar\partial[s\wedge(\bar\partial s)^{n-1}] = [\Delta],$$

so it suffices to show that $\bar\partial A=0$ in the current sense. But since every term in A contains at least one η_j, the singularities which come from the σ_j's and $\bar\partial\sigma_j$'s will be partially cancelled out, and in fact we have $A=\mathcal{O}(|\eta|^{-2n+2})$. A calculation shows also that $\bar\partial A=\mathcal{O}(|\eta|^{-2n+1})$, and it follows that $\bar\partial A=0$ (also cf. the proof of Proposition 2.2).

It should be obvious from the proof that instead of $u=\sigma/\nabla\sigma$, we can choose any u such that $\nabla u=1$ outside Δ and $|u_{k,k-1}|\lesssim|\eta|^{-2k+1}$.

We will obtain more flexible formulas if we use weights.

Definition 3. The section $g\in\mathcal{L}_0$ is a *weight* if we have $\nabla g=0$ and $g_{0,0}(z,z)=1$.

Theorem 3.4 goes through with essentially the same proof if we take

$$K=\int_E u\wedge g\wedge\left(\frac{D\eta}{2\pi i}+\frac{i\widetilde{\Theta}}{2\pi}\right)_n \quad\text{and}\quad P=\int_E g\wedge\left(\frac{D\eta}{2\pi i}+\frac{i\widetilde{\Theta}}{2\pi}\right)_n, \tag{17}$$

as shown by the calculation

$$\bar\partial K = -\int_E \nabla u\wedge g\wedge\left(\frac{D\eta}{2\pi i}+\frac{i\widetilde{\Theta}}{2\pi}\right)_n = -\int_E (g-R)\wedge\left(\frac{D\eta}{2\pi i}+\frac{i\widetilde{\Theta}}{2\pi}\right)_n = [\Delta]-P,$$

which follows from the proof of Theorem 3.4 and the properties of weights. In the next section we will make use of weighted formulas.

Remark 1. If L is a line bundle over X, let L_ζ denote the line bundle over $X_\zeta \times X_z$ defined by $\pi^{-1}(L)$, where $\pi\colon X_\zeta \times X_z \to X_\zeta$. If we want to find formulas for (p,q)-forms $\phi(\zeta)$ taking values in some line bundle L over X, we can use a weight g taking values in $L_z \otimes L_\zeta^*$. In fact, then K and P will also take values in $L_z \otimes L_\zeta^*$, so that $\phi \wedge K$ and $\phi \wedge P$ take values in L_z. Integrating over ζ, we obtain $\phi(z)$ taking values in L.

Remark 2. To obtain more general formulas, one can find forms K and P such that

$$dK = [\Delta] - P \tag{18}$$

by setting $\nabla'_\eta = \delta_\eta - D$ and checking that the corresponding Lemma 3.3 and Theorem 3.4 are still valid. The main difference lies in the fact that since $(\nabla')^2 \neq 0$, we do not have $\nabla' u = 1$ outside Δ, but rather

$$\nabla' u = 1 - \frac{\sigma}{(\nabla'\sigma)^2} \wedge (\nabla')^2 \sigma.$$

A calculation shows that $(\nabla')^2\sigma = \delta_\sigma(D\eta - \widetilde{\Theta})$, where δ_σ operates on sections of E. We have $\delta_\sigma(D\eta - \widetilde{\Theta}) \wedge (D\eta - \widetilde{\Theta})^n = \delta_\sigma(D\eta - \widetilde{\Theta})^{n+1} = 0$ for degree reasons, so that Theorem 3.4 will still hold with ∇ replaced by ∇'. We can use weights in the same way, if we require that a weight g has the property $\nabla' g = 0$ instead of $\nabla g = 0$. In this article we are interested in applications which only require the formulas obtained by using ∇.

In [8] Berndtsson obtains P and K satisfying (18) by different means, resulting in the same formulas, but without weights. Also noteworthy is that ∇' is a superconnection in the sense of Quillen [21], and our ∇ is the $(0,1)$-part of this superconnection. Lemma 3.3 for ∇' is a Bianchi identity for the superconnection.

4. Weighted Koppelman formulas on $\mathbb{P}^n$

We will now apply the method of the previous section to $X = \mathbb{P}^n$. We let $[\zeta] \in \mathbb{P}^n$ denote the equivalence class of $\zeta \in \mathbb{C}^{n+1}$. In order to construct the bundle E, we first let $F' = \mathbb{C}^{n+1} \times (\mathbb{P}^n_{[\zeta]} \times \mathbb{P}^n_{[z]})$ be the trivial bundle of rank $n+1$ over $\mathbb{P}^n_{[\zeta]} \times \mathbb{P}^n_{[z]}$. We next let F be the bundle of rank n over $\mathbb{P}^n_{[\zeta]} \times \mathbb{P}^n_{[z]}$ which has the fiber $\mathbb{C}^{n+1}/(\zeta)$ at the point $([\zeta],[z])$; F is thus a quotient bundle of F'. If α is a section of F', we denote its equivalence class in F with $[\alpha]$. We will not always bother with writing out the brackets, since it will usually be clear from the context whether a section is to be seen as taking values in F' or F. Let L^{-1} denote the *tautological line bundle* of $\mathbb{P}^n$, that is,

$$L^{-1}=\{([\zeta],\xi)\in\mathbb{P}^n\times\mathbb{C}^{n+1}:\xi\in\mathbb{C}\zeta\}$$

We also define $L^{-k}=(L^{-1})^{\otimes k}$, $L^1=(L^{-1})^*$ and $L^k=(L^1)^{\otimes k}$. Finally, let $E=F\otimes L^1_{[z]}\to\mathbb{P}^n_{[\zeta]}\times\mathbb{P}^n_{[z]}$. Observe that E is thus a subbundle of $E'=F'\otimes L^1_{[z]}$. It follows that $E^*=F^*\otimes L^{-1}_{[z]}$, where $F^*=\{\xi\in(F')^*:\xi\cdot\zeta=0\}$. Berndtsson has the same setup in [8, Example 3, p. 337], but does not develop it as much (cf. Remark 2 above).

A remark on notation: we will write a differential form $\alpha([\zeta])$ on $\mathbb{P}^n$ that takes values in L^k as a projective form on $\mathbb{C}^{n+1}$ which is k-homogeneous. That is, α will satisfy $\alpha(\lambda\zeta)=\alpha(\zeta)$, where $\lambda\in\mathbb{C}$, and $\delta_\zeta\alpha=\delta_{\bar\zeta}\alpha=0$, where δ_ζ is contraction with the vector field $\zeta\cdot\partial/\partial\zeta$ and similarly for $\delta_{\bar z}$.

Let $\{e_j\}_{j=1}^n$ be an orthonormal basis of F'. The section η (cf. Section 3) will be $\eta=z\cdot e=z_0e_0+...+z_ne_n$. Note that η takes values in $(F')\otimes L^1_{[z]}$, and will thus define an equivalence class in $F\otimes L^1_{[z]}=E$. The section η defines the diagonal since $[\eta(\zeta,\zeta)]=[\zeta\cdot e]=[0]$, so that η vanishes to the first order on Δ.

We will now choose a metric on E. On F' we choose the trivial metric, which induces the trivial metric also on $(F')^*$ and F^*. For $[\omega]$ taking values in $F=F'/(\zeta)$, the metric induced from F' is $\|[\omega]\|_F=\|\omega-\pi\omega\|_{F'}$, where π is the orthogonal projection $F'\to(\zeta)$. We choose the metric on $E=F\otimes L^1_{[z]}$ to be

$$\|\alpha\otimes[\omega]\|_E=\|\omega-\pi\omega\|_{F'}\frac{|\alpha|}{|z|} \tag{19}$$

for $\alpha\otimes[\omega]\in E$. We introduce the notation $\alpha\cdot\gamma:=\alpha_0\wedge\gamma_0+...+\alpha_n\wedge\gamma_n$, where α and γ are tuples containing differential forms or sections of a bundle.

Proposition 4.1. *Let $\omega\cdot e$ be a section of E. The Chern connection and curvature of E are*

$$D_E(\omega\cdot e)=d\omega\cdot e-\frac{d\zeta\cdot e}{|\zeta|^2}\wedge\bar\zeta\cdot\omega-\partial\log|z|^2\wedge\omega\cdot e, \tag{20}$$

$$\widetilde\Theta_E=\partial\bar\partial\log|z|^2\wedge e^*\cdot e-\bar\partial\frac{\bar\zeta\cdot e^*}{|\zeta|^2}\wedge d\zeta\cdot e, \tag{21}$$

with respect to the metric (19) *and expressed in the frame* $\{e_j\}_{j=1}^n$ *for* F'.

Proof. We begin with finding D_F. Let $\widehat\omega\cdot e=(\omega\cdot\bar\zeta/|\zeta|^2)\zeta\cdot e$ be the projection of $\omega\cdot e$ onto $(\zeta\cdot e)$. Since the Chern connection $D_{F'}$ on F' is just d, it is easy to show that $D_F[\omega\cdot e]=[d(\omega\cdot e-\widehat\omega\cdot e)]$. We have

$$D_F[\omega\cdot e]=[d(\omega\cdot e-\widehat\omega\cdot e)]=\left[d\omega\cdot e-\frac{d\zeta\cdot e}{|\zeta|^2}\wedge\bar\zeta\cdot\omega\right],$$

since if d does not fall on ζ in the second term we get something that is in the zero equivalence class in F. If $\omega\cdot e$ is projective to start with, so will $d\omega\cdot e$ be, and $d\zeta\cdot e$ is a projective form since $\delta_\zeta(d\zeta\cdot e)=\zeta\cdot e=0$ in F.

Since the metric on $L^1_{[z]}$ in the local frame z_0 is $|z_0|^2/|z|^2$, the local connection matrix will be $\partial\log(|z_0|^2/|z|^2)$. If ξ takes values in $L^1_{[z]}$, we get

$$D_{L^1_{[z]}}\xi=\left[d(\xi/z_0)+\frac{\partial\log(|z_0|^2/|z|^2)\xi}{z_0}\right]z_0=d\xi-\partial\log|z|^2\xi.$$

It is easy to see that $d(\xi/z_0)+\partial\log(|z_0|^2/|z|^2)\xi/z_0$ is a projective form, so $d\xi-\partial\log|z|^2\xi$ is also projective. Combining the contributions from $L^1_{[z]}$ and F, we get (20), from which also (21) follows. □

We want to find the solution σ to the equation $\delta_\eta\sigma=1$, such that σ has minimal norm in E^*. It is easy to see that the section $\bar z\cdot e^*/|z|^2$ is the minimal solution to $\delta_\eta v=1$ in the bundle $(E')^*=(F')^*\otimes L^{-1}_{[z]}$. The projection of $\bar z\cdot e^*/|z|^2$ onto the subspace E^* is

$$s=\frac{\bar z\cdot e^*}{|z|^2}-\frac{\bar z\cdot\zeta}{|\zeta|^2|z|^2}\bar\zeta\cdot e^*.$$

Since $\bar z\cdot e^*/|z|^2$ is minimal in $(F')^*\otimes L^{-1}_{[z]}$, s must be the minimal solution in E^*.

Finally, we normalize to get $\sigma=s/\delta_\eta s$. According to the method of the previous section, we can then set $u=\sigma/\nabla\sigma$ and obtain the forms P and K which will give us a Koppelman formula (see Theorem 3.4).

Remark 3. In local coordinates, for example where $\zeta_0, z_0\neq 0$, we have

$$|\eta|^2=\delta_\eta s=\frac{|\zeta|^2|z|^2-|\bar z\cdot\zeta|^2}{|\zeta|^2|z|^2}=\frac{(1+|\zeta'|^2)(1+|z'|^2)-|1+\bar z'\cdot\zeta'|^2}{(1+|\zeta'|^2)(1+|z'|^2)},$$

where $\zeta'=(\zeta_1/\zeta_0,...,\zeta_n/\zeta_0)$ and analogously for z'. For the denominator we locally have $(1+|\zeta'|^2)(1+|z'|^2)\le C$ for some constant C. As for the numerator, we have

$$\begin{aligned}(1+|\zeta'|^2)(1+|z'|^2)-|1+\bar z'\cdot\zeta'|^2&=1+|\zeta'|^2+|z'|^2+|\zeta'|^2|z'|^2-(1+2\operatorname{Re}|\bar z'\cdot\zeta'|+|\bar z'\cdot\zeta'|^2)\\&=|z'-\zeta'|^2+|\zeta'|^2|z'|^2-|\bar z'\cdot\zeta'|^2\ge|z'-\zeta'|^2.\end{aligned}$$

In all, we have $\delta_\eta s\gtrsim|z'-\zeta'|^2$.

To compute integrals of the type (17), we need the following proposition.

Lemma 4.2. *Let* $A\overset{\mathrm{id}}{\hookrightarrow}A'$, *where* A' *is a given vector bundle with a given metric and* $A=\{\xi$ *taking values in* $A'\colon f\cdot\xi=0\}$ *for a fixed* f *taking values in* $(A')^*$. *Let* s *be the dual section to* f, *and* π *be the orthogonal projection* $\pi\colon G_{A'}\to G_A$ *induced* *by the metric on A. If $B'\in G_{A'}$, and $B=\pi B'$, then*

$$\int_A B=\int_{A'} f\wedge s\wedge B'.$$

Proof. We can choose a frame for A' so that $e_0=s$, and then extend it to an orthonormal frame for A', so that $A=\mathrm{Span}(e_1, ..., e_n)$. If we set $e_0^*=f$, we have

$$\int_{A'} f\wedge s\wedge B'=\int_{A'} e_0\wedge e_0^*\wedge \pi B'=\int_A B$$

and we are done, since the integrals are independent of the frame. □

Note that if $E=A\otimes L$, where L is a line bundle, and $B\in G_E$, then $\int_E B=\int_A B$. At least, this is true if we interpret the latter integral to mean that if g is a local frame for L and g^* a local frame for L^*, then g and g^* should cancel out. Since there are as many elements from L as there are from L^*, there will be no line bundle elements left.

We will apply Lemma 4.2 with $A=E$, $A'=E'$ and $f=\zeta\cdot e^*$. We then have

$$P=\int_E \left(\frac{D\eta}{2\pi i}+\frac{i\widetilde{\Theta}}{2\pi}\right)_n=\int_{E'} \frac{\bar{\zeta}\cdot e\wedge\zeta\cdot e^*}{|\zeta|^2}\wedge\left(\frac{D\eta}{2\pi i}+\frac{i\widetilde{\Theta}}{2\pi}\right)_n$$

and similarly for K (this makes it easier to write down P and K explicitly).

By Theorem 3.4, we have

$$\bar{\partial}K=[\Delta]-P.$$

(These K and P are also found at the very end of [8].) We will now modify the method slightly, since in the paper [15] we found formulas for $(0,q)$-forms (derived in a slightly different way) which are more appealing than those we have just found, in that we get better results when we use them to solve $\bar{\partial}$-equations. We would thus like to have formulas for (p,q)-forms that coincide with those of [15] in the $(0,q)$-case.

The bundle F^* is actually isomorphic to $T^*_{1,0}(\mathbb{P}^n_{[\zeta]})$, and an explicit isomorphism is given by $\beta=d\zeta\cdot e$. In fact, if $\xi\cdot e^*$ takes values in F^*, then $\beta(\xi)=d\zeta\cdot\xi$. Since $\xi\cdot\zeta=0$, the contraction of $\beta(\xi)$ with the vector field $\zeta\cdot\partial/\partial\zeta$ will be zero, so $\beta(\xi)\in T^*_{1,0}(\mathbb{P}^n_{[\zeta]})$. If v_{e^*} is a form with values in $\Lambda^n E^*$, then it is easy to see that

$$\int_E v_{e^*}\wedge\beta_n=v_{d\zeta}, \tag{22}$$

where we get $v_{d\zeta}$ by replacing every instance of e_j^* in v_{e^*} with $d\zeta_j$. For example, if $v_{e^*}=f(\zeta,z)e_0^*\wedge...\wedge e_n^*$, then $v_{d\zeta}=f(\zeta,z)d\zeta_0\wedge...\wedge d\zeta_n$. We can use this to construct integral formulas for $(0,q)$-forms with values in $L_{[\zeta]}^{-n}$, by setting

$$K=\int_E u\wedge\beta_n.$$

The formulas we get from this are the same as in [15]. We will now combine these formulas with the ones in (15).

Theorem 4.3. *Let $D\subset\mathbb{P}^n$. If $\phi(\zeta)$ is a (p,q)-form with values in $L_{[\zeta]}^{-n+p}$ and*

$$K_p=\int_E u\wedge\beta_{n-p}\wedge\left(\frac{D\eta}{2\pi i}+\frac{i\widetilde{\Theta}}{2\pi}\right)_p, \tag{23}$$

$$P_p=\int_E \beta_{n-p}\wedge\left(\frac{D\eta}{2\pi i}+\frac{i\widetilde{\Theta}}{2\pi}\right)_p,$$

with $\beta=d\zeta\cdot e^*$, we have the Koppelman formula

$$\phi([z])=\int_{\partial D}\phi K_p\wedge\phi+\int_D\bar\partial K_p\wedge\phi+\bar\partial_{[z]}\int_D K_p\wedge\phi+\int_D P_p\wedge\phi,$$

where the integrals are taken over the $[\zeta]$ variable.

Proof. We have

$$\int_E \bar\partial u\wedge\beta_{n-p}\wedge\left(\frac{D\eta}{2\pi i}+\frac{i\widetilde{\Theta}}{2\pi}\right)_p=[\Delta], \tag{24}$$

where $[\Delta]$ should be integrated against sections of L^{-n+p} with bidegree (p,q). This follows from the proof of Theorem 3.4, since the singularity at Δ comes only from u, and is not affected by exchanging $(D\eta/2\pi i+i\widetilde{\Theta}/2\pi)_{n-p}$ for β_{n-p}.

Using (24), we get

$$\begin{aligned}dK_p&=-\int_E\nabla\left[u\wedge\beta_{n-p}\wedge\left(\frac{D\eta}{2\pi i}+\frac{i\widetilde{\Theta}}{2\pi}\right)_p\right]\\&=-\int_E(\nabla u)\wedge\beta_{n-p}\wedge\left(\frac{D\eta}{2\pi i}+\frac{i\widetilde{\Theta}}{2\pi}\right)_p=[\Delta]-P_p.\end{aligned}$$

The Koppelman formula then follows as in Theorem 2.3. □

To get formulas for other line bundles, we need to use weights (as defined in the previous section). We will use the weight

$$\alpha = \frac{z\cdot\bar{\zeta}}{|\zeta|^2} - 2\pi i\bar{\partial}\frac{\bar{\zeta}\cdot e^*}{|\zeta|^2}.$$

Note that the first term in α takes values in $L^1_{[z]}\otimes L^{-1}_{[\zeta]}$, and the second is a projective form. We then get a Koppelman formula for (p,q)-forms ϕ with values in L^r by using

$$K_{p,r} = \int_E u\wedge\alpha^{n-p+r}\wedge\beta_{n-p}\wedge\left(\frac{D\eta}{2\pi i}+\frac{i\widetilde{\Theta}}{2\pi}\right)_p,$$
$$P_{p,r} = \int_E \alpha^{n-p+r}\wedge\beta_{n-p}\wedge\left(\frac{D\eta}{2\pi i}+\frac{i\widetilde{\Theta}}{2\pi}\right)_p.$$

Remark 4. Let ϕ be a (p,q)-form. Since we cannot raise α to a negative power, how can we get a Koppelman formula if ϕ takes values in L^r where $r<p-n$? In fact, if we look at the proof of the Koppelman formula in Proposition 2.3, we see that the roles of ϕ and ψ are symmetrical: we could just as well use the proof to get a Koppelman formula for the $(n-p,n-q)$-form ψ which takes values in L^{-r}, using the kernels $K_{p,r}$ and $P_{p,r}$ in Theorem 4.3. This is a concrete realization of Serre duality, which in our case says that

$$H^{p,q}(\mathbb{P}^n, L^r)\simeq H^{n-p,n-q}(\mathbb{P}^n, L^{-r}).$$

We will make use of this dual technique when we look at cohomology groups in the next section.

Remark 5. In [7] Berndtsson constructs integral formulas for sections of line bundles over $\mathbb{P}^n$. These formulas coincide with ours in the case $p=0$, but they are not the same in the general (p,q)-case. Nonetheless, they do give the same result as our formulas when used to find the trivial cohomology groups of the line bundles of $\mathbb{P}^n$ (see the next section). More precisely, his formulas can also be used to prove Proposition 5.1 below, but no more, at least not in any obvious way.

5. An application: the cohomology of the line bundles of $\mathbb{P}^n$

Let D in Theorem 4.3 be the whole of $\mathbb{P}^n$; then the boundary integral will disappear. The only obstruction to solving the $\bar{\partial}$-equation is then the term containing $P_{p,r}$. We will use our explicit formula for $P_{p,r}$ to look at the cohomology groups of (p,q)-forms with values in different line bundles, and determine which of them are trivial. We have

$$\begin{aligned} P_{p,r} &= \int_E \beta_{n-p}\wedge\left(\frac{D\eta}{2\pi i}+\frac{i\widetilde{\Theta}}{2\pi}\right)_p\wedge\alpha^{n-p+r} \\ &= \int_{E'} \frac{\bar\zeta\cdot e\wedge\zeta\cdot e^*}{|\zeta|^2}\wedge(d\zeta\cdot e)_{n-p} \\ &\qquad \wedge\left(dz\cdot e-\frac{z\cdot\bar\zeta}{|\zeta|^2}\wedge d\zeta\cdot e-\frac{\partial|z|^2}{|z|^2}z\cdot e+\omega_z e^*\cdot e-\frac{d\bar\zeta\cdot e^*\wedge d\zeta\cdot e}{|\zeta|^2}\right)_p \\ &\qquad \wedge\left(\frac{z\cdot\bar\zeta}{|\zeta|^2}-\bar\partial\frac{\bar\zeta\cdot e^*}{|\zeta|^2}\right)^{n-p+r}, \end{aligned}$$

where $\omega_z=\partial\bar\partial\log|z|^2$. We can now prove the following result.

Proposition 5.1. *From the formula for $P_{p,r}$ above, it follows that the cohomology groups $H^{p,q}(\mathbb{P}^n,L^r)$ are trivial in the following cases*:
(a) $q=p\neq 0,n$ *and* $r\neq 0$*;*
(b) $q=0$*,* $r\leq p$ *and* $(r,p)\neq(0,0)$*;*
(c) $q=n$*,* $r\geq p-n$ *and* $(r,p)\neq(0,n)$*;*
(d) $p<q$ *and* $r\geq -(n-p)$*;*
(e) $p>q$ *and* $r\leq p$*.*

Unfortunately, these are not all the trivial cohomology groups; instead of (d) and (e) we should ideally get that the groups are trivial for $q\neq 0,n,p$ (cf. [12, p. 397]).

Remark 6. Let $\psi(\zeta,z)$ be a differential form in $\mathbb{P}^n_{[\zeta]}\times\mathbb{P}^n_{[z]}$ taking values in some line bundle L_z^k, written as usual in homogeneous coordinates in $\mathbb{C}^{n+1}$. In the proof below, by the integral $\int_{\mathbb{P}^n_\zeta}\psi(\zeta,z)$ we mean the integral over $\mathbb{P}^n_{[\zeta]}$ of the section ψ, regarded as depending on the variables $[\zeta]$ and $[z]$. The point of this notation is to avoid writing out the brackets as far as possible.

Proof. The general strategy is this: we take a $\bar\partial$-closed form $\phi(z)$ of given bidegree and with values in a given line bundle, and then try to show that $\phi(z)$ is exact by means of the Koppelman formula. One possibility of doing this is proving that $\int_{\mathbb{P}^n_\zeta}\phi(\zeta)\wedge P_{p,r}(\zeta,z)=0$, which can be either because the integrand is zero, or because the integrand is $\bar\partial_\zeta$-exact (since then Stokes' formula can be applied). Another possibility is proving that $P_{p,r}$ is $\bar\partial_z$-exact, since then $\int_{\mathbb{P}^n_\zeta}\phi\wedge P_{p,r}$ will be $\bar\partial_z$-exact as well.

(a) Let $r>0$ and $p=q\neq 0,n$; we must then look at the term in $P_{p,r}$ with bidegree (p,p) in z and $(n-p,n-p)$ in ζ, it is equal to

$$C\int_{E'}\frac{\bar\zeta\cdot e\wedge\zeta\cdot e^*}{|\zeta|^2}\wedge(d\zeta\cdot e)_{n-p}\wedge(\omega_z\wedge e^*\cdot e)^p\wedge\left(\frac{z\cdot\bar\zeta}{|\zeta|^2}\right)^r\wedge\left(\frac{d\bar\zeta\cdot e^*}{|\zeta|^2}\right)^{n-p},\tag{25}$$

where C is a constant. We will show that (25) is actually $\bar\partial_z$-exact. The factor in (25) which depends on z is $(z\cdot\bar\zeta)^r\omega_z^p$, which is at least a $\bar\partial_z$-closed form. Can we write $(z\cdot\bar\zeta)^r\omega_z^p=\bar\partial_z g(z)$, where g is a projective form? Actually, we have $\bar\partial_z[(\bar\zeta\cdot z)^r\partial|z|^2/|z|^2\wedge\omega_z^{p-1}]=(z\cdot\bar\zeta)^r\omega_z^p$, but $(\bar\zeta\cdot z)^r\partial|z|^2/|z|^2\wedge\omega_z^{p-1}$ is not a projective form. This can be remedied by adding a $\bar\partial$-closed term $(\bar\zeta\cdot z)^{r-1}(\bar\zeta\cdot dz)\wedge\omega_z^{p-1}$, since then we can take

$$g=(\bar\zeta\cdot z)^{r-1}\left[(\bar\zeta\cdot z)\frac{\partial|z|^2}{|z|^2}-\bar\zeta\cdot dz\right]\wedge\omega_z^{p-1}.$$

As (25) is $\bar\partial_z$-exact, we have proved (a) when $r>0$. If $-r<0$, by Remark 4 in the previous section we must look at $P_{n-p,r}$, which is again $\bar\partial_z$-exact, and then $\int_{\mathbb{P}^n_z}\phi(z)\wedge P_{n-p,r}=0$ by Stokes' theorem.

(b) Note that here we really want to prove that $\phi=0$, since ϕ cannot be $\bar\partial$-exact. To prove this we again use the dual case in Remark 4. We want to show that $\int_{\mathbb{P}^n_z}\phi(z)\wedge P_{n-p,r}(\zeta,z)=0$, when $\phi(z)$ has bidegree $(p,0)$ and takes values in L_z^{-r}. First assume that $p>0$, then we must look at the term in $P_{n-p,r}$ of bidegree $(n-p,n)$ in z. No term in $P_{n-p,r}$ has a higher degree in $d\bar z$ than in dz, so $\int_{\mathbb{P}^n_z}\phi(z)\wedge P_{n-p,r}(\zeta,z)=0$. If $p=0$, then we must look at the term in $P_{n,r}$ with bidegree (n,n) in z and $(0,0)$ in ζ. The z-dependent factor of this term is $(z\cdot\bar\zeta)^r\omega_z^n$, which is $\bar\partial_z$-exact in the same way as in the proof of (a). This proves the case $p=0$, $-r<0$, but the proof breaks down when $r=0$, where there is a non-trivial cohomology.

(c) First let $p<n$. There is no term in $P_{p,r}$ with bidegree (p,n) in z, since there are not enough $d\bar z$'s, so $\int_{\mathbb{P}^n_\zeta}\phi(\zeta)\wedge P_{p,r}(\zeta,z)=0$. If $p=n$, we look at the term in $P_{p,r}$ with bidegree (n,n) in z and $(0,0)$ in ζ. This is dealt with exactly as the case $p=0$ in the proof of (b).

(d) and (e) Let $q\neq 0,n,p$. If $p<q$ and $r\geq -(n-p)$, we look at the term in P_r with bidegree (p,q) in z. It is zero, since we cannot have more $d\bar z$'s than dz's, so $\int_{\mathbb{P}^n_\zeta}\phi(\zeta)\wedge P_{p,r}=0$. Similarly, if $p>q$ we use the dual method: the term in $P_{n-p,r}$ with bidegree $(n-p,n-q)$ in z is zero when $n-p<n-q$ and $r\geq -p$, again since we cannot have more $d\bar z$'s than dz's. This shows that $\int_{\mathbb{P}^n_z}\phi(z)\wedge P_{n-p,r}=0$ for $r\geq -p$, where ϕ takes values in L^{-r} and $-r\leq p$. $\square$

6. Weighted Koppelman formulas on $\mathbb{P}^n \times \mathbb{P}^m$

We will now find integral formulas on $\mathbb{P}^n \times \mathbb{P}^m$. Let $([\zeta],[\tilde{\zeta}],[z],[\tilde{z}])$ be a point in $(\mathbb{P}^n \times \mathbb{P}^m)\times(\mathbb{P}^n \times \mathbb{P}^m)$. The procedure will be quite similar to that of Section 4, but for simplicity we will limit ourselves to the case of $(0,q)$-forms. This corresponds to using only β in the formula (23). According to (22), then, we can construct our kernel directly, without any need to refer to the bundle E, in the following way (see also [15]). Let $\eta_\zeta = 2\pi i z\cdot\partial/\partial\zeta$ and $\eta = \eta_\zeta + \eta_{\tilde{\zeta}}$. We take δ_η to be contraction with η and set $\nabla = \delta_\eta - \bar{\partial}$. Note that $\eta = 0$ on Δ. Now set

$$s_\zeta = \frac{\bar{z}\cdot d\zeta}{|z|^2} - \frac{\bar{z}\cdot\zeta}{|z|^2|\zeta|^2}\bar{\zeta}\cdot d\zeta$$

and then $s = s_\zeta + s_{\tilde{\zeta}}$. Observe that $\delta_\eta s$ is a scalar, which is zero only on Δ.

Proposition 6.1. *If $u = s/\nabla s$, then u satisfies $\nabla u.\phi = (1-[\Delta]).\phi$, where ϕ is a form of bidegree $(n+m, n+m)$ which takes values in $L^{-n}_{[\zeta]} \otimes L^{-m}_{[\tilde{\zeta}]} \otimes L^{n}_{[z]} \otimes L^{m}_{[\tilde{z}]}$ and contains no $d\zeta_i$'s or $d\tilde{\zeta}_i$'s.*

Proof. The restriction on ϕ is another way of saying that our formulas only will work for $(0,q)$-forms. The proposition will follow from Theorem 4.3 if we integrate in $\mathbb{P}^n_{[\zeta]} \times \mathbb{P}^n_{[z]}$ and $\mathbb{P}^m_{[\tilde{\zeta}]} \times \mathbb{P}^m_{[\tilde{z}]}$ separately. $\square$

To obtain weighted formulas, let

$$\alpha = \frac{z\cdot\bar{\zeta}}{|\zeta|^2} + 2\pi i \partial\bar{\partial}\log|\zeta|^2,$$

and let $\widetilde{\alpha}$ be the corresponding form in $([\tilde{\zeta}],[\tilde{z}])$. We have $\nabla\alpha = \nabla\widetilde{\alpha} = 0$, so

$$\nabla(\alpha^{n+k}\wedge\widetilde{\alpha}^{m+l}\wedge u) = \alpha^{n+k}\wedge\widetilde{\alpha}^{m+l}\wedge\nabla u = \alpha^{n+k}\wedge\widetilde{\alpha}^{m+l} - [\Delta],$$

where $[\Delta]$ must be integrated against sections of $L^k_{[\zeta]}\otimes L^l_{[\tilde{z}]}$. The following theorem follows from Theorem 2.3.

Theorem 6.2. *If $K = \alpha^{n+k}\wedge\widetilde{\alpha}^{m+l}\wedge u$ and $P = \alpha^{n+k}\wedge\widetilde{\alpha}^{m+l}$ we get the Koppelman formula*

$$\phi([z],[\tilde{z}]) = \int_{\partial D}\phi([\zeta],[\tilde{\zeta}])\wedge K + \int_D \bar{\partial}\phi([\zeta],[\tilde{\zeta}])\wedge K + (\bar{\partial}_z + \bar{\partial}_{\tilde{z}})\int_D \phi([\zeta],[\tilde{\zeta}])\wedge K + \int_D \phi([\zeta],[\tilde{\zeta}])\wedge P$$

for differential forms $\phi([\zeta],[\tilde{\zeta}])$ on $\mathbb{P}^n\times\mathbb{P}^m$ with bidegree $(0,q)$ which take values in $L^k_{[\zeta]}\otimes L^l_{[\tilde{\zeta}]}$.

Now assume that $\bar\partial\phi=0$. For which q, k and l is ϕ $\bar\partial$-exact? To show that a particular ϕ is $\bar\partial$-exact, we need to show that the term $\int_{\mathbb{P}^n\times\mathbb{P}^m}\phi([\zeta])\wedge P$ either is zero, or is $\bar\partial$-exact. Since P consists of two factors where one depends only on ζ and the other only on $\tilde\zeta$, we can write

$$\int_{\mathbb{P}^n\times\mathbb{P}^m}\phi([\zeta],[\tilde\zeta])\wedge P=\int_{\mathbb{P}^m}\left(\int_{\mathbb{P}^n}\phi([\zeta],[\tilde\zeta])\wedge\alpha^{n+k}\right)\wedge\widetilde{\alpha}^{m+l}. \tag{26}$$

We get the following theorem.

Proposition 6.3. *We look at differential forms $\phi([\zeta],[\tilde\zeta])$ on $\mathbb{P}^n_{[\zeta]}\times\mathbb{P}^m_{[\tilde\zeta]}$ with bidegree $(0,q)$, which take values in the line bundle $L^k_{[\zeta]}\otimes L^l_{[\tilde\zeta]}$. The cohomology groups $H^{(0,q)}(\mathbb{P}^n\times\mathbb{P}^m, L^k_{[\zeta]}\otimes L^l_{[\tilde\zeta]})$ are trivial in the following cases*:

(a) $q\neq 0,n,m,n+m$;
(b) $q=0$ *and* $k<0$ *or* $l<0$;
(c) $q=n$ *and* $l<0$ *or* $k\geq -n$;
(d) $q=m$ *and* $k<0$ *or* $l\geq -m$;
(e) $q=n+m$ *and* $k\geq -n$ *or* $l\geq -m$.

Proof. To determine when (26) is zero, we use Theorem 5.1. Assume that the form ϕ has bidegree $(0,q_1)$ in ζ and $(0,q_2)$ in $\tilde\zeta$ and that $q_1+q_2=q$. If, for some q_1 and k, we know that $H^{(0,q_1)}(\mathbb{P}^n,L^k)$ is trivial, this means either that $\int_{[\zeta]}\phi([\zeta],[\tilde\zeta])\wedge P([\zeta],[z])=0$ or that $\int_{[\zeta]}\phi([\zeta],[\tilde\zeta])\wedge P([\zeta],[z])=\bar\partial_z a([z],[\tilde\zeta])$ for some $a([z],[\tilde\zeta])$. In the first case, it follows that the expression in (26) is also zero. In the second case, we get

$$\begin{aligned}\int_{\mathbb{P}^m}\left(\int_{\mathbb{P}^n}\phi([\zeta],[\tilde\zeta])\wedge\alpha^{n+k}\right)\wedge\widetilde{\alpha}^{m+l}&=\bar\partial_z\int_{\mathbb{P}^m}a([z],[\tilde\zeta])\wedge\widetilde{\alpha}^{m+l}\\&=\bar\partial\int_{\mathbb{P}^m}a([z],[\tilde\zeta])\wedge\widetilde{\alpha}^{m+l}\end{aligned}$$

since the integrand is holomorphic in $[\tilde z]$. The same holds if $H^{(0,q_2)}(\mathbb{P}^m,L^l)$ is trivial. The conclusion is that $H^{(0,q_1+q_2)}(\mathbb{P}^n\times\mathbb{P}^m,L^k_{[\zeta]}\otimes L^l_{[\tilde\zeta]})=0$ either when q_1 and k are such that $H^{(0,q_1)}(\mathbb{P}^n,L^k)=0$, or when q_2 and l are such that $H^{(0,q_2)}(\mathbb{P}^m,L^l)=0$.

Now, we really have a sum

$$\phi=\sum_{q_1+q_2=q}\phi_{q_1,q_2}$$

of terms of the type above. For the cohomology group to be trivial, we must have $\int_{\mathbb{P}^n\times\mathbb{P}^m}\phi_{q_1,q_2}\wedge P=0$ for all of them. We know that $q_2=q-q_1$. If we have either $0<q_1<n$ or $0<q_2<m$ then $\int_{\mathbb{P}^n\times\mathbb{P}^m}\phi_{q_1,q_2}\wedge P=0$ according to Theorem 5.1. The only

ways to avoid this are if $q=q_1=q_2=0$; if $q=q_1=n$ and $q_2=0$; if $q_1=0$ and $q=q_2=m$ or if $q=n+m$, $q_1=n$ and $q_2=m$. Then (a)–(e) follow from Theorem 5.1. □

7. Weighted Koppelman formulas on Stein manifolds

If X is a Stein manifold it is, in general, impossible to find $E\to X\times X$ and η with the desired properties as described in Section 3. What is possible is to find a section η of a bundle E such that η has good properties close to Δ, but then η will in general have other zeroes as well. It turns out that it is possible to work around this and still construct weighted integral formulas. This section relies on the article [18] by Henkin and Leiterer, where such an η was constructed.

More precisely, let π be the projection from $X_\zeta\times X_z$ to X_ζ, and $E=\pi^*(T_{1,0}(X_\zeta))$. Let $\{e_j\}_{j=1}^n$ be a local frame for E. By [18, Section 2.1] we have the following result.

Theorem 7.1. *There exists a holomorphic section η of E such that $\{(\zeta,z):\eta(\zeta,z)=0\}=\Delta\cup F$, where F is closed and $\Delta\cap F=\varnothing$. Close to Δ we have*

$$\eta(\zeta,z)=\sum_{j=1}^{n}[\zeta_j-z_j+\mathcal{O}(|\zeta-z|^2)]e_j. \tag{27}$$

Moreover, there exists a holomorphic function ϕ such that $\phi(z,z)=1$ and $|\phi|\lesssim|\eta|$ on a neighborhood of F.

We define δ_η, ∇ etc. in the same way as in Section 2. Let s taking values in E^* be the section satisfying $\delta_\eta s=1$ outside $\Delta\cup F$ which has pointwise minimal norm, and define $u=s/\nabla s$. If we define

$$K=\int_E \phi^M u\wedge\left(\frac{D\eta}{2\pi i}+\frac{i\widetilde{\Theta}}{2\pi}\right)_n \quad\text{and}\quad P=\int_E \phi^M\left(\frac{D\eta}{2\pi i}+\frac{i\widetilde{\Theta}}{2\pi}\right)_n,$$

where M is large enough so that $\phi^M u$ has no singularities on F, then Theorem 3.4 applies and we have $\bar\partial K=[\Delta]-P$. In this way, we recover the formula found in [8, Example 2], except that our approach also allows for weights. We define weights in the same way as before (note that ϕ is in fact a weight). If g is a weight, we will get a Koppelman formula with

$$K=\int_E \phi^M g\wedge u\wedge\left(\frac{D\eta}{2\pi i}+\frac{i\widetilde{\Theta}}{2\pi}\right)_n \quad\text{and}\quad P=\int_E \phi^M g\wedge\left(\frac{D\eta}{2\pi i}+\frac{i\widetilde{\Theta}}{2\pi}\right)_n. \tag{28}$$

Note that since E is a pullback of a bundle on X_ζ, the connection and curvature forms of E depend only on ζ. Hence $P=c_n(E)$ has bidegree (n,n) in ζ, and we have $\int_{X_\zeta} P(\zeta,z)\wedge\phi(\zeta)=0$ except in the case where ϕ has bidegree $(0,0)$. The last term in the Koppelman formula thus presents no obstruction to solving the $\bar\partial$-equation on X.

Example 3. In [14] there is an example of weighted formulas on Stein manifolds, which we can reformulate to fit into the present formalism. Let $G\subset X$ be a strictly pseudoconvex domain. By [14, Theorem 9] we can find a function ψ defined on a neighborhood U of G which embeds G in a strictly convex set $C\subset\mathbb{C}^n$. If σ is the defining function for C, then $\rho=\sigma\circ\psi$ is a strictly plurisubharmonic defining function for G. On U we introduce the weight

$$g(\zeta,z)=\left(1-\nabla\frac{(\partial\rho(\zeta)/\partial\zeta)\cdot e^*}{2\pi i\rho(\zeta)}\right)^{-\alpha}=\left(-\frac{v}{\rho}-\omega\right)^{-\alpha},$$

where

$$v=\frac{\partial\rho(\zeta)}{\partial\zeta}\cdot\eta-\rho(\zeta)\quad\text{and}\quad\omega=\bar\partial\left[\frac{(\partial\rho(\zeta)/\partial\zeta)\cdot e^*}{2\pi i\rho(\zeta)}\right].$$

Note that g is holomorphic in z. If $\operatorname{Re}\alpha$ is large enough, then $g(\,\cdot\,,\zeta)$ will be zero on ∂G, since $\sigma(\partial C)=0$. This implies that if f is a holomorphic function and P is defined by (28), then we will have

$$f(z)=\int_G f(\zeta)P$$

for $z\in G$, by Koppelman's formula. We also have the estimate

$$-\rho(\zeta)-\rho(z)+\varepsilon|\zeta-z|^2\le 2\operatorname{Re}v(\zeta,z)\le-\rho(\zeta)-\rho(z)+c|\zeta-z|^2,$$

where ε and c are positive and real. By means of this, we can get results in strictly pseudoconvex domains G in Stein manifolds similar to ones which are known in strictly pseudoconvex domains in $\mathbb{C}^n$. For example, one can obtain a direct proof of the Henkin–Skoda theorem which gives L^1-estimates on ∂G for solutions of the $\bar\partial$-equation.

Example 4. We can also solve division problems on X. Let $D\subseteq X$ be a domain, and take $f(\zeta)=(f_1(\zeta),...,f_m(\zeta))$, where $f_j\in\mathcal{O}(\overline{D})$. Assume that f has no common zeroes in D. We want to solve the division problem $\psi=f\cdot p$ in D, where ψ is a given holomorphic function, by means of integral formulas. We do this by a variant of the weights used in [6].

By Cartan's Theorem B, we can find $h(\zeta,z)=(h_1(\zeta,z),...,h_m(\zeta,z))$, where h_j is a holomorphic section of E^*, such that $\delta_\eta h_j(\zeta,z)=\phi(\zeta,z)(f_j(\zeta)-f_j(z))$. We set

$$g_1(\zeta,z)=(\phi-\nabla(h\cdot\sigma(\zeta)))^\mu=(\phi f(z)\cdot\sigma+h\cdot\bar\partial\sigma)^\mu,$$

where $\sigma=\bar f/|f|^2$ and $\mu=\min(m,n+1)$. Then g_1 is a weight. Now, $f(z)$ is a factor in g_1, since $(h\cdot\bar\partial\sigma)^\mu=0$. In fact, we have $(h\cdot\bar\partial\sigma)^{n+1}=0$ for degree reasons, and $(h\cdot\bar\partial\sigma)^m=0$ since $f\cdot\sigma=1$ implies that $f\cdot\bar\partial\sigma=0$, so that $\bar\partial\sigma_1,...,\bar\partial\sigma_m$ are linearly dependent.

By the Koppelman formula we have

$$\psi(z)=\int_{\partial D}\psi\phi^M K+\int_D \psi\phi^M P,$$

where K and P are defined by (28) using the weight g_1. Since $f(z)$ is a factor in g_1, we have $\psi(z)=f(z)\cdot p(z)$, where $p(z)$ will be holomorphic if D is such that we can find u holomorphic in z (for example if D is pseudoconvex).

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References

1. Andersson, M., L_p estimates for the $\bar\partial$-equation in analytic polyhedra in Stein manifolds, in *Several Complex Variables* (*Stockholm, 1987/1988*), Math. Notes **38**, pp. 34–47, Princeton Univ. Press, Princeton, NJ, 1993.
2. Andersson, M., Integral representation with weights. I, *Math. Ann.* **326** (2003), 1–18.
3. Andersson, M., Integral representation with weights. II. Division and interpolation, *Math. Z.* **254** (2006), 315–332.
4. Berenstein, C. A., Gay, R., Vidras, A. and Yger, A., *Residue Currents and Bézout Identities*, Progr. Math. **114**, Birkhäuser, Basel, 1993.
5. Berenstein, C. A. and Yger, A., Effective Bzout identities in $\mathbf{Q}[z_1,...,z_n]$, *Acta Math.* **166** (1991), 69–120.
6. Berndtsson, B., A formula for interpolation and division in $\mathbf{C}^n$, *Math. Ann.* **263** (1983), 399–418.
7. Berndtsson, B., Integral formulas on projective space and the Radon transform of Gindikin–Henkin–Polyakov, *Publ. Mat.* **32** (1988), 7–41.
8. Berndtsson, B., Cauchy–Leray forms and vector bundles, *Ann. Sci. École Norm. Sup.* **24**:3 (1991), 319–337.
9. Berndtsson, B. and Andersson, M., Henkin–Ramírez formulas with weight factors, *Ann. Inst. Fourier* (*Grenoble*) **32** (1982), v–vi, 91–110.
10. Berndtsson, B. and Passare, M., Integral formulas and an explicit version of the fundamental principle, *J. Funct. Anal.* **84** (1989), 358–372.

11. Demailly, J.-P. and Laurent-Thiébaut, C., Formules intégrales pour les formes différentielles de type (p, q) dans les variétés de Stein, *Ann. Sci. École Norm. Sup.* **20** (1987), 579–598.
12. Demailly, J.-P., *Complex Analytic and Differential Geometry*, in preparation. `http://www-fourier.ujf-grenoble.fr/~demailly/books.html`.
13. Dickenstein, A. and Sessa, C., Canonical representatives in moderate cohomology, *Invent. Math.* **80** (1985), 417–434.
14. Fornæss, J. E., Embedding strictly pseudoconvex domains in convex domains, *Amer. J. Math.* **98** (1976), 529–569.
15. Götmark, E., Some applications of weighted integral formulas, Licentiate thesis, 2005.
16. Henkin, G. M., Integral representation of functions in strongly pseudoconvex regions, and applications to the $\bar{\partial}$-problem, *Mat. Sb.* **82(124)** (1970), 300–308 (Russian). English transl.: *Math. USSR-Sb.* **11** (1970), 273–281.
17. Henkin, G. M., H. Lewy's equation, and analysis on a pseudoconvex manifold. II, *Mat. Sb.* **102(144)** (1977), 71–108, 151 (Russian). English transl.: *Math. USSR-Sb.* **102**:(**144**) (1977), 63–94.
18. Henkin, G. M. and Leiterer, J., Global integral formulas for solving the $\bar{\partial}$-equation on Stein manifolds, *Ann. Polon. Math.* **39** (1981), 93–116.
19. Passare, M., Residues, currents, and their relation to ideals of holomorphic functions, *Math. Scand.* **62** (1988), 75–152.
20. Polyakov, P. L. and Henkin, G. M., Homotopy formulas for the $\bar{\partial}$-operator on $\mathbf{C}\mathrm{P}^n$ and the Radon–Penrose transform, *Izv. Akad. Nauk SSSR Ser. Mat.* **50** (1986), 566–597, 639 (Russian). English transl.: *Math. USSR-Izv.* **50** (1986), 555–587.
21. Quillen, D., Superconnections and the Chern character, *Topology* **24** (1985), 89–95.
22. Ramírez de Arellano, E., Ein Divisionsproblem und Randintegraldarstellungen in der komplexen Analysis, *Math. Ann.* **184** (1969/1970), 172–187.
23. Skoda, H., Valeurs au bord pour les solutions de l'opérateur d'' dans les ouverts strictement pseudoconvexes, *C. R. Math. Acad. Sci. Paris, Sér. A–B* **280** (1975), A633–A636.
24. Zériahi, A., Meilleure approximation polynomiale et croissance des fonctions entières sur certaines variétés algébriques affines, *Ann. Inst. Fourier* (*Grenoble*) **37**:2 (1987), 79–104.

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