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The injectivity of the extended Gauss map of general projections of smooth projective varieties

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Abstract. Let X be a smooth n-dimensional projective variety embedded in some projective space $\mathbb{P}^N$ over the field $\mathbb{C}$ of the complex numbers. Associated with the general projection of X to a space $\mathbb{P}^{N-m}$ $(N-m>n+1)$ one defines an extended Gauss map $\overline{\gamma}\colon \overline{X}\to \mathrm{Gr}(n;N-m)$ (in case $N-m>2n-1$ this is the Gauss map of the image of X under the projection). We prove that $\overline{X}$ is smooth. In case any two different points of X do have disjoint tangent spaces then we prove that $\overline{\gamma}$ is injective.

Introduction

0.1. Let X be a smooth n-dimensional projective variety embedded in some projective space $\mathbb{P}^N$ over the field $\mathbb{C}$ of the complex numbers. Associated with a point $x\in X$ there is an embedded tangent space $\mathbb{T}_x(X)$. This is an n-dimensional linear subspace of $\mathbb{P}^N$, hence a point $\gamma(x)$ in the Grassmannian $\mathrm{Gr}(n,N)$. The morphism $\gamma\colon X\to \mathrm{Gr}(n,N)$ is called the *Gauss map*.

In [2] we studied the following question: does there exist a projective embedding $X\subset\mathbb{P}^{2n+1}$ such that the Gauss map is injective? This question is motivated by the following two facts: for each embedding of X in a projective space the Gauss map is generically injective (see [7, I, Corollary 2.8]) and in general $2n+1$ is the smallest possible dimension for a projective space such that X can be embedded in it.

We proved the answer in the affirmative as follows. First we proved that, in case any two different points on X do have disjoint embedded tangent spaces, then a general projection to $\mathbb{P}^{2n+1}$ gives an embedding of X having an injective Gauss map. Next we proved that, starting from an arbitrary embedding of X in some projective space $\mathbb{P}^M$ and composing it with the 3-Veronese embedding of $\mathbb{P}^M$ we obtain an embedding of X such that any two points on X do have disjoint embedded

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tangent spaces. Motivated by this proof, we introduce the following definition: in case any two different points on X have disjoint embedded tangent spaces to X then we say that X has disjoint embedded tangent spaces.

0.2. In this paper, given an embedding of X in $\mathbb{P}^N$ with disjoint embedded tangent spaces, we consider the behavior of the Gauss map using a general projection to some projective space $\mathbb{P}^{N-m}$ with $N-m\leq 2n$. As soon as $N<2n$, in general, the projection is not a local embedding of X at some points, hence the embedded tangent space is not defined at such point. Therefore, the Gauss map associated to the projection $j\colon X\to\mathbb{P}^{N-m}$ is only defined on a non-empty open subset U of X. Let $\overline{X}$ be the closure of the graph of that map in $X\times\mathrm{Gr}(n,N-m)$. The projection $\overline{\gamma}$ of $\overline{X}$ on $\mathrm{Gr}(n,N-m)$ is called the *extended Gauss map* of $j\colon X\to\mathbb{P}^{N-m}$.

0.3. We are going to prove the following theorem.

Theorem. *Let $i\colon X\to\mathbb{P}^N$ be an embedding of a smooth n-dimensional projective variety having disjoint embedded tangent spaces. Assume that Λ is a general $(m-1)$-dimensional linear subspace of $\mathbb{P}^N$. Then the projection with center Λ gives rise to a morphism $j\colon X\to\mathbb{P}^{N-m}$. The extended Gauss map of j is injective if $N-m\geq n+2$.*

In the first part of the paper we give an explicit description of the domain $\overline{X}$ of the extended Gauss map in the situation of a general projection. In particular we will prove that $\overline{X}$ is smooth. In the second part we will prove the theorem.

Part 1

In this part we will prove the following proposition.

Proposition. *Let $X\subset\mathbb{P}^N$ be a smooth n-dimensional projective variety and let Λ be a general $(m-1)$-dimensional linear subspace of $\mathbb{P}^N$. The projection with center Λ gives rise to a morphism $j\colon X\to\mathbb{P}^{N-m}$. Let $\overline{\gamma}\colon\overline{X}\to\mathrm{Gr}(n;N-m)$ be the extended Gauss map. In case $N-m\geq n+2$ then $\overline{X}$ is smooth.*

1.1. First we give a description of the Gauss map of $X\subset\mathbb{P}^N$ using vector bundles.

Let V be a complex vector space of dimension $N+1$ and let $\mathbb{P}^N(V)=\mathrm{Proj}(S^*(V^D))$ be the projective space of 1-dimensional vector subspace of V (we only consider closed points). We omit the vector space V and write $\mathbb{P}^N$. Dualizing the Euler sequence (see [4, Chapter II, Example 8.20.1]) we have a natural exact sequence

$$0\longrightarrow\mathcal{O}_{\mathbb{P}^N}\xrightarrow{\alpha}\mathcal{O}_{\mathbb{P}^N}(1)\otimes_{\mathbb{C}}V\xrightarrow{\beta}T_{\mathbb{P}^N}\longrightarrow 0.$$

A point $x \in \mathbb{P}^N$ corresponds to a 1-dimensional vector subspace $L(x)$ of V and one has $\ker(\beta(x)) = L(x)$ (after twisting by $\mathcal{O}(1)$).

Consider the exact sequence $0 \to T_X \to T_{\mathbb{P}^N}|_X \to N_{X/\mathbb{P}^N} \to 0$. Composing with $\beta|_X$ we obtain an epimorphism of vector bundles $\mathcal{O}_X(1) \otimes V \to N_{X/\mathbb{P}^N}$ on X. Let E_X be its kernel, hence $E_X = (\beta|_X)^{-1}(T_X)$. We obtain the exact sequence:

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\alpha_X} E_X \xrightarrow{\beta_X} T_X \longrightarrow 0.$$

As a vector subbundle of $\mathcal{O}_X(1) \otimes V$, the vector bundle E_X defines a morphism $X \to \mathrm{Gr}(n; N)$; this is the Gauss map γ_X of X; for $x \in X$ one has that $\gamma_X(x)$ corresponds to the embedded tangent space $\mathbb{T}_x(X) \subset \mathbb{P}^N$ of X at x.

1.2. We give a description of a bundle map u that will be used to define the extended Gauss map associated to $j\colon X \to \mathbb{P}^{N-m}$.

A general linear subspace $\Lambda \subset \mathbb{P}^N$ of dimension $m-1$ is a projective space $\mathbb{P}(W)$ for some general m-dimensional vector subspace $W \subset V$. The projection with center Λ gives rise to a morphism $\mathbb{I}\colon \mathbb{P}^N(V) \setminus \Lambda \to \mathbb{P}^{N-m}(V/W) = \mathbb{P}^{N-m}$. The image of a point $x \in \mathbb{P}^N(V) \setminus \Lambda$ is defined as being the 1-dimensional vector subspace $(L(x)+W)/W \subset V/W$. On $\mathbb{P}^{N-m}$ we have the natural exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^{N-m}} \xrightarrow{\alpha'} \mathcal{O}_{\mathbb{P}^{N-m}}(1) \otimes_{\mathbb{C}} (V/W) \xrightarrow{\beta'} T_{\mathbb{P}^{N-m}} \longrightarrow 0.$$

For $x \in \mathbb{P}^N \setminus \Lambda$ the tangent map $d_x\mathbb{I}\colon T_{\mathbb{P}^N,x} \to T_{\mathbb{P}^{N-m},\mathbb{I}(x)}$ lifts to (a multiple of) the natural surjection $V \to V/W$ through β and β' (the multiple depending on a trivialization of $\mathcal{O}_{\mathbb{P}^N}(1)$ and $\mathcal{O}_{\mathbb{P}^{N-m}}(1)$).

Because Λ is general and $N-m > n$ we have $\Lambda \cap X = \varnothing$. Hence the restriction of $\mathbb{I}$ to X is a morphism $j\colon X \to \mathbb{P}^{N-m}$. We have a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_X(1) \otimes_{\mathbb{C}} V & \xrightarrow{\quad \beta_X \quad} & T_{\mathbb{P}^N|_X} \\ \big\downarrow{\scriptstyle \text{natural}} & & \big\downarrow{\scriptstyle d\mathbb{I}|_X} \\ \mathcal{O}_X(1) \otimes_{\mathbb{C}} (V/W) & \xrightarrow{\; j^*(\beta') \;} & j^*(T_{\mathbb{P}^{N-m}}). \end{array}$$

The restriction to the vector bundle $E_X \subset \mathcal{O}_X(1) \otimes_{\mathbb{C}} V$ gives rise to morphisms $u\colon E_X \to \mathcal{O}_X(1) \otimes_{\mathbb{C}} (V/W)$ and $E_X \to j^*(T_{\mathbb{P}^{N-m}})$ of vector bundles on X. Clearly $\alpha_X(\mathcal{O}_X) \subset E_X$ belongs to the kernel of this second morphism, hence it induces a morphism $T_X \to j^*(T_{\mathbb{P}^{N-m}})$. This is the tangent map dj defined by $j\colon X \to \mathbb{P}^{N-m}$. Hence u is a lifting of dj (it induces the identity on $\ker(E_X \to T_X)$).

1.3. Associated with u and dj we can define subschemes of X using rank conditions. Since those subschemes are the same for u and dj we give a description of those subschemes using the morphism u.

We write E (resp. F) instead of E_X (resp. $\mathcal{O}_X(1)\otimes_{\mathbb{C}}(V/W)$). As a set, we define $D_k(u)=\{x\in X : \mathrm{rk}(u(x))\le\dim(X)+1-k\}$, hence $x\in D_k(u)$ if and only if $\dim(\ker(u(x)))\ge k$ (this is equivalent to $\dim(\ker(dj(x)))\ge k$). Locally as a scheme, $D_k(u)$ is defined as follows. Let U be an open neighborhood of x in X such that the restrictions of E and F are trivial. Using trivializations, the morphism u defines a morphism $U\to\mathrm{Hom}(\mathbb{C}^{n+1};\mathbb{C}^{N+1-m})$. The minors of order $n+2-k$ associated with the universal morphism above $\mathrm{Hom}(\mathbb{C}^{n+1};\mathbb{C}^{N+1-m})$ define universal loci $D_k\subset\mathrm{Hom}(\mathbb{C}^{n+1};\mathbb{C}^{N-m+1})$ as subschemes and $D_k(u)\cap U$ is the inverse image of D_k.

Let $x\in D_k(u)\setminus D_{k+1}(u)$ and let $x'\in D_k\setminus D_{k+1}$ be the image of x in the space $\mathrm{Hom}(\mathbb{C}^{n+1};\mathbb{C}^{N+1-m})$ using trivializations as before. The tangent space of $\mathrm{Hom}(\mathbb{C}^{n+1};\mathbb{C}^{N+1-m})$ at x' is $\mathrm{Hom}(\mathbb{C}^{n+1};\mathbb{C}^{N+1-m})$ itself in a natural way. This can be described explicitly as follows. Let v be a tangent vector at x'. This corresponds to a morphism of $\mathbb{C}[\varepsilon]=\mathbb{C}[x]/\langle x^2\rangle$ to $\mathrm{Hom}(\mathbb{C}^{n+1};\mathbb{C}^{N+1-m})$ defined by a $\mathbb{C}$-linear map $\mathbb{C}^{n+1}\to\mathbb{C}^{N+1-m}[\varepsilon]: \underline{t}\mapsto u(x')(\underline{t})+\varepsilon\overline{u}(\underline{t})$ (here $u(x')$ is the linear map defined by x'). Then $\overline{u}$ is the linear map associated with v (and we are going to denote this linear map by v too). The tangent vector v is tangent to D_k if and only if $v(\ker(u(x')))\subset\mathrm{im}(u(x'))$ (see e.g. [3, Example 14.16]). This implies that D_k is smooth at x' and the normal bundle of D_k at x' can be identified with $\mathrm{Hom}(\ker(u(x'));\mathrm{coker}(u(x')))$.

The natural map $T_{x'}(\mathrm{Hom}(\mathbb{C}^{n+1};\mathbb{C}^{N+1-m}))\to N_{D_k;x'}$ is the natural map

$$\mathrm{Hom}(\mathbb{C}^{n+1};\mathbb{C}^{N+1-m})\longrightarrow\mathrm{Hom}(\ker(u(x));\mathrm{coker}(u(x))).$$

The local map $U\to\mathrm{Hom}(\mathbb{C}^{n+1};\mathbb{C}^{N+1-m})$ defined above induces a morphism $T_x(X)\to\mathrm{Hom}(\ker(u(x));\mathrm{coker}(u(x)))$. This map is called the *Kodaira–Spencer map* $\mathrm{RKS}(u;x)$ associated to u at x (see [5, p. 165]). The tangent space $T_x(D_k(u))$ is equal to the kernel of $\mathrm{RKS}(u;x)$.

Concerning the behavior of tangent spaces under general projections, there is an important theorem of Mather (see [6]). There is a discussion of that theorem in the setting of complex algebraic geometry in [1]. Here we only use the easy part of that theorem. It says $D_k(u)\setminus D_{k+1}(u)$ is smooth of codimension

$$(N-n-m+1-1+k)k=(N-m-n+k)k=\dim(\mathrm{Hom}(\ker(u(x));\mathrm{coker}(u(x))).$$

This implies that $\mathrm{RKS}(u;x)$ is surjective.

1.4. Using [5, Section 3.4] at the end of Section 1.3 we obtain that $(D_k(u))_k$ is an R_{N-m-n}-like stratification of X. We recall the concept of R_δ-like stratification of X (see [5]) and we give a general discussion of that concept in case of a stratification defined by means of rank conditions associated to a map between vector bundles.

Consider a sequence $X=Z_0\supset Z_1\supset\ldots\supset Z_m$ of closed subschemes of X. This defines a stratification: X is the disjoint union of locally closed subschemes $Z_i\setminus Z_{i+1}$. This stratification is called *R_δ-like* if for each $x\in Z_i\setminus Z_{i+1}$ there exists an étale neighborhood Z' of X at x and a smooth morphism $Z'\to\mathrm{Hom}(\mathbb{C}^i;\mathbb{C}^{\delta+i})$ such that the inverse image of $D_k\subset\mathrm{Hom}(\mathbb{C}^i;\mathbb{C}^{\delta+i})$ on Z' is equal to the inverse image of $Z_k\subset X$ on Z'.

In case X is a smooth variety; E and F are vector bundles on X of rank e and $e+\delta$, resp.; $u\colon E\to F$ is a morphism such that for each $x\in X$ the map $\mathrm{RKS}(u;x)$ is surjective, then $D_k(u)$ is R_δ-like. In this case, let $x\in D_k(u)\setminus D_{k+1}(u)$. Choose $V\subset E(x)$ (resp. $W\subset F(x)$) of dimension $e-k$ (resp. $\delta+k$) complementary to $\ker(u(x))\subset E(x)$ (resp. $\mathrm{im}(u(x))\subset F(x)$). Using local trivializations of E and F on a neighborhood U of x on X we can consider u as a morphism $U\to\mathrm{Hom}(\ker(u(x))\oplus V;\mathrm{im}(u(x))\oplus W)$. At x this induces an isomorphism $V\to\mathrm{im}(u(x))$, by shrinking U we can assume that it induces an isomorphism at each point of U. Then the rank stratification is induced by $U\to\mathrm{Hom}(\ker(u(x));W)$. This is the morphism $Z'\to\mathrm{Hom}(\mathbb{C}^k;\mathbb{C}^{\delta+k})$ mentioned in the definition of R_δ-like.

Let $\pi_{e;F}\colon\mathrm{Gr}(e;F)\to X$ be the Grassmannian of subbundles of rank e of F and let $0\to E'\to\pi_{e;F}^{-1}(F)\to Q'\to 0$ be the tautological exact sequence. The restriction of u to $X\setminus D_1(u)$ induces a natural section $s\colon X\setminus D_1(u)\to\mathrm{Gr}(e;F)$ such that $s(x)$ corresponds to $\mathrm{im}(u(x))$ for $x\in X\setminus D_1(u)$. Let $\overline{X}\subset\mathrm{Gr}(e;F)$ be the set of points $\overline{x}$ corresponding to a subspace $E_{\overline{x}}\subset F_x$ (here $x=\pi_{e;F}(\overline{x})$) such that $\mathrm{im}(u(x))\subset E_{\overline{x}}$. Clearly $s(X\setminus D_1(u))\subset\overline{X}$, the projection $\overline{\pi}\colon\overline{X}\to X$ is surjective and its fibers are connected. For $\overline{x}\in\overline{X}$ let $\mathrm{RKS}(E_{\overline{x}};x)$ be the composition of $\mathrm{RKS}(u;x)$ and the natural map $\mathrm{Hom}(\ker(u(x));\mathrm{coker}(u(x)))\to\mathrm{Hom}(\ker(u(x));F_x/E_{\overline{x}})$. Clearly for all $\overline{x}\in\overline{X}$ this map is surjective too. From [5, Section 3.5] it follows that $\overline{X}$ is smooth (in [5] one uses a dual description). In particular it follows that $\overline{X}$ is the closure of $s(X/D_1(u))$.

1.5. We return to the situation obtained in Section 1.3 and we apply the conclusion of Section 1.4.

In this case the Grassmannian bundle $\mathrm{Gr}(n+1;F)$ is equal to the product $X\times\mathrm{Gr}(n;\mathbb{P}(V/W)=\mathbb{P}^{N-m})$ and the composition of s with the projection to $\mathrm{Gr}(n;\mathbb{P}^{N-m})$ is the Gauss map of $j\colon X\to\mathbb{P}^{N-m}$. It follows that $\overline{X}$ is the closure of the graph of γ, hence $\overline{\gamma}$ (the projection of $\overline{X}$ to $\mathrm{Gr}(n;\mathbb{P}^{N-m})$) is the extended Gauss map of $j\colon X\to\mathbb{P}^{N-m}$.

Part 2

In this part we are going to prove the theorem from the introduction.

We are going to use induction on m. For each integer m' satisfying $1\le m'\le m$ let $\Lambda_{m'}$ be a general linear subspace of dimension $m'-1$ in $\mathbb{P}^N$. Without loss of generality we can assume that $\Lambda_{m'-1} \subset \Lambda_{m'}$ and once we have chosen a general $\Lambda_{m'-1}$ we can assume that $\Lambda_{m'}$ is general for the condition of containing $\Lambda_{m'-1}$. We write $\Lambda = \Lambda_m$. We obtain morphisms $i_{m'}: X \to \mathbb{P}^{N-m'}$, a Gauss map $\gamma_{m'}$ and an extended Gauss maps $\overline{\gamma}_{m'}$. Clearly $\gamma = \gamma_0 = \overline{\gamma}_0$ is injective and we can assume $\overline{\gamma}_{m-1}$ is injective. We need to prove that $\overline{\gamma}_m = \overline{\gamma}$ is injective.

2.1. First we introduce some notation.

For $q \in X$ we write $\mathbb{T}_q(X, m')$ to denote the linear span of $\mathbb{T}_q(X)$ and $\Lambda_{m'}$. Clearly, the Gauss map $\gamma_{m'}$ is defined at q if and only if $\dim(\mathbb{T}_q(X, m')) = n + m'$. In this case the value of the Gauss map $\gamma_{m'}$ at q corresponds to a linear n-space in $\mathbb{P}^{N-m'}$ denoted by $\mathbb{T}_{q,m'}(X)$ (it is the projection of $\mathbb{T}_q(X, m')$ on $\mathbb{P}^{N-m'}$).

Let $Z_{k,m'}$ be the closure of $X \times X$ of the set of pairs (q, q') with $q \neq q'$ such that $\dim(\mathbb{T}_q(X, m')) = \dim(\mathbb{T}_{q'}(X, m')) = n + m'$ and $\dim(\mathbb{T}_q(X, m') \cap \mathbb{T}_{q'}(X, m')) = m' + k$ (this becomes equivalent to the fact that $\gamma_{m'}$ is defined at the points q and q' and $\dim(\mathbb{T}_{q,m'}(X) \cap \mathbb{T}_{q',m'}(X)) = k$; we use the convention $\dim(\varnothing) = -1$). In case $Z_{n,m} \neq \varnothing$ and (q, q') is a general point of $Z_{n,m}$ then $q \neq q'$; γ_m is defined at the points q and q' but $\mathbb{T}_{q,m}(X) = \mathbb{T}_{q',m}(X)$ In this case the Gauss map γ_m is not injective. So we need to prove that $Z_{n,m} = \varnothing$.

2.2. For each integer $-1 \leq k \leq n$ let $z_{k,m'} := 2n + (k+1)(m' - N + 2n - k)$. We are going to prove the following claim:

If $z_{k,m'} < 0$ then $Z_{k,m'} = \varnothing$.

If $z_{k,m'} \geq 0$ then $\dim(Z_{k,m'}) \leq z_{k,m'}$.

Taking $m' = m$ and $k = n$ we have $z_{n,m} = 2n + (n+1)(m - N + n)$ and using that $N - m \geq n + 2$ we obtain $z_{n,m} \leq 2n + (n+1)(-2) < 0$. Hence proving the claim implies that the Gauss map γ_m is injective. For $k = -1$ the claim is trivial. By assumption (i.e. $X \subset \mathbb{P}^N$ has disjoint embedded tangent spaces) $Z_{k,0} = \varnothing$ for $k \geq 0$. We can assume the claim to be true for $m' = m - 1$.

Remark. For (q, q') general on $X \times X$ one has $\dim(\langle \mathbb{T}_{q,m'}(X) \cup \mathbb{T}_{q',m'}(X) \rangle) \leq N - m'$ hence $\dim(\mathbb{T}_q(X, m') \cap \mathbb{T}_{q'}(X, m')) \geq 2n - N + 2m'$. Therefore $(q, q') \in Z_{k,m'}$ for some $k \geq 2n - N + m'$. This implies that we can always assume that $m' \leq N + k - 2n$. Under this assumption we obtain the natural inequality $z_{k,m'} \leq 2n$.

Proof of the claim. Assume that $Z_{k,m-1} \neq \varnothing$ and let (q, q') be a general element of it. Then $\dim(\langle \mathbb{T}_q(X, m-1) \cup \mathbb{T}_{q'}(X, m-1) \rangle) = 2n + m - 1 - k \leq 2n + N - 2n + k - 1 - k = N - 1$. Since Λ is a general linear $(m-1)$-space containing Λ_{m-1} we find $\Lambda \not\subset \langle \mathbb{T}_q(X, m-1) \cup \mathbb{T}_{q'}(X, m-1) \rangle$, hence $\dim(\langle \mathbb{T}_q(X, m) \cup \mathbb{T}_{q'}(X, m) \rangle) = 2n + m - k$ therefore $\dim(\mathbb{T}_q(X, m) \cap \mathbb{T}_{q'}(X, m)) = m + k$. This implies that $(q, q') \in Z_{k,m}$, and hence $Z_{k,m-1} \subset Z_{k,m}$.

Since $z_{k,m-1}<z_{k,m}$ for $k\geq 0$, we can assume that $Z_{k,m}\neq Z_{k,m-1}$. Let (q,q') be a general point of $Z_{k,m}$, hence $\dim(\langle \mathbb{T}_q(X,m)\cup\mathbb{T}_{q'}(X,m)\rangle)=2n+m-k$. Since $(q,q')\notin Z_{k,m-1}$ we need $\Lambda\subset\langle\mathbb{T}_q(X,m-1)\cup\mathbb{T}_{q'}(X,m-1)\rangle$, hence

$$\langle\mathbb{T}_q(X,m-1)\cup\mathbb{T}_{q'}(X,m-1)\rangle=\langle\mathbb{T}_q(X,m)\cup\mathbb{T}_{q'}(X,m)\rangle$$

and therefore $\dim(\mathbb{T}_q(X,m-1)\cap\mathbb{T}_{q'}(X,m-1))=(m-1)+(k-1)$. This proves that $(q,q')\in Z_{k-1,m-1}$.

In case $z_{k-1,m-1}<0$ we have $Z_{k-1,m-1}=\varnothing$, so this can be excluded. So assume that $z_{k-1,m-1}\geq 0$ and let I be the closure in $Z_{k-1,m-1}\times\mathbb{P}^{N-m+1}$ (with $\mathbb{P}^{N-m+1}$ parameterizing linear $(m-1)$-spaces containing Λ_{m-1} in $\mathbb{P}^N$) of the subset of pairs $((q,q'),\Lambda)$ such that $\Lambda\subset\langle\mathbb{T}_q(X,m-1)\cup\mathbb{T}_{q'}(X,m-1)\rangle$ and $(q,q')\notin Z_{k,m-1}$. Consider the projections $p_1\colon I\to Z_{k-1,m-1}$ and $p_2\colon I\to\mathbb{P}^{N-m+1}$. Components of $Z_{k,m}$ not contained in $Z_{k,m-1}$ are components of $p_1(p_2^{-1}(\Lambda))$. On the other hand, the general fibers of p_1 have dimension $2n-k+1$, hence $\dim(I)\leq z_{k-1,m-1}+(2n-k+1)$, hence $\dim(Z_{k,m})\leq\dim(I)-(N-m+1)\leq z_{k-1,m-1}+(2n-k+1)-(N-m+1)=z_{k,m}$. This finishes the proof of the claim. □

2.3. In order to prove the injectivity of the extended Gauss map we extend the notation from Section 2.1. For integers $-1\leq e_1\leq n$ and $-1\leq e_2\leq n$ let $Z_{k,m',e_1,e_2}\subset X\times X$ be the closure of the subset $Z^0_{k,m',e_1,e_2}\subset X\times X$ equal to the set of the points $(q,q')\in X\times X$ satisfying $q\neq q'$ and

(1) $\dim(\mathbb{T}_q(X,m'))=n+m'-1-e_1$;

(2) $\dim(\mathbb{T}_{q'}(X,m'))=n+m'-1-e_2$;

(3) $\dim(\mathbb{T}_q(X,m')\cap\mathbb{T}_{q'}(X,m'))=m'+k$.

Let $d_q(i_{m'})\colon T_q(X)\to T_{i_{m'}(q)}(\mathbb{P}^{N-m'})$ be the tangent map of $i_{m'}$ at q (and similar for q'). Then condition (1) (resp. (2)) means that $d_q(i_{m'})$ (resp. $d_{q'}(i_{m'})$) has rank $n-1-e_1$ (resp. $n-1-e_2$). In particular $i_{m'}$ is not a local embedding at q (resp. q') if and only if $e_1\geq 0$ (resp. $e_2\geq 0$). Assuming conditions (1) and (2), condition (3) is equivalent to $\dim(\langle\mathbb{T}_q(X,m')\cup\mathbb{T}_{q'}(X,m')\rangle)=2n+m'-2-e_1-e_2-k$. In particular we need $2n+m'-2-e_1-e_2-k\leq N$ hence we can always assume that $m'\leq N-2n+2+e_1+e_2+k$.

Using the description of $\overline{X}$ and $\overline{\gamma}$ in Part 1, we obtain a contradiction to the injectivity of the extended Gauss map $\overline{\gamma}_{m'}$ if and only if we find two different points q and q' on X and an $(n+m')$-dimensional linear subspace Γ of $\mathbb{P}^N$ such that $\Gamma\supset\langle\mathbb{T}_q(X,m')\cup\mathbb{T}_{q'}(X,m')\rangle$. In case $(q,q')\in Z^0_{k,m',e_1,e_2}$ then such a subspace Γ exists if and only if $n+m'\geq 2n+m'-2-e_1-e_2-k$, hence $n\leq e_1+e_2+k+2$.

So in order to prove the injectivity of the extended Gauss map $\overline{\gamma}_{m'}$ we need to prove that $Z_{k,m',e_1,e_2}=\varnothing$ in case $n\leq e_1+e_2+k+2$. In case $m'=0$ then this condition is satisfied. We are going to assume that this condition holds for $m'=m-1$ and we are going to prove that it holds for $m'=m$.

In the proof we can assume that $e_1 \geq e_2$. In case $e_1 = e_2 = -1$ the condition becomes $Z_{k,m} = \varnothing$ for $n \leq k$. This is proved in Section 2.2, so we can assume that $e_1 \geq 0$.

Using the induction we can assume that Λ_{m-1} is a general hyperplane in Λ. Let (q, q') be a general element of Z_{k,m,e_1,e_2}. Since Z_{k,m,e_1,e_2} does not depend on the choice of Λ_{m-1} inside Λ we can consider Λ_{m-1} to be general in Λ independent of (q, q'). In particular, since $e_1 \geq 0$ we have $\dim(\mathbb{T}_q(X) \cap \Lambda) \geq 0$ and so

$$\dim(\mathbb{T}_q(X) \cap \Lambda_{m-1}) = \dim(\mathbb{T}_q(X) \cap \Lambda) - 1$$

and also

$$\dim(\langle \mathbb{T}_q(X) \cup \mathbb{T}_{q'}(X) \rangle \cap \Lambda_{m-1}) = \dim(\langle \mathbb{T}_q(X) \cup \mathbb{T}_{q'}(X) \rangle \cap \Lambda) - 1.$$

This implies that $\mathbb{T}_q(X, m) = \mathbb{T}_q(X, m-1)$ and

$$\langle \mathbb{T}_q(X, m) \cup \mathbb{T}_{q'}(X, m) \rangle = \langle \mathbb{T}_q(X, m-1) \cup \mathbb{T}_{q'}(X, m-1) \rangle.$$

2.4. In the induction argument we are going to distinguish between two cases and in both cases we are going to prove a dimension inequality between varieties Z for the values m and $m-1$.

(A) In case $e_2 \geq 0$ we also have $\mathbb{T}_{q'}(X, m) = \mathbb{T}_{q'}(X, m-1)$. In particular

$$\mathbb{T}_q(X, m) \cap \mathbb{T}_{q'}(X, m) = \mathbb{T}_q(X, m-1) \cap \mathbb{T}_{q'}(m-1).$$

It follows that $(q, q') \in Z^0_{k+1,m-1,e_1-1,e_2-1}$.

In this case we have $\Lambda \subset \mathbb{T}_q(X, m-1) \cap \mathbb{T}_{q'}(X, m-1)$.

(B) In case $e_2 = -1$ we have $\dim(\mathbb{T}_{q'}(X, m)) = \dim(\mathbb{T}_{q'}(X, m-1)) + 1$. In particular $\dim(\mathbb{T}_q(X, m-1) \cap \mathbb{T}_{q'}(X, m-1)) = \dim(\mathbb{T}_q(X, m) \cap \mathbb{T}_{q'}(X, m)) - 1$. It follows that $(q, q') \in Z^0_{k,m-1,e_1-1,-1=e_2}$.

In this case we have $\Lambda \not\subset \mathbb{T}_{q'}(X, m-1)$.

Let T be an irreducible component of Z_{k,m,e_1,e_2} and let τ_1 and τ_2 be the restrictions to T of the projections of $X \times X$ on X. Let $c = \dim(T)$ and $c_i = \dim(\tau_i(T))$ for $i = 1, 2$.

In case (A) there exists an irreducible component T' of $Z_{k+1,m-1,e_1-1,e_2-1}$ such that $T \subset \{(q, q') \in T' : \Lambda \subset \mathbb{T}_q(X, m-1) \cap \mathbb{T}_{q'}(X, m-1)\}$. Instead of starting with $\Lambda = \Lambda_m$ and considering Λ_{m-1} as a general hyperplane in Λ, we can start with a general linear subspace Λ_{m-1} of dimension $m-2$ in $\mathbb{P}^N$ and consider $\Lambda = \Lambda_m$ as a general element of the space $\mathbb{P}^{N-m+1}$ of $(m-1)$-dimensional linear subspaces of $\mathbb{P}^N$ containing Λ_{m-1}. Let T' be an irreducible component of $Z_{k+1,m-1,e_1-1,e_2-1}$ and let (q, q') be a general point of T'. The space of $(m-1)$-dimensional linear subspaces of $\mathbb{T}_q(X, m-1) \cap \mathbb{T}_{q'}(X, m-1)$ containing Λ_{m-1} has dimension $k+1$. For a suitable component T' the union of these spaces has to dominate $\mathbb{P}^{N-m+1}$ and T is a component of a general fiber of this union above $\mathbb{P}^{N-m+1}$. Writing $c'=\dim(T')$ we conclude that $c'+k+1\geq N-m+1$, i.e. $m\geq N-c'-k$. Under this condition we find that $c=c'+k-N+m$.

In case (B) there exists an irreducible component T' of $Z_{k,m-1,e_1-1,-1}$ such that $T\subset\{(q,q')\in T':\Lambda\subset\mathbb{T}_q(X,m-1)\}$. Again, starting with a general linear subspace Λ_{m-1} of dimension $m-2$ in $\mathbb{P}^N$ we consider an irreducible component T' of $Z_{k,m-1,e_1-1,-1}$. Let $\mathbb{P}^{N-m+1}$ be as before. Let c' (resp. c'_1) be the dimension of T' (resp. $\tau'_1(T')\subset X$; the projection on the first factor). For (q,q') general on T' the $(m-1)$-dimensional linear subspaces of $\mathbb{T}_q(X,m-1)$ containing Λ_{m-1} give rise to a linear subspace of $\mathbb{P}^{N-m+1}$ of dimension $\dim(\mathbb{T}_q(X,m-1))-(m-1)=n-e_1$. For a suitable component T' the union of these spaces has to dominate $\mathbb{P}^{N-m+1}$ and T is a component of a general fiber of this union above $\mathbb{P}^{N-m+1}$. This implies that $c'_1+n-e_1\geq N-m+1$ and $c_1\leq c'_1+n-e_1+m-N-1$.

In this situation, for q general on $\tau_1(T)\subset\tau'_1(T')$ we find that $\tau_1^{-1}(q)=\tau_1'^{-1}(q)$. If we take Λ general in $\mathbb{P}^{N-m+1}$ and q general in $\tau_1(T)$, then q is general in $\tau'_1(T')$. This implies that $\dim(\tau_1'^{-1}(q))=c'-c'_1$. Hence

$$c=c_1+\dim(\tau_1^{-1}(q))\leq c'_1+n-e_1+m-N-1+c'-c'_1=c'+n-e_1+m-N-1.$$

2.5. From the inequalities between c and c' we are going to finish the proof of the theorem.

To make the computations easier, from now on we write $N-m=2n-t$, and hence $m=N+t-2n$. Since we only have to consider Z_{k,m,e_1,e_2} in the case $(e_1,e_2)\neq(-1,-1)$ and since i_m is a local embedding at each point of X in case $N-m\geq 2n$ we can assume that $t\geq0$. Also, for $t=0$ we know that $Z_{k,N-2n,e_1,e_2}=\varnothing$ if $(e_1,e_2)\neq(-1,-1)$. We already proved that $Z_{k,m,e_1,-1}\subset Z_{k,m-1,e_1-1,-1}$ for $e_1\geq0$ and $Z_{k,m,e_1,e_2}\subset Z_{k+1,m-1,e_1-1,e_2-1}$ for $e_2\geq0$, hence $Z_{k,N-2n+t,e_1,e_2}=\varnothing$ for $e_1\geq t$. So we only have to consider $Z_{k,N-2n+t,e_1,e_2}$ for $t\geq0$ and $e_1\leq t-1$.

Claim. For $-1\leq e_1\leq t-1$ one has

$$\begin{aligned}\dim(Z_{k,N+t-2n,e_1,-1})&\leq -k^2+k(t-2-e_1)-(e_1+2)(e_1+1)-(e_1-1)n+(e_1+2)t\\&=z_{k,t,e_1,-1}.\end{aligned}$$

Proof. In case $e_1=-1$ one has $z_{k,t,-1,-1}=z_{k,N-2n+t}$ and we already have proved the claim in this case. This proves the claim if $t=0$. So we can use induction on t.

Assume that $t>0$ and $e_1\geq0$. From case (B) of Section 2.4 we concluded that

$$\dim(Z_{k,N-2n+t,e_1,-1})\leq\dim(Z_{k,N-2n+(t-1),e_1-1,-1})+n-e_1+N-2n+t-N-1.$$

Using the induction hypothesis we find that

$$\dim(Z_{k,N-2n+t,e_1,-1}) \le z_{k,t-1,e_1-1,-1}+t-n-e_1-1=z_{k,t,e_1,-1}.$$

Now we conclude that the sets $Z_{k,m,e_1,-1}$ cannot give a contradiction to the injectivity of the extended Gauss map. From Section 2.3 we know that we need to prove that $z_{k,t,e_1,-1}<0$ if $n\le e_1+k+1$. Also $2n-t\ge n+2$, hence $t\le n-2$. So $n\le e_1+k+1$ implies that $k\ge t-e_1+1$.

Consider $\phi(k)=-z_{k,t,e_1,-1}$. From $e_1\le t-1$ we obtain $t-e_1+1\ge(t-e_1-2)/2$, hence $\phi(k)\ge\phi(t-e_1+1)$. One has $\phi(t-e_1+1)=(e_1-1)(e_1+1+n-t)+6$. So in case $e_1\ge1$, since $t\le n-2$, we find that $\phi(t-e_1+1)>0$. In case $e_1=0$ we have $\phi(k)=k^2-(t-2)k+2-n+2t$. We need to prove that $\phi(k)>0$ if $n\le k+1$, hence $k\ge n-1$. Since $n-1\ge(t-2)/2$ we find that $\phi(k)\ge\phi(n-1)$. But $\phi(n-1)=(n-1)^2-(t-2)(n-1)+2-n-2t$ and $t\le n-2$ hence, $\phi(n-1)\ge(n-1)^2-(n-4)(n-1)+2-n-2n+4=3>0$.

From case (A) we know that

$$\dim(Z_{k,N-2n+t,e_1,0}) \le \dim(Z_{k+1,N-2n+t-1,e_1-1,-1})+k-2n+t.$$

From the previous claim we obtain that

$$\begin{aligned}\dim(Z_{k,N-2n+t,e_1,0}) &\le z_{k+1,t-1,e_1-1,-1}+k-2n+t\\ &=-k^2+k(t-3-e_1)+(e_1+3)t-e_1n-e_1(e_1+3)-4:=-\psi(k).\end{aligned}$$

For $x>0$, in order for $Z_{k-x,N-2n+t+x,e_1+x,x}$ to be non-empty we need $Z_{k,N-2n+t,e_1,0}$ to be non-empty. From Section 2.3 we know that the injectivity of the extended Gauss map would be contradicted by the non-emptiness of $Z_{k-x,N-2n+t+x,e_1+x,x}$ if and only if $n\le(e_1+x)+x+(k-x)+2=x+k+e_1+2$. On the other hand $m\le N-(n+2)$ implies that $2n-t-x\ge n+2$, hence $x\le n-t-2$. Thus, in case we obtain a contradiction to the injectivity of the extended Gauss map we obtain $n\le n-t-2+k+e_1+2$, hence $k\ge t-e_1$.

It is enough to prove that $\psi(k)>0$ if $k\ge t-e_1$. One computes $\psi(t-e_1)=e_1(n-t+e_1)+4$. Since $t\le n-2$ one finds that $\psi(t-e_1)\ge4$. On the other hand, from $e_1\le t-1$ it also follows that $t-e_1\ge(t-3-e_1)/2$, hence $\psi(k)\ge\psi(t-e_1)$ for $k\ge t-e_1$ and so $\psi(k)>0$.

This finishes the proof of the injectivity of the extended Gauss map for $N-m\ge n+2$. □

References

1. Alzati, A. and Ottaviani, G., The theorem of Mather on generic projections in the setting of algebraic geometry, *Manuscripta Math.* **74** (1992), 391–412.
2. Coppens, M. and De Volder, C., The existence of embeddings for which the Gauss map is an embedding, *Ann. Mat. Pura Appl.* **181** (2002), 453–462.

3. Harris, J., *Algebraic Geometry*, Grad. Texts in Math. **133**, Springer, New York, 1992.
4. Hartshorne, R., *Algebraic Geometry*, Grad. Texts in Math. **52**, Springer, New York, 1977.
5. Hirschowitz, A., Rank techniques and jump stratifications, in *Vector Bundles on Algebraic Varieties (Bombay, 1984)*, Tata Inst. Fund. Res. Stud. Math. **11**, pp. 159–205, Tata Inst. Fund. Res., Bombay, 1987.
6. Mather, J. N., Generic projections, *Ann. of Math.* **98** (1973), 226–245.
7. Zak, F. L., *Tangents and Secants of Algebraic Varieties*, Transl. Math. Monogr. **127**, Amer. Math. Soc., Providence, RI, 1993.

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