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Geometry of spaces of compact operators

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Abstract. We introduce the notion of compactly locally reflexive Banach spaces and show that a Banach space X is compactly locally reflexive if and only if $\mathcal{K}(Y, X^{**}) \subseteq \mathcal{K}(Y, X)^{**}$ for all reflexive Banach spaces Y. We show that X^* has the approximation property if and only if X has the approximation property and is compactly locally reflexive. The weak metric approximation property was recently introduced by Lima and Oja. We study two natural weak compact versions of this property. If X is compactly locally reflexive then these two properties coincide. We also show how these properties are related to the compact approximation property and the compact approximation property with conjugate operators for dual spaces.

1. Introduction

Let X and Y be Banach spaces. We denote by $\mathcal{L}(Y, X)$ the Banach space of all bounded linear operators from Y to X, and $\mathcal{F}(Y, X)$, $\mathcal{K}(Y, X)$, and $\mathcal{W}(Y, X)$ denote the subspaces of finite-rank, compact, and weakly compact operators respectively.

Recall that a Banach space X is said to have the *approximation property* (AP) if there exists a net $(S_\alpha) \subset \mathcal{F}(X, X)$ such that $S_\alpha \to I_X$ uniformly on compact sets in X. If (S_α) can be chosen with $\sup_\alpha \|S_\alpha\| \leq 1$, then X is said to have the *metric* AP (MAP). The weaker properties *compact* AP (CAP) and *metric* CAP (MCAP) are defined similarly but with the net $(S_\alpha) \subset \mathcal{K}(X, X)$. The dual space X^* has the CAP *with conjugate operators* (CAPconj) if there is a net $(S_\alpha) \subset \mathcal{K}(X, X)$ such that $S_\alpha^* \to I_{X^*}$ uniformly on compact sets in X^*.

Let us also recall that a linear subspace E of a Banach space F is an *ideal* in F if $E^\perp$ is the kernel of a norm-one projection on F^*. The notion of an ideal was introduced and studied by Godefroy, Kalton, and Saphar in [6].

A linear operator $\varphi: E^* \to F^*$ is called a *Hahn–Banach extension operator* if $(\varphi e^*)(e) = e^*(e)$ and $\|\varphi e^*\| = \|e^*\|$ for all $e \in E$ and $e^* \in E^*$. Let us denote the set of all Hahn–Banach extension operators $\varphi: E^* \to F^*$ by $\mathrm{HB}(E, F)$. It is well known (and straightforward to verify) that $\mathrm{HB}(E, F) \neq \emptyset$ if and only if E is an ideal in F.

There is also a local characterization of ideals which predates the term ideal (see Fakhoury [4], or e.g. Kalton [13] or Lima [14]).

If Y is a separable subspace of a Banach space X, then by a result of Sims and Yost [31] there exists a separable subspace Z of X such that $Y \subseteq Z$ and $\mathrm{HB}(Z,X) \neq \varnothing$. Such a subspace Z will be called a *separable ideal* in X.

Approximation properties involving compact operators behave differently in certain respects from those involving only finite-rank operators. A Banach space X has the AP if and only if $\mathcal{F}(Y,X)$ is an ideal in $\mathcal{W}(Y,X)$ for every Banach space Y, see [16] and [15]. In [15] there is also an example of a Banach space X without the CAP such that $\mathcal{K}(Y,X)$ is trivially an ideal in $\mathcal{W}(Y,X)$ for every Y. Something can be done for the CAP however. X has the CAP if and only if for every Banach space Y and $T \in \mathcal{W}(Y,X)$ there is a net $(S_\alpha) \subset \mathcal{K}(X,X)$ with $\sup_\alpha \|S_\alpha T\| \leq \|T\|$ such that $S_\alpha \to I_X$ in the strong operator topology [25, Theorem 2.3]. In this paper we will study these "local", the sense that the operators are factorized, and "global", non-factorized, aspects of compact operators.

Related to the "local" characterization of the CAP is the weak version of the MAP introduced by Lima and Oja in [20]. A Banach space X has the *weak metric approximation property* (weak MAP) if for every Banach space Y and every operator $T \in \mathcal{W}(X,Y)$ there exists a net $(S_\alpha) \subset \mathcal{F}(X,X)$ with $\sup_\alpha \|TS_\alpha\| \leq \|T\|$ such that $S_\alpha \to I_X$ uniformly on compact sets in X. Just like the AP the weak MAP can be characterized "globally"; Lima [23] showed that X has the weak MAP if and only if $\mathcal{F}(Y,X)$ is an ideal in $\mathcal{W}(Y,X^{**})$ for every Banach space Y.

In [20, Theorem 3.2] Lima and Oja showed that a Banach space X has the weak MAP if and only if the trace mapping $V\colon Y^* \widehat{\otimes}_\pi X \to \mathcal{F}(X,Y)^*$ is isometric for every reflexive Banach space Y. Recall also Grothendieck's result that X has the MAP if and only if the trace mapping $V\colon Y \widehat{\otimes}_\pi X \to \mathcal{F}(X,Y^*)^*$ is isometric for every Banach space Y [8, Chapter 1, § 5, p. 179] (see also [30, Theorem 4.14] and [21, Corollary 2.10]). We study the related properties for compact operators in Sections 2 and 3. In Section 2 we consider the space of compact operators with range X or its bidual X^{**}, and in Section 3 we let X be the domain space of the compact operators. The main types of questions we consider are the following: When is the trace mapping into the dual of the space of compact operators isometric? When can compact or weakly compact operators into X^{**} be "well" approximated by compact operators into X? When is the space of compact operators an ideal in a larger space of compact or weakly compact operators?

In Section 2 we introduce the notion of compactly locally reflexive Banach spaces. In Proposition 2.2 we give a list of equivalent formulations of this property. We give examples of spaces which are not compactly locally reflexive. In Proposition 2.5 we prove that a dual space X^* has the AP if and only if X has the AP and X is compactly locally reflexive, so we have a plentiful supply of spaces which are compactly locally reflexive.

In Sections 4 and 5 we introduce and study the weak MCAP and its (not just formally, see Remark 5.1) weaker cousin the very weak MCAP. Characterizations of these properties similar to what can be found for the weak MAP in [20] and [23] are proved. In particular, we show how these properties are related to the existence of certain approximable Hahn–Banach extension operators in $\mathrm{HB}(X, X^{**})$ (see Theorems 4.3 and 5.3). These characterizations are similar to the one given by Grothendieck for the AP.

We also relate our new approximation properties to the classical ones. In Theorem 4.9 we prove that X^* has the CAPconj property if and only if X has the weak MCAP in every equivalent renorming and in Theorem 5.6 we show that X^* has the CAP if and only if X has the very weak MCAP in every equivalent renorming.

We conclude the paper with a section on open problems where we also try to give an overview over the connection between the results in the previous sections.

One of our main tools will be the isometric version of the famous Davis–Figiel–Johnson–Pełczyński factorization lemma [2] due to Lima, Nygaard, and Oja [15]. In this paper, it will be called the *factorization lemma*. If K is a closed absolutely convex subset of the unit ball B_X of a Banach space X, we shall write

$$[Z, J] = \mathrm{DFJP}(K),$$

where Z is the Banach space constructed from K in the factorization lemma and $J\colon Z \to X$ is the norm-one identity embedding of Z into X (see [15, Lemma 1.1]).

We will also repeatedly be using the following results (FS) by Feder and Saphar [5, Theorem 1] and (GS) by Godefroy and Saphar [7, Proposition 1.1] which we cite for easy reference.

Theorem 1.1. *Let X and Y be Banach spaces and assume that Y^{**} or X^* has the Radon–Nikodým property.*

(FS) *The trace mapping $V\colon X^*\widehat{\otimes}_\pi Y^{**} \to \mathcal{K}(Y, X)^*$, defined by $\langle V(u), T\rangle = \mathrm{tr}(T^{**}u)$ where $u \in X^*\widehat{\otimes}_\pi Y^{**}$ and $T \in \mathcal{K}(Y, X)$, is a quotient map. Moreover, for all $\phi \in \mathcal{K}(Y, X)^*$, there exists $u \in X^*\widehat{\otimes}_\pi Y^{**}$ such that $\phi = V(u)$ and $\|\phi\| = \|u\|_\pi$.*

(GS) *$\mathcal{K}(Y, X)^{**}$ is isometrically isomorphic to the weak*-closure of the space $Z = \{T^{**} : T \in \mathcal{K}(Y, X)\}$ in $(X^*\widehat{\otimes}_\pi Y^{**})^* = \mathcal{L}(Y^{**}, X^{**})$.*

Let us fix some notation. In a normed linear space X, we denote the closed unit ball by B_X. If A is a subset of X, $\overline{A}$ denotes the closure of A, $\mathrm{span}(A)$ denotes the linear span of A, and $\mathrm{conv}(A)$ is the convex hull of A. The identity operator on X is denoted I_X and the natural embedding of X into its bidual X^{**} is denoted k_X.

We only consider Banach spaces over the real scalar field.

2. $\mathcal{K}(Y,X)$ as a subspace of $\mathcal{K}(Y,X^{**})$ and $\mathcal{W}(Y,X^{**})$

In [21, Proposition 2.9], Lima and Oja proved the following result.

Proposition 2.1. *Let X and Y be Banach spaces. Then there exists a Hahn–Banach extension operator $\Phi\colon \mathcal{F}(Y,X)^*\to\mathcal{F}(Y,X^{**})^*$ such that $W=\Phi\circ V$, where $V\colon X^*\widehat{\otimes}_\pi Y^{**}\to\mathcal{F}(Y,X)^*$ and $W\colon X^*\widehat{\otimes}_\pi Y^{**}\to\mathcal{F}(Y,X^{**})^*$ are the trace mappings. In particular, $\|Wu\|=\|Vu\|$ for all $u\in X^*\widehat{\otimes}_\pi Y^{**}$.*

This result, or more precisely a version of its corollary [21, Corollary 2.11], was used as a starting point by Oja and Põldvere for a new proof of the "principle of local reflexivity" (see [29, Lemma 1.1]). Considering compact operators instead of finite-rank operators in Proposition 2.1 the statement we get is not true in general. In the next proposition we characterize this property. We shall say that a Banach space X is *compactly locally reflexive* (CLR) if it satisfies the statements in Proposition 2.2.

Proposition 2.2. *Let X be a Banach space. The following statements are equivalent*:

(a) *For every Banach space Y, there is a Hahn–Banach extension operator $\Phi\colon \mathcal{K}(Y,X)^*\to\mathcal{K}(Y,X^{**})^*$ such that $W=\Phi\circ V$, where $V\colon X^*\widehat{\otimes}_\pi Y^{**}\to\mathcal{K}(Y,X)^*$ and $W\colon X^*\widehat{\otimes}_\pi Y^{**}\to\mathcal{K}(Y,X^{**})^*$ are the trace mappings. In particular, $\|Wu\|=\|Vu\|$ for all $u\in X^*\widehat{\otimes}_\pi Y^{**}$.*

(b) *For every reflexive Banach space Y, $\|Wu\|=\|Vu\|$ for all $u\in X^*\widehat{\otimes}_\pi Y$, where $V\colon X^*\widehat{\otimes}_\pi Y\to\mathcal{K}(Y,X)^*$ and $W\colon X^*\widehat{\otimes}_\pi Y\to\mathcal{K}(Y,X^{**})^*$ are the trace mappings.*

(c) *For every reflexive Banach space Y, $\ker V=\ker W$, where $V\colon X^*\widehat{\otimes}_\pi Y\to\mathcal{K}(Y,X)^*$ and $W\colon X^*\widehat{\otimes}_\pi Y\to\mathcal{K}(Y,X^{**})^*$ are the trace mappings.*

(d) *For every Banach space Y and every $T\in\mathcal{K}(Y,X^{**})$, there is a net $(S_\alpha)\subset\mathcal{K}(Y,X)$ with $\sup_\alpha\|S_\alpha\|\le\|T\|$ such that $S_\alpha^*\to T^*|_{X^*}$ in the strong operator topology.*

(e) *For every reflexive Banach space Y and every $T\in\mathcal{K}(Y,X^{**})$, there is a net $(S_\alpha)\subset\mathcal{K}(Y,X)$ with $\sup_\alpha\|S_\alpha\|\le\|T\|$ such that $S_\alpha^*\to T^*|_{X^*}$ in the strong operator topology.*

(f) *$\mathcal{K}(Y,X^{**})\subseteq\mathcal{K}(Y,X)^{**}$ for every reflexive Banach space Y.*

(g) *$\mathcal{K}(Y,\widehat{X})$ is an ideal in $\mathcal{K}(Y,\widehat{X}^{**})$ for every Banach space Y and every equivalent renorming $\widehat{X}$ of X.*

Proof. We will need the following additional statement:

(a′) For every reflexive Banach space Y, there is a Hahn–Banach extension operator $\Phi\colon \mathcal{K}(Y,X)^*\to\mathcal{K}(Y,X^{**})^*$ such that $W=\Phi\circ V$, where $V\colon X^*\widehat{\otimes}_\pi Y^{**}\to\mathcal{K}(Y,X)^*$ and $W\colon X^*\widehat{\otimes}_\pi Y^{**}\to\mathcal{K}(Y,X^{**})^*$ are the trace mappings.

(a)⇒(b)⇒(c) and (d)⇒(e) are trivial.

(c)⇒(b) follows by applying the theorem of Feder and Saphar cited in (FS) in Theorem 1.1.

(b)⇒(a′). From (FS) in Theorem 1.1 it follows that $\Phi = W \circ V^{-1} : \mathcal{K}(Y,X)^* \to \mathcal{K}(Y,X^{**})^*$ is a Hahn–Banach extension operator.

(a)⇒(d) and (a′)⇒(e) follow by using Goldstine's theorem.

(e)⇒(f) follows from the result by Godefroy and Saphar cited in (GS) in Theorem 1.1.

(f)⇒(g). Note that a Banach space is always an ideal in its bidual space. Since (f) is true for all equivalent renormings of X, (g) follows from (f) for Y reflexive. To show that this implies (g) for all Banach spaces Y, we use the local characterization of ideals and the factorization lemma. (See Theorem 3.1 in [15] for details.)

(g)⇒(a). This follows from [22, Theorem 4.4]. □

Remark 2.1. From Proposition 2.2 (g) it is immediate that if X is CLR then every equivalent renorming of X is CLR.

We do not know whether or not all ideals in X are CLR whenever X is, but let us prove the converse.

Corollary 2.3. *Let X be a Banach space. If every separable ideal in X is* CLR*, then X is* CLR.

Proof. Let Y be a reflexive Banach space. Let $u = \sum_{n=1}^{\infty} x_n^* \otimes y_n \in X^* \widehat{\otimes}_\pi Y$ such that $u \in \ker V$ and $T \in \mathcal{K}(Y, X^{**})$. By Proposition 2.2 (c) it is enough to show that $\langle u, T\rangle = 0$.

By [9, Lemma III.4.3] there is a separable ideal $Z \subseteq X$ with $\psi \in \mathrm{HB}(Z,X)$ such that $\{x_n^*\}_{n=1}^{\infty} \subset \psi(Z^*)$. Find $z_n^* \in Z^*$ such that $x_n^* = \psi(z_n^*)$ for all n and define $v = \sum_{n=1}^{\infty} z_n^* \otimes y_n$. Then $u = (\psi \otimes I_Y)(v)$. Let $i_Z : Z \to X$ be the natural embedding. We have $i_Z^* \psi(z^*) = z^*$ for all $z^* \in Z^*$. Assume that $S \in \mathcal{K}(Y,Z)$. Then $i_Z S \in \mathcal{K}(Y,X)$ and

$$0 = \langle u, i_Z S\rangle = \langle (\psi \otimes I_Y)(v), i_Z S\rangle = \langle v, S\rangle$$

and thus, by assumption,

$$0 = \langle v, \psi^* T\rangle = \langle u, T\rangle$$

as desired. □

Corollary 2.4. *Let X be a Banach space. Assume that X has the* AP *and is* CLR*. Then every ideal in X is* CLR.

Proof. Let $Z \subseteq X$ be an ideal with $\psi \in \mathrm{HB}(Z,X)$ and let Y be a reflexive Banach space.

Assume that $v \in Z^* \widehat{\otimes}_\pi Y$ is such that $\langle v, S\rangle = 0$ for all $S \in \mathcal{K}(Y,Z)$. Let $u = (\psi \otimes I_Y)(v) \in X^* \widehat{\otimes}_\pi Y$ and $T \in \mathcal{K}(Y,X)$. Since X has the AP we may assume that $T = y^* \otimes x$. We get $\langle u, T\rangle = \langle v, \psi^* T\rangle$, where $\psi^* T \in \mathcal{F}(Y, Z^{**})$. Proposition 2.1 tells us that $\langle v, \psi^* T\rangle = 0$ and so, by assumption, $\langle u, S\rangle = 0$ for all $S \in \mathcal{K}(Y, X^{**})$. Let

$T\in\mathcal{K}(Y,Z^{**})$. Then

$$0=\langle u,i_Z^{**}T\rangle=\langle v,T\rangle,$$

so that Z is CLR by Proposition 2.2 (c). $\square$

Remark 2.2. Assume that a Banach space X is CLR and that Z is an ideal in X. Using trace duality and the local characterization of ideals we can show that Z is CLR if and only if $\mathcal{K}(Y,Z)$ is an ideal in $\mathcal{K}(Y,X)$ for every reflexive Banach space Y.

By using [26, Theorem 1.e.4 (v)], [22, Theorem 4.5], Corollary 2.4 and the fact that a Banach space X has the AP if and only if every separable ideal in X has the AP we get the following result.

Proposition 2.5. *Let X be a Banach space. The following statements are equivalent*:

(a) X^* *has the* AP.

(b) X *has the* AP *and is* CLR.

(c) *Every separable ideal in X has the* AP *and is* CLR.

From [18, Theorem 4.6] it follows that if X^* has the CAPconj, then X is CLR. This was used in the proof of the following result which is from [25, Theorem 3.8]. The MCAP part of the proposition is similar to Corollary 3.6 in [21].

Proposition 2.6. *Let X be a Banach space. The following statements are equivalent*:

(a) X^* *has the* CAPconj (*resp.* MCAPconj).

(b) X^* *has the* CAP (*resp.* MCAP) *and X is* CLR.

(c) X^* *has the* CAP (*resp.* MCAP) *and $\mathcal{K}(X,\widehat{X})$ is an ideal in $\mathcal{K}(X,\widehat{X}^{**})$ for every equivalent renorming $\widehat{X}$ of X.*

Remark 2.3. There are a number of spaces which are not CLR. These include the Casazza–Jarchow space [1, Theorem 1] (cf. Example 1.2 in [18]), the Johnson–Schechtman space [12, Corollary JS], and the space $\ell_2\widehat{\otimes}_\pi\ell_2$. In fact, any Banach space with the AP such that the dual does not have the AP is not CLR by Proposition 2.5.

Consider $X=\ell_2\widehat{\otimes}_\pi\ell_2$. If X was CLR then, by Proposition 2.5, $X^*=\mathcal{L}(\ell_2,\ell_2)$ would have the AP contradicting Szankowski [32]. From Example 3.9 in [22] we see that there exists a $\varphi\in\mathrm{HB}(X,X^{**})$, with $\varphi\neq k_{X^*}$, such that for every reflexive Banach space Y and every $T\in\mathcal{K}(Y,X^{**})$ there is a net $(S_\alpha)\subset\mathcal{K}(Y,X)$ with $\sup_\alpha\|S_\alpha\|\le\|T\|$ such that $S_\alpha\to T^*\varphi$ in the strong operator topology.

In the two following propositions we consider Proposition 2.2 in light of these Hahn–Banach extension operators. In the first, which is a continuation of Proposition 2.2, we allow renorming of the space but in the second proposition we do not.

Proposition 2.7. *Let X be a Banach space. The following statements are equivalent*:

(a) *X is* CLR.

(b) *For every $\varphi\in\mathrm{HB}(X,X^{**})$ and every reflexive Banach space Y, we have*

$$\{\varphi^*T : T\in\mathcal{K}(Y,X^{**})\}\subset\mathcal{K}(Y,X)^{**}.$$

(c) *For every $\varphi\in\mathrm{HB}(X,X^{**})$ and every Banach space Y, there exists an isometric embedding $U\colon\mathcal{K}(Y,X^{**})\to\mathcal{K}(Y,X)^{**}$ such that $\langle u,U(T)\rangle=\langle u,\varphi^*T\rangle$ for all $T\in\mathcal{K}(Y,X^{**})$ and $u\in X^*\widehat{\otimes}_\pi Y^{**}$.*

(d) *For every Banach space Y, there exists an isometric embedding*

$$U\colon\mathcal{K}(Y,X^{**})\longrightarrow\mathcal{K}(Y,X)^{**}$$

such that $\langle u,U(T)\rangle=\langle u,T\rangle$ for all $T\in\mathcal{K}(Y,X^{**})$ and $u\in X^*\widehat{\otimes}_\pi Y^{**}$.

Proof. (c)⇒(d) is trivial with $\varphi=k_{X^*}$.

(a)⇒(b) and (d)⇒(a) are straightforward using Proposition 2.2 (f).

(b)⇒(c). Let $\varphi\in\mathrm{HB}(X,X^{**})$ and let Y be a Banach space. Let $H\subset\mathcal{K}(Y,X^{**})$ be a finite-dimensional subspace. Using the factorization lemma, see [15], we find a reflexive Banach space Z "sitting inside" X^{**} and a compact operator $J\in\mathcal{K}(Z,X^{**})$ with $\|J\|\le1$ such that every $T\in H$ has a factorization

$$Y\xrightarrow{\widehat{T}}Z\xrightarrow{J}X^{**},$$

with $T=J\circ\widehat{T}$ and $\|T\|=\|\widehat{T}\|$. By assumption $\varphi^*J\in\mathcal{K}(Z,X)^{**}$. Thus we can find a net $(J_\alpha)\subset\mathcal{K}(Z,X)$ such that $\sup_\alpha\|J_\alpha\|\le1$ and $J_\alpha\to\varphi^*J$ weak* in $\mathcal{K}(Z,X)^{**}$.

Define linear operators $U_\alpha\colon H\to\mathcal{K}(Y,X)$ by $U_\alpha(T)=J_\alpha\circ\widehat{T}$. Then we get a net $(U_\alpha)\subset\mathcal{L}(H,\mathcal{K}(Y,X))$ with $\sup_\alpha\|U_\alpha\|\le1$. Going to a subnet, we may assume that $U_H=\omega^*\text{-}\lim_\alpha U_\alpha$ exists in $\mathcal{L}(H,\mathcal{K}(Y,X))^{**}=\mathcal{L}(H,\mathcal{K}(Y,X)^{**})$.

Let $T\in H$ and $u=\sum_{n=1}^\infty x_n^*\otimes y_n^{**}\in X^*\widehat{\otimes}_\pi Y^{**}$. Then $u\otimes T\in\mathcal{K}(Y,X)^*\widehat{\otimes}_\pi H=\mathcal{L}(H,\mathcal{K}(Y,X))^*$. We get

$$\begin{aligned}\langle u,U_H(T)\rangle&=\langle u\otimes T,U_H\rangle=\lim_\alpha\langle u\otimes T,U_\alpha\rangle=\lim_\alpha\langle u,U_\alpha(T)\rangle\\&=\lim_\alpha\sum_{n=1}^\infty(J_\alpha^*x_n^*)(\widehat{T}^{**}y_n^{**})=\langle u,\varphi^*T\rangle.\end{aligned}$$

Now we use the Lindenstrauss compactness argument to complete the proof. □

When we do not include renorming we get the following result, which can be seen as an extension of Theorem 2.3 in [19].

Proposition 2.8. *Let X be a Banach space. The following statements are equivalent*:

(a) $\mathcal{K}(Y,X)$ *is an ideal in* $\mathcal{K}(Y,X^{**})$ *for every reflexive Banach space* Y.

(b) *There exists* $\varphi\in\mathrm{HB}(X,X^{**})$ *such that*

$$\{\varphi^*T : T\in\mathcal{K}(Y,X^{**})\}\subset\mathcal{K}(Y,X)^{**}$$

for every reflexive Banach space Y.

(c) *There exists* $\varphi\in\mathrm{HB}(X,X^{**})$ *such that for every Banach space* Y, *there exists an isometric embedding* $U\colon\mathcal{K}(Y,X^{**})\to\mathcal{K}(Y,X)^{**}$ *such that* $\langle u,U(T)\rangle=\langle u,\varphi^*T\rangle$ *for all* $T\in\mathcal{K}(Y,X^{**})$ *and* $u\in X^*\widehat{\otimes}_\pi Y^{**}$.

Remark 2.4. Proposition 2.8 remains true if we replace $\mathcal{K}(Y,X^{**})$ by $\mathcal{W}(Y,X^{**})$ everywhere. But note that (a) in Proposition 2.8 does not imply that $\mathcal{K}(Y,X)$ is an ideal in $\mathcal{W}(Y,X^{**})$ for every Banach space Y. To see this just take X reflexive without the CAP and confer Theorem 1.1 in [24].

Proof. (a)$\Rightarrow$(b). Let Y be reflexive. From [17] it follows that there exists $\varphi\in\mathrm{HB}(X,X^{**})$ such that for every reflexive Banach space Y, there exists a Hahn–Banach extension operator

$$\Phi\colon\mathcal{K}(Y,X)^*\longrightarrow\mathcal{K}(Y,X^{**})^*$$

with $\Phi(x^*\otimes y)=(\varphi x^*)\otimes y$ for all $y\in Y$ and $x^*\in X^*$. Assume that $u=\sum_{n=1}^\infty x_n^*\otimes y_n\in X^*\widehat{\otimes}_\pi Y$ and that $u=0$ on $\mathcal{K}(Y,X)$. Then $\Phi(u)=0$ on $\mathcal{K}(Y,X^{**})$. Thus for every $T\in\mathcal{K}(Y,X^{**})$,

$$0=(\Phi(u))(T)=\sum_{n=1}^\infty x_n^*(\varphi^*Ty_n)=\langle u,\varphi^*T\rangle.$$

From this it follows that $\{\varphi^*T:T\in\mathcal{K}(Y,X^{**})\}\subset\overline{\mathcal{K}(Y,X)}^{w^*}=\mathcal{K}(Y,X)^{**}$.

(b)$\Rightarrow$(c) is contained in the proof of (b)$\Rightarrow$(c) in Proposition 2.7.

(c)$\Rightarrow$(b). This follows from the characterization of $\mathcal{K}(Y,X)^{**}$ given by Godefroy and Saphar (see Theorem 1.1). $\square$

Now we want to look at $\mathcal{K}(Y,X)$ as a subspace of $\mathcal{W}(Y,X^{**})$. (d) in the next proposition should be compared with (g) in Proposition 2.2. Also note that when replacing $\mathcal{K}(Y,\widehat{X})$ with $\mathcal{F}(Y,\widehat{X})$ in Proposition 2.9 (d) and Proposition 2.2 (g) we get two statements which both are equivalent to the AP for X^* (see [22, Theorem 4.5]).

Proposition 2.9. *Let X be a Banach space. The following statements are equivalent*:

(a) *The trace mapping $V\colon X^*\widehat{\otimes}_\pi Y\to\mathcal{K}(Y,X)^*$ is isometric for every reflexive Banach space Y.*

(b) *The trace mapping $V\colon X^*\widehat{\otimes}_\pi Y\to\mathcal{K}(Y,X)^*$ is one-to-one for every reflexive Banach space Y.*

(c) *For every reflexive Banach space Y, we have $\mathcal{K}(Y,X)^{**}=\mathcal{W}(Y,X^{**})$.*

(d) *$\mathcal{K}(Y,\widehat{X})$ is an ideal in $\mathcal{W}(Y,\widehat{X}^{**})$ for every Banach space Y and every equivalent renorming $\widehat{X}$ of X.*

(e) *For every Banach space Y and every $T\in\mathcal{W}(Y,X^{**})$, there is a net $(S_\alpha)\subset\mathcal{K}(Y,X)$ with $\sup_\alpha\|S_\alpha\|\le\|T\|$ such that $S_\alpha^*\to T^*|_{X^*}$ in the strong operator topology.*

(f) *For every Banach space Y, there is an isometric embedding $U\colon\mathcal{W}(Y,X^{**})\to\mathcal{K}(Y,X)^{**}$ with $V^*(U(T))=T$ for $T\in\mathcal{W}(Y,X^{**})$, where $V\colon X^*\widehat{\otimes}_\pi Y^{**}\to\mathcal{K}(Y,X)^*$.*

Proof. (a)⇒(c) and (b)⇒(a). These implications follow from (GS) and (FS) in Theorem 1.1, respectively.

(d)⇒(b) and (d)⇒(e) follow from [22, Theorem 4.4].

(c)⇒(d). Equality in (c) is preserved when renorming X. To pass from reflexive to general Y we use the local characterization of ideals and the factorization lemma.

(e)⇒(f). Let $\mathcal{F}_Z$ denote the set of finite-dimensional subspaces of a Banach space Z. Let Y be a Banach space. Define an index set $I=\mathcal{F}_{\mathcal{W}(Y,X^{**})}\times\mathcal{F}_{X^*\widehat{\otimes}_\pi Y^{**}}\times(0,\infty)$. I becomes a directed set with the order $(H,G,\varepsilon)\le(\widehat{H},\widehat{G},\hat{\varepsilon})$ if $H\subset\widehat{H}$, $G\subset\widehat{G}$, and $\hat{\varepsilon}\le\varepsilon$. Let $\mathcal{U}$ be an ultrafilter on I refining the order filter on I.

Let $i=(H,G,\varepsilon)\in I$. By the factorization lemma (cf. Theorem 2.3 in [15]) there exist a reflexive Banach space Z, a $J\in\mathcal{W}(Z,X^{**})$ with norm one, and an isometric embedding $\Phi\colon H\to\mathcal{W}(Y,Z)$ such that we have a factorization

$$Y\xrightarrow{\Phi(\,\cdot\,)}Z\xrightarrow{J}X^{**},\quad T=J\circ\Phi(T)\text{ for all }T\in H.$$

Choose an ε-net $\{u_i\}_{i=1}^k$ for the unit sphere of G. Choose representations $u_i=\sum_{n=1}^\infty x_{i,n}^*\otimes y_{i,n}^{**}$ for $i=1,...,k$. We may assume that $\|y_{i,n}^{**}\|=1$ for all i and n. There is an N such that $\sum_{n>N}\|x_{i,n}^*\|<\varepsilon/3$ for $i=1,...,k$.

By (e), we can find $J_H\in\mathcal{K}(Z,X)$ with $\|J_H\|\le1$ such that

$$\|J_H^*x_{i,n}^*-J^*x_{i,n}^*\|<\varepsilon/3N$$

for $i=1,...,k$ and $n=1,...,N$. Define operators $U_H\colon H\to\mathcal{K}(Y,X)$ by $U_H(T)=J_H\circ\Phi(T)$. Clearly, $\|U_H\|\le1$ and $|\langle V(u_i),U_H(T)\rangle-\langle V(u_i),T\rangle|<\varepsilon\|T\|$ for $i=1,...,k$.

Thus we get

$$|\langle V(u), U_H(T)\rangle - \langle V(u), T\rangle| < 3\varepsilon\|u\|\,\|T\|$$

for all $T\in H$ and $u\in G$.

We obtain an isometric embedding $U\colon \mathcal{W}(Y,X^{**})\to\mathcal{K}(Y,X)^{**}$ by letting $U=(U_H)_{\mathcal{U}}\circ J_{\mathcal{W}(Y,X^{**})}$, where $J_{\mathcal{W}(Y,X^{**})}$ is an isometric embedding of $\mathcal{W}(Y,X^{**})$ into the ultraproduct $(\prod_I H)_{\mathcal{U}}$ of finite-dimensional subspaces of $\mathcal{W}(Y,X^{**})$ (cf. e.g. [10] or [3, Theorem 8.10]).

(f)$\Rightarrow$(a). Let Y be reflexive and let $u\in X^*\widehat{\otimes}_\pi Y$. Choose $T\in\mathcal{W}(Y,X^{**})$ with $\|T\|=1$ such that $\|u\|_\pi=\langle u,T\rangle$. Then $U(T)\in\mathcal{K}(Y,X)^{**}$, $\|U(T)\|\le 1$ and $\|u\|_\pi=\langle Vu, U(T)\rangle$. This shows that the trace mapping is isometric. □

3. $\mathcal{K}(X,Y)$ as a subspace of $\mathcal{W}(X,Y)$

In this section we shall look at the space $\mathcal{K}(X,Y)$ as a subspace of $\mathcal{W}(X,Y)$, in particular when Y is reflexive. Note that when Y is reflexive, then $\mathcal{W}(X,Y)=(Y^*\widehat{\otimes}_\pi X)^*$. We will also use Godefroy and Saphar's identification of $\mathcal{K}(X,Y)^{**}$ cited in (GS) in Theorem 1.1.

In Theorem 3.2 in [20] Lima and Oja showed that Proposition 3.1 (a) is equivalent to the weak MAP if we replace $\mathcal{K}(X,Y)^*$ with $\mathcal{F}(X,Y)^*$. This should be compared to Proposition 5.8 below where we relate Proposition 3.1 to the compact approximation properties we will introduce in Sections 4 and 5. See also Theorem 5.3 and Corollary 5.4 for "local" versions of the next proposition.

Proposition 3.1. *Let X be a Banach space. The following statements are equivalent*:

(a) *The trace mapping $V\colon Y^*\widehat{\otimes}_\pi X\to\mathcal{K}(X,Y)^*$ is isometric for every reflexive Banach space Y.*

(b) *The trace mapping $V\colon Y^*\widehat{\otimes}_\pi Z\to\mathcal{K}(Z,Y)^*$ is isometric for every separable ideal Z in X and every reflexive Banach space Y.*

(c) *For every Banach space Y, there exists an isometric embedding*

$$U\colon \mathcal{W}(X,Y)\longrightarrow\mathcal{K}(X,Y)^{**}$$

such that $V^*\circ U=I_{\mathcal{W}(X,Y)}$, where V is the trace mapping.

(d) *There exists $\varphi\in\mathrm{HB}(X,X^{**})$ such that for every reflexive Banach space Y,*

$$\{T^{**}\varphi^*|_{X^{**}} : T\in\mathcal{W}(X,Y)\}\subset\mathcal{K}(X,Y)^{**}\subseteq\mathcal{W}(X^{**},Y).$$

(e) *There exists $\varphi\in\mathrm{HB}(X,X^{**})$ such that for every Banach space Y and every $T\in\mathcal{W}(X,Y)$, there exists $S\in\mathcal{K}(X,Y)^{**}$ with $\|S\|\le\|T\|$ and $\langle T^{**}\varphi^*|_{X^{**}},u\rangle=\langle S,u\rangle$ for every $u\in Y^*\widehat{\otimes}_\pi X^{**}$.*

(f) *For every Banach space* Y *and every* $T\in\mathcal{W}(X,Y)$, *there exists a net* $(S_\alpha)\subset\mathcal{K}(X,Y)$ *with* $\sup_\alpha\|S_\alpha\|\le\|T\|$ *such that* $S_\alpha\to T$ *in the strong operator topology.*

Proof. (a)⇒(b). Let Z be a separable ideal in X. For every reflexive Banach space Y we have that $Y^*\widehat{\otimes}_\pi Z$ is a subspace of $Y^*\widehat{\otimes}_\pi X$ (cf. e.g. [27, Theorem 3.1]). Let $u\in Y^*\widehat{\otimes}_\pi Z$ and $\varepsilon>0$. Choose $S\in\mathcal{K}(X,Y)$ with $\|S\|=1$ and $\operatorname{tr}(Su)>1-\varepsilon$. Then $S|_Z\in\mathcal{K}(Z,Y)$ and $\operatorname{tr}(S|_Z u)>1-\varepsilon$ and we have (b).

(b)⇒(a). Let $u\in Y^*\widehat{\otimes}_\pi X$. There is a separable ideal Z in X so that $u\in Y^*\widehat{\otimes}_\pi Z$, a subspace of $Y^*\widehat{\otimes}_\pi X$. Let $\varphi\in\mathrm{IB}(Z,X)$ and $\varepsilon>0$. Choose $S\in\mathcal{K}(Z,Y)$ with $\|S\|=1$ and $\operatorname{tr}(Su)>1-\varepsilon$. Letting $T=S^{**}\circ\varphi^*|_X\in\mathcal{K}(X,Y)$ we see that $\operatorname{tr}(Tu)>1-\varepsilon$ and (a) follows.

(a)⇒(f). Let Y be a Banach space and let $T\in\mathcal{W}(X,Y)$. By factorizing T through a reflexive Banach space (using the factorization lemma), we may assume that Y is reflexive. Since V is isometric, there exists $S\in\mathcal{K}(X,Y)^{**}$ with $\|S\|=\|T\|$ such that $V^*(S)=T$. Let (S_α) be a net in $\mathcal{K}(X,Y)$ with $\sup_\alpha\|S_\alpha\|\le\|S\|$ such that $S_\alpha\to S$ weak*. Then $S_\alpha\to T$ in the weak operator topology. By taking a new net from $\operatorname{conv}(S_\alpha)$, we may assume that $S_\alpha\to T$ in the strong operator topology.

(f)⇒(c) is similar to the proof of (e)⇒(f) in Proposition 2.9 but we use Corollary 2.4 from [15] instead of Theorem 2.3 from [15].

(c)⇒(a). We have $V^*\colon\mathcal{K}(X,Y)^{**}\to\mathcal{W}(X,Y)$. Hence, if $u\in Y^*\widehat{\otimes}_\pi X$, we find $T\in\mathcal{W}(X,Y)$ with $\|T\|=1$ such that $\|u\|_\pi=\operatorname{tr}(Tu)$. Then $U(T)\in\mathcal{K}(X,Y)^{**}$ and $\|u\|_\pi=\langle U(T),V(u)\rangle$. From this we get (a).

(c)⇒(d). Consider the collection of weakly compact subsets of the dual unit ball. Let

$$\mathfrak{K}=\{K\subset B_{X^*}: K\text{ is absolutely convex and weakly compact}\}$$

and let $\mathcal{F}_{X^*}$ denote the set of finite-dimensional subspaces of X^*. Define an index set $I=\mathcal{F}_{X^*}\times\mathfrak{K}$. I becomes a directed set with the order $(F,K)\le(\widehat{F},\widehat{K})$ if $F\subset\widehat{F}$ and $K\subset\widehat{K}$. Let $\mathcal{U}$ be an ultrafilter on I which refines the order filter on I.

For $K\in\mathfrak{K}$ we let $[Z,J]=\mathrm{DFJP}(K)$. Then Z is reflexive and $J\in\mathcal{W}(Z,X^*)$ has norm one. Since $J^*|_X\in\mathcal{W}(X,Z^*)$ we use (c) to find an isometric embedding $U\colon\mathcal{W}(X,Z^*)\to\mathcal{K}(X,Z^*)^{**}$ such that $V^*U=I_{\mathcal{W}(X,Z^*)}$, where $V\colon Z\widehat{\otimes}_\pi X\to\mathcal{K}(X,Z^*)^*$ is the trace mapping.

If $x^*=Jz$ for some $z\in Z$, then

$$\langle U(J^*|_X),z\otimes x\rangle=\langle J^*|_X(x),z\rangle=J(z)(x)=x^*(x)\tag{3.1}$$

for all $x\in X$. Note that $\|U(J^*|_X)\|\le1$ and, by (GS) in Theorem 1.1, $U(J^*|_X)\in\mathcal{K}(X,Z^*)^{**}\subseteq\mathcal{W}(X^{**},Z^*)$. Thus $U(J^*|_X)^*\in\mathcal{W}(Z,X^{***})$. Note that if $F\subset X^*$ is a finite-dimensional subspace such that $B_F\subset K$, then for every $x^*=Jz\in F$ we have $\|z\|=\|x^*\|$.

X^* is isometrically isomorphic to a subspace of the ultraproduct $(\prod_I F)_{\mathcal{U}}$ of its finite-dimensional subspaces (cf. e.g. [3, Theorem 8.8]). For $i=(F,K)\in I$, let $U_i=U(J^*|_X)^*$ if $B_F\subset K$ and $U_i=0$ otherwise. We get a linear map from the ultraproduct into X^{***}. Combine this with the isometric embedding and we get a map $\varphi: X^*\to X^{***}$ (cf. e.g. [3, Theorem 8.10]). We see that φ is linear with norm less than one. By (3.1), φ is a Hahn–Banach extension operator.

Let Y be a reflexive Banach space and let $T\in\mathcal{W}(X,Y)$. We may assume that $\|T\|=1$. We need to show that $T^{**}\varphi^*|_{X^{**}}\in\mathcal{K}(X,Y)^{**}\subseteq\mathcal{W}(X^{**},Y)$. Let $u\in Y^*\widehat{\otimes}_\pi X^{**}$ be such that $u=0$ on $\mathcal{K}(X,Y)$.

Choose a representation $u=\sum_{n=1}^\infty y_n^*\otimes x_n^{**}$. We may assume that $\|y_n^*\|=1$ for all n. Factorize $T^*=J\circ\widehat{T^*}$ through Z and for $n=1,2,\dots$ let $z_n=\widehat{T^*}y_n^*$. Let $\varepsilon>0$ and choose N such that $\sum_{n=N+1}^\infty\|x_n^{**}\|<\varepsilon/3$. Choose $K\in\mathfrak{K}$ such that $T^*(B_{Y^*})\subset K$ and

$$|\varphi(x_n^*)(x_n^{**})-U(J^*|_X)^*z_n(x_n^{**})|<\varepsilon/3N,$$

where $x_n^*=T^*y_n^*=Jz_n\in K$ for $n=1,\dots,N$ and $[Z,J]=\mathrm{DFJP}(K)$.

Let $v=\widehat{T^*}u$. Then $v=0$ on $\mathcal{K}(X,Z^*)$. Since $U(J^*|_X)\in\mathcal{K}(X,Z^*)^{**}$ we have

$$0=\langle v,U(J^*|_X)\rangle=\sum_{n=1}^\infty(U(J^*|_X)^*z_n)(x_n^{**})$$

and so

$$\begin{aligned}\left|\sum_{n=1}^\infty(\varphi T^*y_n^*)(x_n^{**})\right| &= \left|\sum_{n=1}^\infty(\varphi T^*y_n^*)(x_n^{**})-\sum_{n=1}^\infty(U(J^*|_X)^*z_n)(x_n^{**})\right|\\ &\le\sum_{n=1}^N|(\varphi T^*y_n^*-U(J^*|_X)^*z_n)(x_n^{**})|\\ &\quad+\sum_{n=N+1}^\infty\|\varphi T^*y_n^*-U(J^*|_X)^*z_n\|\|x_n^{**}\|\\ &<\varepsilon/3+2\varepsilon/3\\ &=\varepsilon.\end{aligned}$$

Since ε was arbitrary this shows that $\langle u,T^{**}\varphi^*|_{X^{**}}\rangle=0$ so that $T^{**}\varphi^*|_{X^{**}}\in\mathcal{K}(X,Y)^{**}$.

(d)$\Rightarrow$(e). Let Y be a Banach space and let $T\in\mathcal{W}(X,Y)$ with $\|T\|=1$. Let $K=T^*(B_{Y^*})\subset B_{X^*}$, and let $[Z,J]=\mathrm{DFJP}(K)$. We have $J^*|_X\in\mathcal{W}(X,Z^*)$. By (d), there exists $\varphi\in\mathrm{HB}(X,X^{**})$ such that $J^*\varphi^*|_{X^{**}}\in\mathcal{K}(X,Z^*)^{**}$,

$$X^{**}\xrightarrow{\varphi^*|_{X^{**}}}X^{**}\xrightarrow{J^*}Z^*\xrightarrow{(\widehat{T^*})^*}Y.$$

Let $(S_\alpha)\subset\mathcal{K}(X,Z^*)$ be such that $S_\alpha\to J^*\varphi^*|_{X^{**}}$ weak* and $\sup_\alpha\|S_\alpha\|\le 1$, and define $T_\alpha=(\widehat{T^*})^*\circ S_\alpha\in\mathcal{K}(X,Y)$. Then $\sup_\alpha\|T_\alpha\|\le\|T\|$. Going to a subnet, we may assume that $T_\alpha\to S\in\mathcal{K}(X,Y)^{**}$ weak*. From this (e) follows.

(e)⇒(f). This follows from Goldstine's theorem and by using that $\varphi^*x=x$ for all $x\in X$. □

The next result shows what we get if we allow renorming of X in Proposition 3.1.

Proposition 3.2. *Let X be a Banach space. The following statements are equivalent*:

(a) *The trace mapping $V: Y^*\widehat{\otimes}_\pi\widehat{X}\to\mathcal{K}(\widehat{X},Y)^*$ is isometric for every reflexive Banach space Y and every equivalent renorming $\widehat{X}$ of X.*

(b) *$\mathcal{W}(X,Y)\subseteq\mathcal{K}(X,Y)^{**}$ for every reflexive Banach space Y.*

(c) *$\mathcal{K}(X,Y)$ is an ideal in $\mathcal{W}(X,Y)$ for every Banach space Y.*

(d) *For every Banach space Y, there exists an isometric embedding U: $\mathcal{W}(X,Y)\to\mathcal{K}(X,Y)^{**}$ such that $\langle u,T\rangle=\langle u,U(T)\rangle$ for all $u\in Y^*\widehat{\otimes}_\pi X^{**}$.*

(e) *For every Banach space Y and every operator $T\in\mathcal{W}(X,Y)$, there exists a net $(S_\alpha)\subset\mathcal{K}(X,Y)$ with $\sup_\alpha\|S_\alpha\|\le\|T\|$ such that $S^*_\alpha\to T^*$ in the strong operator topology.*

Proof. (a)⇒(b). From (a)⇒(d) in Proposition 3.1 we get that for all equivalent renormings X_F of X there exists $\varphi_F\in\mathrm{HB}(X_F,X_F^{**})$ such that

$$\begin{aligned}\{T^{**}\varphi_F^*|_{X^{**}}:T\in\mathcal{W}(X,Y)\}&=\{T^{**}\varphi_F^*|_{X_F^{**}}:T\in\mathcal{W}(X_F,Y)\}\\&\subset\mathcal{K}(X_F,Y)^{**}=\mathcal{K}(X,Y)^{**}.\end{aligned}$$

For a finite-dimensional subspace $F\subset X^*$ let X_F be an equivalent renorming of X which is $1+1/\dim F$ close to the original norm and such that the norm on X_F^* is locally uniformly rotund on F (cf. [24, Lemma 2.4]). For the above corresponding $\varphi_F\in\mathrm{HB}(X_F,X_F^{**})$ we have $\varphi_F(x^*)=x^*$ for all $x^*\in F$.

By going to a subnet we may assume that $\varphi=\omega^*\text{-}\lim\varphi_F$ for some operator $\varphi\in\mathcal{L}(X^*,X^{***})$. Clearly we must have $\varphi=k_{X^*}$. If $T\in\mathcal{W}(X,Y)$ and $u\in Y^*\widehat{\otimes}_\pi X^{**}$ then $(T^*\otimes I)(u)\in X^*\widehat{\otimes}_\pi X^{**}$, hence $T^{**}\varphi_F^*|_{X^{**}}\to T^{**}$ in the weak* topology in $\mathcal{W}(X^{**},Y)$ which is enough by the above set inclusion.

(b)⇒(a) is contained in (d)⇒(a) in Proposition 3.1, with the natural embedding as the extension operator, since (b) remains true when we renorm X.

(b)⇒(c) is trivial when Y is reflexive. The general case follows by using the factorization lemma and local characterization of ideals (see Theorem 3.1 in [15] for details).

(c)⇒(e) is contained in [15, Corollary 4.3].

(e)⇒(b). Let Y be reflexive and let $T \in \mathcal{W}(X,Y)$. Let $(S_\alpha) \subset \mathcal{K}(X,Y)$ be a net as in (e). For every $u = \sum_{n=1}^{\infty} y_n^* \otimes x_n^{**} \in Y^* \widehat{\otimes}_\pi X^{**}$, we get $\langle u, T\rangle = \lim_\alpha \langle u, S_\alpha\rangle$. Hence $T \in \overline{\mathcal{K}(Y,X)}^{w^*} = \mathcal{K}(X,Y)^{**} \subseteq \mathcal{W}(X^{**},Y)$.

(d)⇒(b). Let Y be reflexive and let the embedding U be as in (d). Assume that $u = \sum_{n=1}^{\infty} y_n^* \otimes x_n^{**} \in Y^* \widehat{\otimes}_\pi X^{**}$ is 0 on $\mathcal{K}(X,Y)$. Then $u=0$ on $\mathcal{K}(X,Y)^{**}$. For every $T \in \mathcal{W}(X,Y)$, we have $U(T) \in \mathcal{K}(X,Y)^{**}$ and

$$\langle u, T\rangle = \langle u, U(T)\rangle = 0.$$

This shows that $\mathcal{W}(X,Y) \subseteq \overline{\mathcal{K}(X,Y)}^{w^*} = \mathcal{K}(X,Y)^{**}$.

(a)⇒(d). As in (a)⇒(b), by renorming X, we can assume that the extension operator in Proposition 3.1 (e) satisfies $\varphi = k_{X^*}$. Thus (a) implies the following statement:

(d′) For every Banach space Y and every operator $T \in \mathcal{W}(X,Y)$, there exists $S \in \mathcal{K}(X,Y)^{**}$ with $\|S\| \le \|T\|$ such that $\langle u, T\rangle = \langle u, S\rangle$ for all $u \in Y^* \widehat{\otimes}_\pi X^{**}$.

We cannot be sure that the map $T \mapsto S$ in (d′) is linear, so let us prove that we may assume it is.

Let $H \subset \mathcal{W}(X,Y)$ be a finite-dimensional subspace. We now use the factorization lemma to produce a reflexive Banach space Z and a norm-one operator $J \in \mathcal{W}(X,Z)$ such that for every $T \in H$ there is an operator $T_H \in \mathcal{W}(Z,Y)$ with $\|T\| = \|T_H\|$ such that $T = T_H J$,

$$X \xrightarrow{J} Z \xrightarrow{T_H} Y.$$

By (d′) there exists $J_H \in \mathcal{K}(X,Z)^{**}$ such that $\|J_H\| \le 1$ and $\langle u, J\rangle = \langle u, J_H\rangle$ for all $u \in Z^* \widehat{\otimes}_\pi X^{**}$.

Let $(J_\alpha) \subset \mathcal{K}(X,Z)$ be a net such that $\sup_\alpha \|J_\alpha\| \le 1$ and $J_\alpha \to J_H$ weak*. Since H has finite dimension, we may assume that $\omega^*\text{-}\lim_\alpha T_H J_\alpha$ exists in $\mathcal{K}(X,Y)^{**}$ for every $T \in H$. Let us define a map $U_H \colon H \to \mathcal{K}(X,Y)^{**}$ by $U_H(T) = \omega^*\text{-}\lim_\alpha T_H J_\alpha$. Clearly U_H is a linear operator and $\|U_H\| \le 1$.

If $u = \sum_{n=1}^{\infty} y_n^* \otimes x_n^{**} \in Y^* \widehat{\otimes}_\pi X^{**}$, and $T \in H$, we define $v = \sum_{n=1}^{\infty} (T_H^* y_n^*) \otimes x_n^{**} \in Z^* \widehat{\otimes}_\pi X^{**}$. Then we get

$$\langle u, T\rangle = \langle v, J\rangle = \lim_\alpha \langle v, J_\alpha\rangle = \lim_\alpha \langle u, T_H J_\alpha\rangle = \langle u, U_H(T)\rangle.$$

Now it only remains to use Lindenstrauss' compactness argument to prove the existence of the operator $U \colon \mathcal{W}(X,Y) \to \mathcal{K}(X,Y)^{**}$. □

For comparison with what happens when dealing with finite-rank operators instead of compact operators we state the following result where only (c) seems to be new. The other parts can be found in [15, Theorem 3.4], [20, Theorems 3.2, 3.6 and 4.2] and [22, Theorem 4.5] (see also [23, Section 3]). In particular, we see that for finite-rank operators Propositions 2.9 and 3.2 are equivalent.

Theorem 3.3. *Let X be a Banach space. The following statements are equivalent*:

(a) $\widehat{X}$ *has the weak* MAP *for every equivalent renorming* $\widehat{X}$ *of* X.

(b) X^* *has the* AP.

(c) $\mathcal{W}(X,Y)\subseteq\mathcal{F}(X,Y)^{**}$ *for every reflexive Banach space* Y.

(d) $\mathcal{F}(X,Y)$ *is an ideal in* $\mathcal{W}(X,Y)$ *for every Banach space* Y.

(e) $\mathcal{F}(Y,\widehat{X})$ *is an ideal in* $\mathcal{W}(Y,\widehat{X}^{**})$ *for every Banach space* Y *and every equivalent renorming* $\widehat{X}$ *of* X.

(f) *The trace mapping* $V\colon Y^*\widehat{\otimes}_\pi\widehat{X}\to\mathcal{F}(\widehat{X},Y)^*$ *is isometric for every reflexive Banach space* Y *and every equivalent renorming* $\widehat{X}$ *of* X.

(g) *The trace mapping* $V\colon X^*\widehat{\otimes}_\pi Y\to\mathcal{F}(Y,X)^*$ *is isometric for every reflexive Banach space* Y.

4. Weak MCAP

In this section we will study a natural compact companion to the weak MAP introduced and studied by Lima and Oja in [20].

Definition 4.1. A Banach space X has the *weak metric compact approximation property* (*weak* MCAP) if for every Banach space Y and every operator $T\in\mathcal{W}(X,Y)$ there exists a net $(S_\alpha)\subset\mathcal{K}(X,X)$ with $\sup_\alpha\|TS_\alpha\|\le\|T\|$ such that $S_\alpha\to I_X$ uniformly on compact sets in X.

Theorem 2.4 in [20] translates into the following theorem for the weak MCAP.

Theorem 4.1. *Let X be a Banach space. The following statements are equivalent*:

(a) X *has the weak* MCAP.

(b) *For every separable reflexive Banach space* Y *and operator* $T\in\mathcal{K}(X,Y)$ *there exists a net* $(S_\alpha)\subset\mathcal{K}(X,X)$ *with* $\sup_\alpha\|TS_\alpha\|\le\|T\|$ *such that* $S_\alpha\to I_X$ *in the strong operator topology.*

(c) *For every separable reflexive Banach space* Y *and operator* $T\in\mathcal{K}(X,Y)$ *there exists a net* $(S_\alpha)\subset\mathcal{K}(X,X)$ *with* $\sup_\alpha\|TS_\alpha\|\le\|T\|$ *such that* $TS_\alpha\to T$ *in the strong operator topology.*

(d) *For every Banach space Y, every operator $T \in \mathcal{W}(X,Y)$ with $\|T\|=1$, and all sequences $\{x_n\}_{n=1}^\infty \subset X$ and $\{y_n^*\}_{n=1}^\infty \subset Y^*$ with $\sum_{n=1}^\infty \|x_n\|\,\|y_n^*\| < \infty$ we have*

$$\left|\sum_{n=1}^\infty y_n^*(Tx_n)\right| \le \sup_{\substack{\|TS\|\le 1\\ S\in\mathcal{K}(X,X)}} \left|\sum_{n=1}^\infty y_n^*(TSx_n)\right|.$$

Proof. The proof is similar to the proof of Theorem 2.4 in [20]. $\square$

Recall from the introduction that a separable subspace Y of a Banach space X such that $\mathrm{IB}(Y,X)\neq\varnothing$ is called a *separable ideal*.

It is well-known, and not difficult to show, that a Banach space X has the (M)AP if and only if every separable ideal in X has the (M)AP (cf. [13, Theorem 5.1], [14, Corollary 2] and [27, Proposition 2.1]). The corresponding result for the weak MAP is also true. In fact, one can show this by arguing as in Theorem 5.3 below. For the weak MCAP we have the following result.

Proposition 4.2. *Let X be a Banach space such that every separable ideal in X has the weak* MCAP*. Then X itself has the weak* MCAP*.*

Proof. Let Y be a Banach space, let $T\in\mathcal{W}(X,Y)$, and let $\{x_n\}_{n=1}^\infty\subset X$ and $\{y_n^*\}_{n=1}^\infty \subset Y^*$ be sequences with $\sum_{n=1}^\infty \|x_n\|\,\|y_n^*\|<\infty$.

Let $Z\subseteq X$ be a separable ideal such that $\{x_n\}_{n=1}^\infty\subset Z$ and let $\varphi\in\mathrm{IB}(Z,X)$. Then $T_Z=T|_Z\in\mathcal{W}(Z,Y)$. Moreover if $S\in\mathcal{K}(Z,Z)$ with $\|T_ZS\|\le\|T\|$ then $i_Z\circ S^{**}\circ \varphi^*|_X\in\mathcal{K}(X,X)$ and $\|Ti_ZS^{**}\varphi^*|_X\|\le\|T_ZS\|\le\|T\|$. Since Z has the weak MCAP we use (d) from Theorem 4.1 and get

$$\begin{aligned}\left|\sum_{n=1}^\infty y_n^*(Tx_n)\right| = \left|\sum_{n=1}^\infty y_n^*(T_Zx_n)\right| &\le \sup_{\substack{\|T_ZS\|\le\|T_Z\|\\ S\in\mathcal{K}(Z,Z)}} \left|\sum_{n=1}^\infty y_n^*(T_ZSx_n)\right| \\ &\le \sup_{\substack{\|TS\|\le\|T\|\\ S\in\mathcal{K}(X,X)}} \left|\sum_{n=1}^\infty y_n^*(TSx_n)\right|.\end{aligned}$$

Using (d) in Theorem 4.1 again we see that X has the weak MCAP. $\square$

Similarly it is not difficult to show that X has the CAP (resp. MCAP) if every separable ideal in X has the CAP (resp. MCAP). If X is CLR the converse is true for the CAP, MCAP, and weak MCAP. The converse is open in general.

Next we have several equivalent formulations of the weak MCAP similar to the characterizations of the weak MAP in Theorem 2.6 and Proposition 3.1 in [23]. Characterizing the weak MCAP in terms of ideals of operators is not as simple as for the weak MAP. A similar contrast can be found between characterizing the AP [15] and the CAP [25].

Theorem 4.3. *Let X be a Banach space. The following statements are equivalent*:

(a) *X has the weak* MCAP.

(b) *There exists $\varphi\in\mathrm{HB}(X,X^{**})$ such that*

$$\varphi^*|_{X^{**}}\in\overline{\mathcal{K}(X,X)}^{w^*}$$

in $(X^*\widehat{\otimes}_\pi X^{**})^*=\mathcal{L}(X^{**},X^{**})$.

(c) *For every Banach space Y and every operator $T\in\mathcal{W}(Y,X^{**})$,*

$$\mathfrak{E}=\{S^{**}T:S\in\mathcal{K}(X,X)\}$$

is an ideal in $\mathfrak{F}=\mathrm{span}(\mathfrak{E},\{T\})$.

(d) *For every separable reflexive Banach space Y and operator $T\in\mathcal{K}(Y,X^{**})$,*

$$\mathfrak{E}=\{S^{**}T:S\in\mathcal{K}(X,X)\}$$

is an ideal in $\mathfrak{F}=\mathrm{span}(\mathfrak{E},\{T\})$.

(e) *There exists $\varphi\in\mathrm{HB}(X,X^{**})$ such that for every reflexive Banach space Y and every $T\in\mathcal{W}(Y,X^{**})$ we have $\varphi^*T\in\mathfrak{E}^{**}$, where $\mathfrak{E}=\{S^{**}T:S\in\mathcal{K}(X,X)\}$.*

(f) *There exists $\varphi\in\mathrm{HB}(X,X^{**})$ such that for every separable reflexive Banach space Y and every $T\in\mathcal{K}(Y,X^{**})$ we have $\varphi^*T\in\mathfrak{E}^{**}$, where $\mathfrak{E}=\{S^{**}T:S\in\mathcal{K}(X,X)\}$.*

(g) *There exists $\varphi\in\mathrm{HB}(X,X^{**})$ such that for all sequences $\{x_n^*\}_{n=1}^\infty\subset X^*$ and $\{x_n^{**}\}_{n=1}^\infty\subset X^{**}$ with $\sum_{n=1}^\infty\|x_n^*\|\,\|x_n^{**}\|<\infty$ and $\sum_{n=1}^\infty x_n^{**}(S^*x_n^*)=0$, for all $S\in\mathcal{K}(X,X)$, we have $\sum_{n=1}^\infty x_n^{**}(\varphi x_n^*)=0$.*

Proof. (c)⇒(d), (e)⇒(f), and (g)⇒(b) are trivial.

(a)⇒(b) is similar to the proof of Proposition 2.5 in [23].

(b)⇒(a) is similar to the proof of Proposition 2.3 in [23].

(b)⇒(c). Assume first that Y is reflexive and that $T\in\mathcal{W}(Y,X^{**})$. By assumption there exists a $\varphi\in\mathrm{HB}(X,X^{**})$ in the weak* closure of $\mathcal{K}(X,X)$ in $\mathcal{L}(X^{**},X^{**})$. Choose a net $(S_\alpha)\subset\mathcal{K}(X,X)$ such that $\omega^*\text{-}\lim_\alpha S_\alpha=\varphi^*|_{X^{**}}$. In particular $S_\alpha\to I_X$ uniformly on compact sets in X.

Define $\mathfrak{E}=\{S^{**}T:S\in\mathcal{K}(X,X)\}$. Let $v\in\mathfrak{E}^*$ and $\varepsilon>0$. As $\mathfrak{E}\subseteq\mathcal{K}(Y,X)$, by (FS) in Theorem 1.1, v has a representation $v=\sum_{n=1}^\infty x_n^*\otimes y_n$ with $\sum_{n=1}^\infty\|x_n^*\|\,\|y_n\|<\|v\|+\varepsilon$. Define a Hahn–Banach extension operator $\Phi\colon\mathfrak{E}^*\to\mathrm{span}(\mathfrak{E},\{T\})^*$ by $\Phi(v)=\sum_{n=1}^\infty\varphi x_n^*\otimes y_n$. Since

$$|\Phi(v)(T)-\mathrm{tr}(S_\alpha^{**}Tv)|=\left|\sum_{n=1}^\infty\langle x_n^*,(\varphi^*|_{X^{**}}-S_\alpha^{**})Ty_n\rangle\right|\to0,$$

Φ is well defined.

For general Y we use the factorization lemma and the local characterization of ideals. (See the proof of Theorem 2.2 in [25] for details.)

(c)⇒(e). Let Y be reflexive and $T\in\mathcal{W}(Y,X^{**})$ with $\|T\|=1$. Put $\mathfrak{E}=\{S^{**}T: S\in\mathcal{K}(X,X)\}$. We will use Godefroy and Saphar's identification of $\mathcal{K}(Y,X)^{**}$ as the weak* closure of $\mathcal{K}(Y,X)$ in $\mathcal{W}(Y,X^{**})$ cited in Theorem 1.1. In particular we have

$$\mathfrak{E}^{**}=\overline{\mathfrak{E}}^{w^*}\subseteq\mathcal{K}(Y,X)^{**}\subseteq(X^*\widehat{\otimes}_\pi Y)^*=\mathcal{W}(Y,X^{**}).$$

Let $K=\overline{T(B_Y)}\subset B_{X^{**}}$. Next we use the factorization lemma on the weakly compact set K, $[Z,J]=\mathrm{DFJP}(K)$. Z is reflexive and we get a factorization $T=J\circ\widehat{T}$.

By (c), $\mathfrak{E}_Z=\{S^{**}J:S\in\mathcal{K}(X,X)\}$ is an ideal in $\mathfrak{F}_Z=\mathrm{span}(\mathfrak{E}_Z,\{J\})$. Let $\Phi\in \mathrm{HB}(\mathfrak{E}_Z,\mathfrak{F}_Z)$ be the extension operator. By Theorem 2.3 in [17] there is a $\varphi\in \mathrm{HB}(X,X^{**})$ such that

$$\langle x^*\otimes z,\Phi^*(J)\rangle=\Phi(x^*\otimes z)(J)=(\varphi x^*\otimes z)(J)=\langle x^*\otimes z,\varphi^*J\rangle$$

for all $x^*\in X^*$ and $z\in Z$. (Note that $\mathcal{F}(Z,X)\subseteq\mathfrak{E}_Z$.)

Let $u\in X^*\widehat{\otimes}_\pi Y$ and choose a representation $u=\sum_{n=1}^\infty x_n^*\otimes y_n$. If $u=0$ on $\mathfrak{E}$ then

$$0=\sum_{n=1}^\infty x_n^*(S^{**}Ty_n)=\sum_{n=1}^\infty x_n^*(S^{**}J\widehat{T}y_n)=\langle\widehat{T}u,S^{**}J\rangle$$

for all $S\in\mathcal{K}(X,X)$, so that $\widehat{T}u=0$ on $\mathfrak{E}_Z$, where $\widehat{T}u\in X^*\widehat{\otimes}_\pi Z$ is given by $\widehat{T}u=\sum_{n=1}^\infty x_n^*\otimes\widehat{T}y_n$. Since $\Phi^*(J)\in\mathfrak{E}_Z^{**}$ we get

$$0=\langle\widehat{T}u,\Phi^*(J)\rangle=\langle\widehat{T}u,\varphi^*J\rangle=\sum_{n=1}^\infty x_n^*(\varphi^*J\widehat{T}y_n)=\sum_{n=1}^\infty x_n^*(\varphi^*Ty_n)$$

so that $u=0$ on φ^*T. Thus $\varphi^*T\in\mathfrak{E}^{**}$.

(d)⇒(f). This is similar to the proof of (c)⇒(e).

(f)⇒(g). Let $\{x_n^*\}_{n=1}^\infty\subset X^*$ and $\{x_n^{**}\}_{n=1}^\infty\subset X^{**}$ with $\sum_{n=1}^\infty\|x_n^*\|\,\|x_n^{**}\|<\infty$ and $\sum_{n=1}^\infty x_n^{**}(S^*x_n^*)=0$ for all $S\in\mathcal{K}(X,X)$. We may assume that $1\geq\|x_n^{**}\|\to0$. Let $K=\overline{\mathrm{conv}}\{\pm x_n^{**}\}_{n=1}^\infty$ and $[Z,J]=\mathrm{DFJP}(K)$. Then $J\in\mathcal{K}(Z,X^{**})$ and Z is separable and reflexive. Choose $z_n\in B_Z$ such that $J(z_n)=x_n^{**}$ for all n. Then $v=\sum_{n=1}^\infty x_n^*\otimes z_n\in\mathfrak{E}^*$, where $\mathfrak{E}$ is defined as in (f), and

$$\mathrm{tr}(Sv)=\sum_{n=1}^\infty x_n^*(S^{**}Jz_n)=\sum_{n=1}^\infty x_n^*(S^{**}x_n^{**})=0$$

for all $S\in\mathcal{K}(X,X)$ so that by (f) there exists a $\varphi\in\mathrm{HB}(X,X^{**})$ with

$$0=\operatorname{tr}(\varphi^* Jv)=\sum_{n=1}^{\infty} x_n^*(\varphi^* Jz_n)=\sum_{n=1}^{\infty} x_n^{**}(\varphi x_n^*)$$

as desired. □

From Theorem 4.3 (b) and [25] the following corollary is immediate.

Corollary 4.4. *Let X be a Banach space. If X has the weak* MCAP *then X has the* CAP.

Corollary 4.5. *Let X be a Banach space. If X^* has the* CAPconj *then X has the weak* MCAP.

Proof. The CAPconj for X^* implies (d) in Theorem 4.3 by using Lemma 3.5 in [25]. □

Recall that a Banach space X is said to have the *unique extension property* if the only operator $T\in\mathcal{L}(X^{**},X^{**})$ such that $\|T\|\le 1$ and $T|_X=I_X$ is $T=I_{X^{**}}$. This is equivalent to $\mathrm{HB}(X,X^{**})$ consisting of a single element: the canonical embedding $k_{X^*}: X^*\to X^{***}$.

Corollary 4.6. *Let X be a Banach space with the unique extension property. If X has the weak* MCAP *then X^* has the* CAPconj.

Proof. The extension operator in Theorem 4.3 (g) is the natural inclusion. □

Remark 4.1. Godefroy and Saphar proved in [7, Theorem 2.2] that X^* has the MCAPconj whenever X has the MCAP and the unique extension property. See [23, Theorem 2.9] for a similar result for the weak MAP and also Corollary 5.5 below.

In the presence of the Radon–Nikodým property for the first or second dual the weak MCAP is no longer weak. The same is also true for the weak MAP as proved by Oja in [28].

Theorem 4.7. *Let X be a Banach space such that either X^* or X^{**} has the Radon–Nikodým property. If X has the weak* MCAP *then X has the* MCAP.

Proof. Let $\varphi\in\mathrm{HB}(X,X^{**})$ be as in Theorem 4.3 (b). From (GS) in Theorem 1.1 we have $\varphi^*|_{X^{**}}\in\mathcal{K}(X,X)^{**}$ and thus by Goldstine's theorem and (FS) in Theorem 1.1 there is a net $(S_\alpha)\subseteq\mathcal{K}(X,X)$ with $\sup_\alpha\|S_\alpha\|\le 1$ such that $S_\alpha\to\varphi^*|_{X^{**}}$ in the weak* topology in $\mathcal{L}(X^{**},X^{**})$, and in particular $S_\alpha\to I_X$ uniformly on compact sets in X. □

As the CAP comes in two flavors for dual spaces we could expect the same to be the case for the weak MCAP. We are tempted to state the following definition.

"Definition". The dual X^* of a Banach space X has the *weak metric compact approximation property with conjugate operators* (*weak* MCAPconj) if for every Banach space Y and every operator $T\in\mathcal{W}(X^*,Y)$ there exists a net $(S_\alpha)\subset\mathcal{K}(X,X)$ with $\sup_\alpha \|TS_\alpha^*\|\le\|T\|$ such that $S_\alpha^*\to I_{X^*}$ uniformly on compact sets in X^*.

However, arguing as in Theorem 4.1 we see that the "weak MCAPconj" is equivalent to the following statement:

> For every Banach space Y and every $T\in\mathcal{W}(Y,X^{**})$ there exists a net $(S_\alpha)\subset\mathcal{K}(X,X)$ such that $\sup_\alpha \|S_\alpha^{**}T\|\le\|T\|$ and $S_\alpha^*\to I_{X^*}$ in the strong operator topology.

This statement is equivalent to the CAPconj as shown in [25, Theorem 3.6].

We saw in Section 2 that compact local reflexivity provided a link between CAP and CAPconj for dual spaces. The same proposition also gives us a link between the weak MCAP and "weak MCAPconj" (CAPconj) for dual spaces. Indeed, if the dual of a Banach space has the weak MCAP then in particular it has the CAP. Proposition 2.6 gives us the following result.

Proposition 4.8. *Let X be a Banach space. If X^* has the weak* MCAP *and X is* CLR *then X^* has the* CAPconj.

We conclude this section with a theorem similar to Theorem 4.2 in [20].

Theorem 4.9. *Let X be a Banach space. The following statements are equivalent*:

(a) *X has the weak* MCAP *in every equivalent norm.*

(b) *X^* has the* CAPconj.

Proof. (a)⇒(b). This is essentially proved in Remark 3.1 in [25], but we include a short proof here for completeness. Let $\{x_n^*\}_{n=1}^\infty\subset X^*$ and $\{x_n^{**}\}_{n=1}^\infty\subset X^{**}$ with $\sum_{n=1}^\infty \|x_n^*\|\,\|x_n^{**}\|<\infty$ and $\sum_{n=1}^\infty x_n^{**}(S^*x_n^*)=0$ for all $S\in\mathcal{K}(X,X)$. We may assume that $\|x_n^*\|=1$ for all n.

Let $F_m=\operatorname{span}\{x_n^*\}_{n=1}^m$. There exists a renorming X_m of X such that X_m^* is locally uniformly rotund on F_m and is arbitrarily close to the original norm (cf. [24, Lemma 2.4]). By Theorem 4.3 (g) there exists $\varphi_m\in\mathrm{HB}(X_m,X_m^{**})$ such that $\sum_{n=1}^\infty x_n^{**}(\varphi_m x_n^*)=0$. But $\varphi_m x_n^*=x_n^*$ for $n=1,\dots,m$, since the norm on F_m is locally uniformly rotund, and thus $\sum_{n=1}^\infty x_n^{**}(x_n^*)=0$.

(b)⇒(a). If X^* has the CAPconj then $\widehat{X}^*$ has the CAPconj for every equivalent renorming $\widehat{X}$ of X.

Let Y be a separable reflexive space and let $T\in\mathcal{K}(\widehat{X},Y)$. We use Theorem 3.4 in [25] to find a net $(S_\alpha)\subset\mathcal{K}(\widehat{X},\widehat{X})$ with $\sup_\alpha\|TS_\alpha\|\le\|T\|$ such that $S_\alpha^*T^*\to T^*$ in the strong operator topology. Theorem 4.1 (c) shows that $\widehat{X}$ has the weak MCAP. $\square$

Remark 4.2. The CAP does not imply the weak MCAP. Indeed, let X be the Casazza–Jarchow space [1, Theorem 1]. Then X has the AP and the dual is separable, but X does not have the MCAP. Hence X has the CAP but cannot have the weak MCAP, by Theorem 4.7.

Remark 4.3. In [11, Theorem 4] Johnson proved that if a Banach space has the MAP in every equivalent norm then X^* has the MAP. From [24, Theorems 1.1 and 1.2] we see that if X has the MCAP in every equivalent norm then X^* has the MCAPconj, and in [20, Theorem 4.2] Lima and Oja proved that if X has the weak MAP in every equivalent norm then X^* has the AP (AP and weak MAP are equivalent for dual spaces). Theorem 4.9 above is the corresponding result for the weak MCAP. See also Theorem 5.6 below.

5. Very weak MCAP

We saw in Section 4 how the weak MCAP was connected to the CAPconj for the dual. When the dual X^* of a Banach space X has the CAP but not the CAPconj then we cannot replace the approximating operators in $\mathcal{K}(X^*,X^*)$ with conjugates of operators in $\mathcal{K}(X,X)$. The space $\mathcal{K}(X^*,X^*)$ is isometrically isomorphic to the space $\mathcal{K}(X,X^{**})$. It is the latter viewpoint we take when introducing the following approximation property which we will later connect to the CAP for the dual.

Definition 5.1. A Banach space X has the *very weak metric compact approximation property* (*very weak* MCAP) if for every Banach space Y and every operator $T\in\mathcal{W}(X,Y)$ there exists a net $(S_\alpha)\subset\mathcal{K}(X,X^{**})$ with $\sup_\alpha\|T^{**}S_\alpha\|\le\|T\|$ such that $\lim_\alpha \operatorname{tr}(S_\alpha u)=\operatorname{tr}(I_X u)$ for every $u\in X^*\widehat{\otimes}_\pi X$.

In the above definition we do not use uniform convergence on compact sets simply because $\mathcal{K}(X,X^{**})$ is not a subspace of $\mathcal{L}(X,X)$. In this case it is more natural to regard $\mathcal{K}(X,X^{**})$ as a subspace of $(X^*\widehat{\otimes}_\pi X)^*=\mathcal{L}(X,X^{**})$.

First we prove a theorem similar to Theorem 4.1 which shows that it is enough to consider reflexive spaces and compact operators only in the definition of the very weak MCAP.

Theorem 5.1. *Let X be a Banach space. The following statements are equivalent*:

(a) *X has the very weak* MCAP.

(b) *For every reflexive Banach space Y and every operator $T\in\mathcal{W}(X,Y)$ there exists a net $(S_\alpha)\subset\mathcal{K}(X,X^{**})$ with $\sup_\alpha\|T^{**}S_\alpha\|\le\|T\|$ such that $T^{**}S_\alpha\to T$ in the strong operator topology.*

(c) *For every reflexive Banach space Y and every operator $T\in\mathcal{K}(X,Y)$ there exists a net $(S_\alpha)\subset\mathcal{K}(X,X^{**})$ with $\sup_\alpha\|T^{**}S_\alpha\|\le\|T\|$ such that $T^{**}S_\alpha\to T$ in the strong operator topology.*

(d) *For every Banach space Y, every operator $T\in\mathcal{W}(X,Y)$ with $\|T\|=1$, and all sequences $\{x_n\}_{n=1}^\infty\subset X$ and $\{y_n^*\}_{n=1}^\infty\subset Y^*$ with $\sum_{n=1}^\infty\|x_n\|\,\|y_n^*\|<\infty$ we have*

$$\left|\sum_{n=1}^\infty y_n^*(Tx_n)\right|\le\sup_{\substack{\|T^{**}S\|\le1\\ S\in\mathcal{K}(X,X^{**})}}\left|\sum_{n=1}^\infty y_n^*(T^{**}Sx_n)\right|.$$

Proof. (a)⇒(b)⇒(c) is trivial.

(c)⇒(a). Let Y be a reflexive space and let $T\in\mathcal{W}(X,Y)$. Let $a_1,\dots,a_m\in X$ and let $\varepsilon>0$. Define

$$C=\{T^{**}S:S\in\mathcal{K}(X,X^{**})\text{ and }\|T^{**}Sa_i-Ta_i\|<\varepsilon\text{ for }i=1,\dots,m\}$$

and argue as in the proof of (a′)⇒(a) in Theorem 2.4 in [20].

(a)⇒(d). Let Y be a Banach space and let $T\in\mathcal{W}(X,Y)$ with $\|T\|=1$. Choose $(S_\alpha)\subseteq\mathcal{K}(X,X^{**})$ with $\sup_\alpha\|T^{**}S\|\le\|T\|$ and $\lim_\alpha\operatorname{tr}(S_\alpha u)=\operatorname{tr}(I_Xu)$ for all $u\in X^*\widehat{\otimes}_\pi X$. For $v\in Y^*\widehat{\otimes}_\pi X$ we have $T^*v\in X^*\widehat{\otimes}_\pi X$ and thus

$$\langle v,T^{**}S_\alpha\rangle=\langle T^*v,S_\alpha\rangle\xrightarrow{\alpha}\langle T^*v,I_X\rangle=\langle v,T\rangle$$

and (d) follows.

(d)⇒(b). Let Y be a Banach space and let $T\in\mathcal{W}(X,Y)$. (d) says that

$$T\in\overline{\{T^{**}S:S\in\mathcal{K}(X,X^{**})\text{ and }\|T^{**}S\|\le\|T\|\}}^{\tau},$$

where τ is the topology of uniform convergence on compact sets in X (see [26] for the representation of the dual of $(\mathcal{L}(X,Y),\tau)$). Thus we can find a net $(S_\alpha)\subset\mathcal{K}(X,X^{**})$ with $\sup_\alpha\|T^{**}S_\alpha\|\le\|T\|$ such that $T^{**}S_\alpha\to T$ in the τ-topology. But then $T^{**}S_\alpha\to T$ in the weak operator topology and by taking convex combinations, if necessary, we may assume that we have convergence in the strong operator topology. □

From [25, Theorem 3.2], the factorization lemma and Feder and Saphar's characterization of $\mathcal{K}(X,Y)^*$, see the (FS) part of Theorem 1.1, we get the following theorem.

Theorem 5.2. *Let X be a Banach space. Then X^* has the* CAP *if and only if for every Banach space Y and every $T\in\mathcal{K}(X,Y)$ there exists a net $(S_\alpha)\subset\mathcal{K}(X,X^{**})$ such that $T^{**}S_\alpha\to T$ in norm.*

From Theorems 5.2 and 5.1 we see that X has the very weak MCAP whenever X^* has the CAP. With $X=Y$, the approximating net in the above theorem can be thought of as a "right approximate identity". But note that we have to go out of the Banach algebra $\mathcal{K}(X,X)$ to find this net. (See e.g. [1] for references on approximate identities.)

The next theorem is similar to Theorem 4.3 and give characterizations of the very weak MCAP involving Hahn–Banach extension operators. Note that in the case of the very weak MCAP we are in fact able to show that a Banach space has this property if and only if every separable ideal in the space has this property. In the case of the weak MCAP we have only been able to prove one implication (see Proposition 4.2). As we will see in the proof below, the problem is that we end up with operators into the bidual instead of into the space itself.

Theorem 5.3. *Let X be a Banach space. The following statements are equivalent*:

(a) *X has the very weak* MCAP.

(b) *There exists $\varphi\in\mathrm{HB}(X,X^{**})$ such that*

$$\varphi^*|_{X^{**}}\in\overline{\mathcal{K}(X,X^{**})}^{w^*}$$

in $(X^*\widehat{\otimes}_\pi X^{**})^*=\mathcal{L}(X^{**},X^{**})$.

(c) *There exists $\varphi\in\mathrm{HB}(X,X^{**})$ such that for every reflexive Banach space Y and every $T\in\mathcal{W}(X,Y)$ we have $T^{**}\varphi^*|_{X^{**}}\in\mathfrak{E}^{**}$, where*

$$\mathfrak{E}=\{T^{**}S:S\in\mathcal{K}(X,X^{**})\}$$

and $\mathfrak{E}^{**}=\overline{\mathfrak{E}}^{w^*}\subseteq\mathcal{K}(X,Y)^{**}=\overline{\mathcal{K}(X,Y)}^{w^*}\subseteq\mathcal{W}(X^{**},Y)$.

(d) *There exists $\varphi\in\mathrm{HB}(X,X^{**})$ such that for every reflexive Banach space Y and every $T\in\mathcal{K}(X,Y)$ there exists a net $(S_\alpha)\subset\mathcal{K}(X,X^{**})$ with $\sup_\alpha\|T^{**}S_\alpha\|\le\|T\|$ such that $\omega^*\text{-}\lim_\alpha S_\alpha^*T^*y^*=\varphi T^*y^*$ in X^{***} for all $y^*\in Y^*$.*

(e) *There exists $\varphi\in\mathrm{HB}(X,X^{**})$ such that for every reflexive Banach space Y and operator $T\in\mathcal{K}(X,Y)$ there exists a net $(S_\alpha)\subset\mathcal{K}(X,X^{**})$ with $\sup_\alpha\|T^{**}S_\alpha\|\le\|T\|$ such that $T^{**}S_\alpha^{**}\to T^{**}\varphi^*$ in the strong operator topology.*

(f) *Every separable ideal in X has the very weak* MCAP.

(g) *There exists $\varphi\in\mathrm{HB}(X,X^{**})$ such that for all sequences $\{x_n^*\}_{n=1}^\infty\subset X^*$ and $\{x_n^{**}\}_{n=1}^\infty\subset X^{**}$ with $\sum_{n=1}^\infty\|x_n^*\|\,\|x_n^{**}\|<\infty$ and $\sum_{n=1}^\infty x_n^{**}(S^*x_n^*)=0$, for all $S\in\mathcal{K}(X,X^{**})$, we have $\sum_{n=1}^\infty x_n^{**}(\varphi x_n^*)=0$.*

Proof. (d)⇒(e) and (g)⇒(b) are trivial.

(a)⇒(b) is similar to the proof of Proposition 2.5 in [23].

(b)⇒(a) is similar to the proof of Proposition 2.3 in [23].

(b)⇒(c). Let Y be a reflexive Banach space and let $T\in\mathcal{W}(X,Y)$. There is a $\varphi\in\mathrm{HB}(X,X^{**})$ and a net $(S_\alpha)\subset\mathcal{K}(X,X^{**})$ such that $S_\alpha\to_\alpha\varphi^*|_{X^{**}}$ in the weak* topology. Let $v\in Y^*\widehat{\otimes}_\pi X^{**}$ and consider $T^*v\in X^*\widehat{\otimes}_\pi X^{**}$. Then we have

$$\langle T^{**}S_\alpha, v\rangle = \langle S_\alpha, T^*v\rangle \to \langle \varphi^*|_{X^{**}}, T^*v\rangle = \langle T^{**}\varphi^*|_{X^{**}}, v\rangle$$

as desired.

(c)⇒(d) follows from Goldstine's theorem.

(e)⇒(a) follows from Theorem 5.1 (c) and the fact that $\varphi^*x=x$ for all $x\in X$.

(e)⇒(f). Let $Z\subseteq X$ be a separable ideal and let $\psi\in\mathrm{HB}(Z,X)$. Let Y be a reflexive Banach space and let $T\in\mathcal{K}(Z,Y)$. Then $T^{**}\psi^*|_X\in\mathcal{K}(X,Y)$ and by assumption there is a net $(S_\alpha)\subset\mathcal{K}(X,X^{**})$ with $\sup_\alpha\|T^{**}\psi^*S_\alpha\|\le\|T^{**}\psi^*|_X\|\le\|T\|$ such that $\lim_\alpha T^{**}\psi^*S_\alpha^{**}x^{**}=T^{**}\psi^*\varphi^*x^{**}$ for all $x^{**}\in X^{**}$.

Let $T_\alpha=\psi^*S_\alpha|_Z\in\mathcal{K}(Z,Z^{**})$. From the above we get $\sup_\alpha\|T^{**}T_\alpha\|\le\|T\|$ and $\lim_\alpha T^{**}T_\alpha^{**}z^{**}=T^{**}\psi^*\varphi^*z^{**}$ for all $z^{**}\in Z^{**}$. Since also $\psi^*\varphi^*|_{Z^{**}}\in\mathrm{HB}(Z,Z^{**})$ we see that Z satisfies (e) itself and by what we have already proved Z has the very weak MCAP.

(f)⇒(g). Consider the collection of all separable ideals in X,

$$\mathfrak{Z}=\{Z: Z \text{ is a separable subspace of } X \text{ and there exists } \psi\in\mathrm{HB}(Z,X)\}.$$

Let $\mathcal{U}$ be an ultrafilter refining the order filter on $\mathfrak{Z}$. Let $Z\in\mathfrak{Z}$. Let $i_Z\colon Z\to X$ be the natural inclusion and let $\varphi_Z\in\mathrm{HB}(Z,X)$. By assumption there exists a $\psi_Z\in\mathrm{HB}(Z,Z^{**})$ such that $(\psi_Z)^*|_{Z^{**}}\in\overline{\mathcal{K}(Z,Z^{**})}^{w^*}$ in $\mathcal{L}(Z^{**},Z^{**})$.

Using weak* compactness we can define $\varphi\colon X^*\to X^{***}$ by taking limits along $\mathcal{U}$,

$$\varphi(x^*)=\omega^*\text{-}\lim_{\mathcal{U}}(\varphi_Z^{**}\circ\psi_Z\circ i_Z^*)(x^*),$$

and get a well-defined linear operator with $\|\varphi\|\le1$. This φ is a Hahn–Banach extension operator. Indeed, let $x^*\in X^*$ and $x\in X$, then

$$\langle\varphi x^*, x\rangle=\lim_{\mathcal{U}}\langle(\varphi_Z^{**}\psi_Z i_Z^*)(x^*), x\rangle=\lim_{\mathcal{U}}\langle\psi_Z i_Z^*x^*, \varphi_Z^* x\rangle=\lim_{\mathcal{U}}\langle\psi_Z i_Z^* x^*, x\rangle=x^*(x)$$

since we have $x\in Z$ when Z is large enough.

Let $\{x_n^*\}_{n=1}^\infty\subset X^*$ and $\{x_n^{**}\}_{n=1}^\infty\subset X^{**}$ with $\sum_{n=1}^\infty\|x_n^*\|\,\|x_n^{**}\|<\infty$ such that $\sum_{n=1}^\infty x_n^{**}(S^*x_n^*)=0$ for all $S\in\mathcal{K}(X,X^{**})$.

Let $T\in\mathcal{K}(Z,Z^{**})$. Then $i_Z^{**}\circ T^{**}\circ\varphi_Z^*|_X\in\mathcal{K}(X,X^{**})$ and so

$$0=\sum_{n=1}^\infty x_n^{**}(\varphi_Z T^* i_Z^* x_n^*)=\left\langle\sum_{n=1}^\infty(i_Z^*x_n^*)\otimes(\varphi_Z^*x_n^{**}), T\right\rangle.$$

Since $\sum_{n=1}^{\infty}(i_Z^*x_n^*)\otimes(\varphi_Z^*x_n^{**})\in Z^*\widehat{\otimes}_\pi Z^{**}$ and $(\psi_Z)^*|_{Z^{**}}\in\overline{\mathcal{K}(Z,Z^{**})}^{w^*}$ we have

$$0=\left\langle\sum_{n=1}^{\infty}(i_Z^*x_n^*)\otimes(\varphi_Z^*x_n^{**}),\psi_Z\right\rangle=\sum_{n=1}^{\infty}(\varphi_Z^{**}\psi_Z i_Z^*x_n^*)(x_n^{**}).$$

This holds for all $Z\in\mathfrak{Z}$ and thus $\sum_{n=1}^{\infty}x_n^{**}(\varphi x_n^*)=0$ as desired. □

We will need the following corollary to relate the weak MCAP and the very weak MCAP in Proposition 5.7.

Corollary 5.4. *Let X be a Banach space. The following statements are equivalent*:

(a) *X has the weak* MCAP.

(b) *There exists $\varphi\in\mathrm{HB}(X,X^{**})$ such that for every reflexive Banach space Y and every $T\in\mathcal{W}(X,Y)$ we have $T^{**}\varphi^*|_{X^{**}}\in\mathfrak{E}^{**}$, where*

$$\mathfrak{E}=\{TS:S\in\mathcal{K}(X,X)\}$$

and $\mathfrak{E}^{**}=\overline{\mathfrak{E}}^{w^*}\subseteq\mathcal{K}(X,Y)^{**}=\overline{\mathcal{K}(X,Y)}^{w^*}\subseteq\mathcal{W}(X^{**},Y)$.

Proof. The proof of (a)⇒(b) is similar to the proof of (b)⇒(c) in Theorem 5.3.

(b)⇒(a). Proceed as in Theorem 5.3 to show a statement similar to Theorem 5.3 (e) but with the net $(S_\alpha)\subset\mathcal{K}(X,X)$. From this and Theorem 4.1 (c) it follows that X has the weak MCAP. □

Remark 5.1. Using trace duality and the factorization lemma we can show that a Banach space X has the CAP if and only if for every reflexive Banach space Y and every $T\in\mathcal{K}(X,Y)$ we have

$$T\in\overline{\{TS:S\in\mathcal{K}(X,X)\}}^{w^*}\tag{5.1}$$

in $(Y^*\widehat{\otimes}_\pi X)^*=\mathcal{W}(X,Y)$.

Let X be the Casazza–Jarchow space [1, Theorem 1]. X has the CAP but not the MCAP although X^* has the MCAP. From Theorem 5.3 (b) it is clear that X has the very weak MCAP (consider the weak* limit of a bounded approximating net on X^*) so, by Theorem 5.1 (d), for every reflexive Banach space Y and operator $T\in\mathcal{K}(X,Y)$ there is a net $(S_\alpha)\subset\mathcal{K}(X,X^{**})$ with $\sup_\alpha\|T^{**}S_\alpha\|\le\|T\|$ such that $\omega^*\text{-}\lim_\alpha T^{**}S_\alpha=T$. By (5.1) there is also a net $(T_\beta)\subset\mathcal{K}(X,X)$ such that $\omega^*\text{-}\lim_\beta TT_\beta=T$.

But there is a renorming X_1 of X such that X_1 does not have the weak MCAP. Thus we cannot obtain the norm bound on the net (TT_β).

We are now ready to relate the very weak MCAP to the CAP for the dual as promised in the introduction of this section.

Corollary 5.5. *Let X be a Banach space with the unique extension property. If X has the very weak* MCAP *then X^* has the* CAP.

Proof. The extension operator in Theorem 5.3 is the identity. □

Theorem 5.6. *Let X be a Banach space. The following statements are equivalent*:

(a) *X has the very weak* MCAP *in every equivalent norm.*

(b) *X^* has the* CAP.

Proof. To prove (a)⇒(b) we argue as in the proof of (a)⇒(b) from Theorem 4.9.

(b)⇒(a). See the remarks following Theorem 5.2 and note that $\widehat{X}^*$ has the CAP for every equivalent renorming $\widehat{X}$ of X. □

From Theorem 5.3 (b) we see why there is no "very weak MAP". Using the principle of local reflexivity we can show that the weak* closures of $\mathcal{F}(X,X)$ and $\mathcal{F}(X,X^{**})$ in $\mathcal{L}(X^{**},X^{**})$ are the same.

The following proposition is the best we can do connecting the weak MCAP and the very weak MCAP.

Theorem 5.7. *Let X be a Banach space and let $V\colon X^*\widehat{\otimes}_\pi X^{**}\to\mathcal{K}(X,X)^*$ and $W\colon X^*\widehat{\otimes}_\pi X^{**}\to\mathcal{K}(X,X^{**})^*$ be the trace mappings. The following statements are equivalent*:

(a) *X has the weak* MCAP.

(b) *X has the very weak* MCAP *and there exists $\varphi\in\mathrm{HB}(X,X^{**})$ such that $(I_{X^*}\otimes\varphi^*|_{X^{**}})(\ker V)\subset\ker W$.*

In particular, if X has the very weak MCAP *and X is* CLR *then X has the weak* MCAP.

Proof. (a)⇒(b). If X has the weak MCAP then it has the very weak MCAP.

Let $S\in\mathcal{K}(X,X^{**})$ with $\|S\|=1$. Let further $K=S^*|_{X^*}(B_{X^*})\subset B_{X^*}$ and $[Z,J]=\mathrm{DFJP}(K)$. Factorize $S^*=J\circ S_1$ through Z. Now we have a reflexive space Z and an operator $J^*|_X\in\mathcal{K}(X,Z^*)$. By Corollary 5.4 there exists $\varphi\in\mathrm{HB}(X,X^{**})$ such that $J^*\varphi^*|_{X^{**}}\in\mathfrak{C}^{**}$, where $\mathfrak{C}=\{J^*|_X S{:}S\in\mathcal{K}(X,X)\}$. Assume that $u\in\ker V$. Use S_1 to map u to $S_1u\in Z\widehat{\otimes}_\pi X^{**}$,

$$0=\langle S_1u,J^*\varphi^*|_{X^{**}}\rangle=\langle (I_{X^*}\otimes\varphi^*|_{X^{**}})(u),S\rangle.$$

(b)⇒(a). Let $\varphi\in\mathrm{HB}(X,X^{**})$ be as given in (b). If $u=\sum_{n=1}^\infty x_n^*\otimes x_n^{**}\in\ker V$ then $\sum_{n=1}^\infty x_n^*\otimes\varphi^*(x_n^{**})\in\ker W$. By Theorem 5.3 (g) there exists a $\psi\in\mathrm{HB}(X,X^{**})$

such that $\sum_{n=1}^{\infty}\varphi^*(x_n^{**})\psi(x_n^*)=0$. Now $\psi^*\varphi^*$ has norm one and acts as the identity when restricted to X so it is a Hahn–Banach extension operator for which Theorem 4.3 (g) holds. $\square$

Proposition 5.8. *Let X be a Banach space. The statements below are related as follows*: (a)$\Rightarrow$(b)$\Rightarrow$(c).

(a) *X has the weak* MCAP.

(b) *X has the very weak* MCAP.

(c) *The trace mapping $V\colon Y^*\widehat{\otimes}_\pi X\to\mathcal{K}(X,Y)^*$ is isometric for every reflexive Banach space Y.*

Proof. (a)$\Rightarrow$(b) is trivial and (b)$\Rightarrow$(c) follows from the identity $(Y^*\widehat{\otimes}_\pi X)^*=\mathcal{W}(X,Y)$ and (b) in Theorem 5.1. $\square$

Remark 5.2. Using Theorem 5.6 we see that the Banach space in Remark 4.2 is also an example of a space without weak MCAP which has the very weak MCAP (since the dual has the MCAP).

We do not know whether or not the implication (b)$\Rightarrow$(c) in Proposition 5.8 can be reversed.

As was the case for the weak MCAP there is no conjugate version of the very weak MCAP for dual spaces. The natural definition of a conjugate version of the very weak MCAP for the dual of a Banach space X is obviously just the definition of weak MCAP for X^*.

6. Questions and comments

Throughout this section we will assume that X and Y are Banach spaces. Summarizing the previous sections we have the following implications.

$$\begin{array}{ccc}
X^* \text{ CAPconj} & \Longrightarrow & X \text{ weak MCAP} \\
\Downarrow & & \Downarrow \\
X^* \text{ CAP} & \Longrightarrow & X \text{ very weak MCAP} \\
\Downarrow & & \Downarrow \\
\mathcal{K}(X,Y) \overset{\text{ideal}}{\subseteq} \mathcal{W}(X,Y) \text{ for all } Y & \Longrightarrow & V\colon Y^*\widehat{\otimes}_\pi X\to\mathcal{K}(X,Y)^* \text{ is isometric} \\
 & & \text{for all reflexive } Y.
\end{array}$$

Also considering equivalent renormings $\widehat{X}$ of X we have

$$\begin{array}{ccc}
X^* \text{ CAPconj} & \Longleftrightarrow & \widehat{X} \text{ weak MCAP for all } \widehat{X} \\
\Downarrow & & \Downarrow \\
X^* \text{ CAP} & \Longleftrightarrow & \widehat{X} \text{ very weak MCAP for all } \widehat{X} \\
\Downarrow & & \Downarrow \\
\mathcal{K}(X,Y) \overset{\text{ideal}}{\subseteq} \mathcal{W}(X,Y) \text{ for all } Y & \Longleftrightarrow & Y^* \widehat{\otimes}_\pi \widehat{X} \xrightarrow{V} \mathcal{K}(\widehat{X},Y)^* \text{ is isometric} \\
& & \text{for all } \widehat{X} \text{ and all reflexive } Y.
\end{array}$$

We close this paper by collecting some open problems concerning compact approximation properties.

In Section 4 we observed that the "weak MCAPconj" is equivalent to the CAPconj for dual spaces.

Question 6.1. Is the weak MCAP for X^* equivalent to the CAP for X^*?

We saw in Section 5 that if X is CLR then X has the weak MCAP, and in particular the CAP, whenever it has the very weak MCAP.

Question 6.2. If X has the very weak MCAP does it have the CAP?

Finally a question related to Proposition 4.2. We know that e.g. c_0 contains a subspace without the CAP, but can we choose this subspace to be an ideal in c_0? More generally we ask the following question.

Question 6.3. Does there exist X with CAP (or weak MCAP or MCAP) and a separable ideal Z in X which does not share this property?

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