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Slices in the unit ball of the symmetric tensor product of $\mathcal{C}(K)$ and $L_1(\mu)$

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Dedicated to Angel Rodríguez Palacios on the occasion of his 60th birthday.

Abstract. We prove that for the cases $X=\mathcal{C}(K)$ (K infinite) and $X=L_1(\mu)$ (μ σ-finite and atomless) it holds that every slice of the unit ball of the N-fold symmetric tensor product of X has diameter two. In fact, we prove more general results for weak neighborhoods relative to the unit ball. As a consequence, we deduce that the spaces of N-homogeneous polynomials on those classical Banach spaces have no points of Fréchet differentiability.

1. Introduction

The characterization of the Radon–Nikodým property of a Banach space in terms of the existence of arbitrarily small slices of any bounded, closed and convex subset of it is well-known. But many of the classical Banach spaces lacks the Radon–Nikodým property. In such cases, a modulus of non-dentability introduced by Schachermayer, Sersouri and Werner [SSW] can be useful. For a bounded, closed and convex subset C of a Banach space, this number is given by $\delta_1(C):=\inf\{\operatorname{diam} S: S \text{ is a slice of } C\}$. These authors proved that any Banach space X without the Radon–Nikodým property satisfies that for every $\varepsilon>0$ there is subset C of X with $\operatorname{diam} C=1$ and such that $\delta_1(C)>1-\varepsilon$. For certain classical Banach spaces (endowed with their usual norm), there are results stating that an appropriate ball satisfies the above condition. We will state some of them. Nygaard and Werner [NW] proved that every infinite-dimensional uniform algebra satisfies that every nonempty relatively weak open set of the unit ball does have diameter two. Shvydkoy [Sh]

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obtained that every Banach space with the Daugavet property also satisfies the same condition. Becerra and López [BL] showed the previous result for spaces of vector-valued continuous functions defined on an infinite compact space K (see also [Ra]). They also characterized the spaces $L_1(\mu, X)$ where the above phenomena happens. Becerra, López and Rodríguez showed that every non-empty weak open set of the unit ball of any infinite-dimensional C^*-algebra also has diameter two [BLR] (see also [BLPR]).

Before stating the main results of the paper, we recall some basic definitions and well-known facts. For a (real or complex) Banach space X, S_X and B_X will denote the unit sphere and the unit ball of X, respectively. X^* will be the topological dual of X. We will consider the symmetric projective tensor product $\bigotimes_{\pi,s}^N X$. This space is the completion of the linear space generated by $\{x\otimes \overset{N}{\ldots} \otimes x : x\in X\}$ under the norm given by

$$\|z\| = \inf\left\{ \sum_{i=1}^{N} |\lambda_i| : z = \sum_{i=1}^{N} \lambda_i x_i \otimes \overset{N}{\ldots} \otimes x_i, \lambda_i \in \mathbb{K} \text{ and } x_i \in S_X \text{ for all } i \in \{1, \ldots, N\} \right\},$$

where $\mathbb{K}$ is the scalar field. Its topological dual can be identified with the space of all N-homogeneous (and bounded) polynomials on X, denoted by $\mathcal{P}^N(X)$. Every polynomial $P\in\mathcal{P}^N(X)$ acts as a linear functional $\widehat{P}$ on the N-fold symmetric tensor product through its associated symmetric N-linear form $\overline{P}$ and satisfies $P(x)=\overline{P}(x, \overset{N}{\ldots}, x)=\widehat{P}(x\otimes \overset{N}{\ldots} \otimes x)$ for every element $x\in X$.

The dual norm of the symmetric tensor product is the usual polynomial norm, given by

$$\|P\| = \sup\{|P(x)| : x \in X, \|x\| \le 1\}, \quad P \in \mathcal{P}^N(X).$$

Now we recall some basic notions that will be used. A *slice* of B_X (X is a Banach space) is a subset of the form

$$S(B_X, x^*, \alpha) := \{x \in B_X : \operatorname{Re} x^*(x) > 1-\alpha\},$$

where $x^*\in S_{X^*}$ and $0<\alpha<1$. For an element u in S_X, the *roughness* of X at u, $\eta(X, u)$ is given by

$$\eta(X, u) := \limsup_{\|h\|\to 0} \frac{\|u+h\|+\|u-h\|-2}{\|h\|}.$$

Clearly $0\le\eta(X, u)\le 2$. It is well-known that $\eta(X, u)=0$ if and only if the norm of X is Fréchet differentiable at u. We will also use throughout the paper the equality

$$\eta(X^*, x^*) = \inf\{\operatorname{diam} S(B_X, x^*, \alpha) : 0 < \alpha < 1\}$$

valid for $x^*\in S_{X^*}$ (see for instance [DGZ, Proposition I.1.11]).

In this paper we obtain the same kind of results that we stated at the beginning for the N-fold symmetric projective tensor product of some classical Banach spaces including $\mathcal{C}(K)$ and $L_1(\mu)$. Let us point out that there are just a few results on isometric properties of the symmetric tensor product of Banach spaces (see [RT], [BR], [ACKM] and [GGM]).

It is well-known that in the finite-dimensional case, the unit ball has arbitrarily small slices. Let Ω be a Hausdorff locally compact topological space. In the Section 2, we prove that every infinite-dimensional space $C_0(\Omega)$ (real or complex) satisfies that every nonempty relatively weak open set of the unit ball of $\bigotimes_{\pi,s}^N X$ has diameter two. As a consequence, it can be obtained that the space $\mathcal{P}^N(C_0(\Omega))$ has no points of Fréchet differentiability for every infinite and Hausdorff topological space Ω. Indeed, it is obtained that the modulus of roughness of every polynomial is as large as it can be. Let us remark that Boyd and Ryan proved that the space $\mathcal{P}^N(X)$ is never smooth, for every Banach space X with dimension greater or equal to two and $N\geq 2$ [BR, Proposition 17] (see also [RT, Corollary 7]).

Section 3 is devoted to $L_1(\mu)$. If μ has atoms, then $L_1(\mu)$ has strongly exposed points and so, the unit ball of the symmetric N-fold tensor product of $L_1(\mu)$ has slices with arbitrarily small diameter. We prove that if μ is a σ-finite and atomless measure and N is odd, then every weak open subset of the unit ball of the symmetric N-fold projective tensor product of $L_1(\mu)$ (real case) has diameter two. We also obtain that every slice of the unit ball of $\bigotimes_{\pi,s}^N L_1(\mu)$ has diameter two, for every σ-finite measure μ without atoms, and for every N. Hence under the previous assumptions, the space $\mathcal{P}^N(L_1(\mu))$ has no points of Fréchet differentiability of the norm.

2. Results for $\mathcal{C}(K)$

As usual, we denote by $C_0(\Omega)$ the Banach space of all the scalar-valued continuous functions on Ω vanishing at infinity, where Ω is a Hausdorff locally compact topological space. We consider the real as well as the complex case.

As we already noticed, if Ω is finite, then the unit ball of $\bigotimes_{\pi,s}^N C_0(\Omega)$ has arbitrarily small slices. Our aim is to show that this is the only case where the above condition happens. For such a purpose we will use the following easy and technical result.

Lemma 2.1. *Let Ω be a locally compact and Hausdorff infinite topological space. Then there are two sequences of non-empty open and relatively compact sets $\{V_n\}_{n=1}^{\infty}$ and $\{U_n\}_{n=1}^{\infty}$ satisfying*

$$\overline{V}_n \subset U_n \text{ and } U_n \cap U_m = \varnothing, \quad n \neq m,$$

and

(i) *two sequences of functions* $\{g_n\}_{n=1}^{\infty}$ *and* $\{h_n\}_{n=1}^{\infty}$ *in* $C_0(\Omega)$ *satisfying*

$$g_n \xrightarrow{w} 0, \quad h_n \xrightarrow{w} 0,$$

and also for every positive integer n,

$$0 \le g_n, h_n \le 1, \quad \operatorname{supp} h_n \subset V_n, \quad \|h_n\| = 1, \quad \operatorname{supp} g_n \subset U_n \quad \textit{and} \quad g_n(V_n) = \{1\};$$

(ii) *sequences of continuous functions* $\{f_n\}_{n=1}^{\infty}$, $\{g_n\}_{n=1}^{\infty}$ *and* $\{h_n\}_{n=1}^{\infty}$ *in* $C_0(\Omega)$ *satisfying*

$$f_n \xrightarrow{w} 0, \quad g_n \xrightarrow{w} 0, \quad h_n \xrightarrow{w} 0,$$

and for all n

$$\operatorname{supp} f_n, \operatorname{supp} h_n \subset V_n, \quad f_n(\Omega), h_n(\Omega) \subset [-1, 1],$$
$$\textit{there exist } s_n, t_n \in V_n \textit{ such that } f_n(s_n) = f_n(t_n) = 1 = h_n(t_n) = -h_n(s_n),$$
$$0 \le g_n \le 1, \quad \operatorname{supp} g_n \subset U_n \quad \textit{and} \quad g_n(V_n) = \{1\}.$$

Proof. Since Ω is infinite, we will prove that there is a sequence of disjoint nonempty open sets $\{U_n\}_{n=1}^{\infty}$ whose closure is compact. In order to show that, we start with a sequence $\{t_n\}_{n=1}^{\infty}$ of distinct points in Ω.

If $\{t_n\}_{n=1}^{\infty}$ has no cluster points, then one can choose a compact neighborhood C_1 of t_1 such that it does not contain any other term of the sequence. By using that $t_2 \notin C_1$, and that t_2 is not a cluster point of the sequence, then there is a new compact neighborhood C_2 of t_2 such that C_2 does not contain more terms of the sequence and such that $C_2 \cap C_1 = \varnothing$. By continuing in the same way, a sequence of sets $\{C_n\}_{n=1}^{\infty}$ is produced such that $\{U_n\}_{n=1}^{\infty} := \{\mathring{C}_n\}_{n=1}^{\infty}$ is a sequence of pairwise nonempty open sets whose closure is compact.

In the case that the sequence $\{t_n\}_{n=1}^{\infty}$ has a cluster point t_0, we can clearly assume that $t_0 \neq t_n$ for $n \geq 1$ and we proceed in the following manner. Since Ω is Hausdorff and locally compact, we can choose compact subsets F_1 and C_1 of Ω satisfying

$$t_1 \in \mathring{C}_1, \quad t_0 \in \mathring{F}_1 \quad \text{and} \quad C_1 \cap F_1 = \varnothing.$$

Since t_0 is a cluster point of the sequence and $t_0 \in \mathring{F}_1$, F_1 contains infinitely many t_n's. We choose a new element $t_{\sigma(2)} \in \mathring{F}_1$ and, by using again that Ω is Hausdorff and locally compact, we can choose two disjoint compact neighborhoods of $t_{\sigma(2)}$ and t_0, respectively. By intersecting both sets with F_1, we then obtain compact subsets C_2 and F_2 satisfying

$$t_{\sigma(2)} \in \mathring{C}_2, \quad t_0 \in \mathring{F}_2, \quad C_2 \cap F_2 = \varnothing \quad \text{and} \quad C_1 \cap C_2 = \varnothing.$$

Assume that we have obtained n closed and disjoint subsets $C_1, ..., C_n$ of K, whose interiors are non-empty and satisfying $t_0 \notin \bigcup_{i=1}^{n} C_i$, then we choose $t_{\sigma(n+1)}$ such that the element $t_{\sigma(n+1)} \notin \bigcup_{i=1}^{n} C_i$. We now choose a compact neighborhood C_{n+1} of $t_{\sigma(n+1)}$ not containing t_0 and such that $C_{n+1} \cap (\bigcup_{i=1}^{n} C_i) = \varnothing$. Again by using that every point has a basis of compact neighborhoods, there are nonempty open sets V_n such that $\overline{V}_n \subset \mathring{C}_n$. We just write $U_n := \mathring{C}_n$.

Finally, by Urysohn's lemma, there are continuous functions g_n and h_n on Ω satisfying

$$0 \le g_n, h_n \le 1, \quad \operatorname{supp} h_n \subset V_n, \quad \|h_n\| = 1, \quad \operatorname{supp} g_n \subset U_n, \quad g_n(V_n) = \{1\},$$

and so that g_n and h_n belong to $C_0(\Omega)$. Since the subsets $\{U_n\}_{n=1}^{\infty}$ are disjoint, the sequences $\{g_n\}_{n=1}^{\infty}$ and $\{h_n\}_{n=1}^{\infty}$ are equivalent to the usual basis of c_0, and so, they converge to zero in the weak topology of $C_0(\Omega)$. We already proved (i). In order to obtain (ii) it suffices to use $U_{2n-1} \cup U_{2n}$ instead of U_n, and $V_{2n-1} \cup V_{2n}$ instead of V_n, and the functions $g_{2n-1} + g_{2n}$, $h_{2n-1} + h_{2n}$, $h_{2n-1} - h_{2n}$ satisfy the conditions required of g_n, f_n and h_n, respectively, in (ii). □

Proposition 2.2. *Let Ω be a locally compact and Hausdorff topological space and $N \in \mathbb{N}$. If there exists a weak neighborhood relative to the unit ball of $\bigotimes_{\pi,s}^{N} C_0(\Omega)$ with diameter less than two, then Ω is finite.*

Proof. Assume that Ω is infinite. We will write $X := \bigotimes_{\pi,s}^{N} C_0(\Omega)$. Let W be a non-empty weak open set relative to the unit ball of X. Since X is infinite-dimensional, W contains an element in the unit sphere of X. In fact we can assume that W contains an element x that can be expressed as

$$x = \sum_{i=1}^{s} t_i y_i^N,$$

where $t_i \in \mathbb{K}$, $\sum_{i=1}^{s} |t_i| = 1$, $y_i \in C_0(\Omega)$, $\|y_i\| = 1$, and $y^N := y \otimes \overset{N}{\ldots} \otimes y$.

In the complex case we can assume that $t_i \in \mathbb{R}^+$. In this case, we use Lemma 2.1 (i) and define the two sequences $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ in the N-fold symmetric tensor product of $C_0(\Omega)$ by

$$u_n = \sum_{i=1}^{s} t_i (y_i(1-g_n) + h_n)^N, \quad \text{and} \quad v_n = \sum_{i=1}^{s} t_i (y_i(1-g_n) + w h_n)^N,$$

where w is a complex number satisfying $w^N = -1$. Let us notice that for every $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, and every $s \in \Omega$, depending on if $s \in V_n$ or $s \notin V_n$, one has

$$|(y_i(1-g_n) + \lambda h_n)(s)| = |h_n(s)| \le 1 \text{ or } |(y_i(1-g_n) + \lambda h_n)(s)| = |(y_i(1-g_n))(s)| \le 1,$$

hence $\|u_n\|, \|v_n\| \leq 1$. In this case we take the norm-one polynomial on $C_0(\Omega)$ given by $Q_N = \delta_t^N$, where $t \in \Omega$ satisfies $h_n(t) = 1$. This polynomial clearly satisfies $\widehat{Q}_N(u_n) - \widehat{Q}_N(v_n) = 2$.

In the real case, if N is odd, then again by Lemma 2.1 (i) we define the sequences in the following way

$$u_n = \sum_{i \in P} t_i (y_i(1-g_n) + h_n)^N + \sum_{i \in N} t_i (y_i(1-g_n) - h_n)^N,$$
$$v_n = \sum_{i \in P} t_i (y_i(1-g_n) - h_n)^N + \sum_{i \in N} t_i (y_i(1-g_n) + h_n)^N,$$

where $P := \{i : t_i \geq 0\}$ and $N := \{i : t_i < 0\}$. By the same argument as above we obtain that $\|u_n\|, \|v_n\| \leq 1$. In this case we consider the N-homogeneous polynomial on $C_0(\Omega)$ given by $Q_N = \delta_t^N$, for some $t \in \Omega$ satisfying $h_n(t) = 1$, that clearly satisfies $\|Q_N\| \leq 1$ and $\widehat{Q}_N(u_n) - \widehat{Q}_N(v_n) = 2$.

Finally, if the space is real and N is even, we use Lemma 2.1 (ii) and consider the elements

$$u_n = \sum_{i \in P} t_i (y_i(1-g_n) + f_n)^N + \sum_{i \in N} t_i (y_i(1-g_n) + h_n)^N,$$
$$v_n = \sum_{i \in P} t_i (y_i(1-g_n) + h_n)^N + \sum_{i \in N} t_i (y_i(1-g_n) + f_n)^N,$$

where $P := \{i : t_i \geq 0\}$ and $N := \{i : t_i < 0\}$. It is immediate to check that u_n and v_n belong to B_X. In this case, we consider the N-homogeneous polynomial given by

$$Q_N = \delta_{s_n}^{N-1} \delta_{t_n}$$

which clearly satisfies $\|Q_N\| \leq 1$ and

$$\begin{aligned}
\widehat{Q}_N(u_n) - \widehat{Q}_N(v_n) &= \sum_{i \in P} t_i f_n(s_n)^{N-1} f_n(t_n) + \sum_{i \in N} t_i h_n(s_n)^{N-1} h_n(t_n) \\
&\quad - \sum_{i \in P} t_i h_n(s_n)^{N-1} h_n(t_n) - \sum_{i \in N} t_i f_n(s_n)^{N-1} f_n(t_n) \\
&= \sum_{i \in P} t_i + \sum_{i \in N} t_i (-1)^{N-1} - \sum_{i \in P} t_i (-1)^{N-1} - \sum_{i \in N} t_i \\
&= 2 \sum_{i=1}^{s} |t_i| \\
&= 2.
\end{aligned}$$

In any of the above cases, the sequences $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ belong to X. Since $\{f_n\}_{n=1}^{\infty}$, $\{g_n\}_{n=1}^{\infty}$ and $\{h_n\}_{n=1}^{\infty}$ are sequences weakly convergent to zero, and $C_0(\Omega)$

has the Dunford–Pettis property, then it has the polynomial Dunford–Pettis property [Ry, Theorem 2.1] and so the sequences $\{u_n\}_{n=1}^\infty$, $\{v_n\}_{n=1}^\infty$ converge weakly to x. Hence $u_n, v_n \in W$ for n large enough. In any case,

$$|\widehat{Q}_N(u_n) - \widehat{Q}_N(v_n)| = 2,$$

and so $2 \le \|u_n - v_n\| \le \operatorname{diam} W$.

Hence, if there is a (non-empty) weak open set relative to the unit ball of X with diameter less than two, then Ω is finite. □

In [Fe, Proposition 4.1], Ferrera proved that $\mathcal{P}^N(\mathcal{C}(K))$ has no points of Fréchet differentiability for any infinite compact K and $N \in \mathbb{N}$ (see also [BR, Proposition 17]). As a consequence of the previous result and the facts already recalled in the introduction, we obtain the largest possible modulus of roughness.

Corollary 2.3. *If K is any infinite compact topological space, $N \in \mathbb{N}$ and P is any homogeneous polynomial on $\mathcal{C}(K)$, then $\eta(\mathcal{P}^N(\mathcal{C}(K)), P) = 2$. As a consequence, the norm is not Fréchet differentiable at any homogeneous polynomial.*

By using the same technique as before, we also deduce the following result, valid in the real as well as in the complex case. Choi, García, Maestre and Martín recently obtained a result along the same line (see [CGMM, Corollary 2.10 and Theorem 1.1]).

Proposition 2.4. *Let K be an infinite compact Hausdorff topological space and $N \in \mathbb{N}$. Given an N-homogeneous polynomial P on $\mathcal{C}(K)$ and $\delta > 0$, there are functions $f, g \in S_{\mathcal{C}(K)}$ such that*

$$|\operatorname{Re} P(f)| \ge \|P\| - \delta, \quad |\operatorname{Re} P(g)| \ge \|P\| - \delta,$$
$$\operatorname{Re} P(f) \operatorname{Re} P(g) > 0, \quad \|f - g\| = \|f + g\| = 2.$$

Proof. We can clearly assume (by changing the sign of the polynomial) that $\|P\| = \sup_{f \in B_{\mathcal{C}(K)}} \operatorname{Re} P(f)$. Assume that $f_0 \in S_{\mathcal{C}(K)}$ satisfies $\operatorname{Re} P(f_0) > \|P\| - \delta$.

By Lemma 2.1, there are sequences of continuous functions $\{g_n\}_{n=1}^\infty$ and $\{h_n\}_{n=1}^\infty$ on K satisfying

$$g_n \xrightarrow{w} 0 \quad \text{and} \quad h_n \xrightarrow{w} 0$$

and sequences of disjoint open sets, $\{V_n\}_{n=1}^\infty$ and $\{U_n\}_{n=1}^\infty$, satisfying

$$0 \le g_n, h_n \le 1, \quad \operatorname{supp} h_n \subset V_n, \quad \|h_n\| = 1, \quad \operatorname{supp} g_n \subset U_n, \quad g_n(V_n) = \{1\}. \tag{2.1}$$

Since $\{g_n\}_{n=1}^\infty$ and $\{h_n\}_{n=1}^\infty$ converge weakly to zero, the sequences

$$\{u_n\}_{n=1}^\infty = \{f_0(1 - g_n) + h_n\}_{n=1}^\infty \quad \text{and} \quad \{v_n\}_{n=1}^\infty = \{f_0(1 - g_n) - h_n\}_{n=1}^\infty$$

converge weakly to f_0. Also, both sequences are in the unit ball of the space. We check this assertion. If $n \in \mathbb{N}$ and $t \in K$, by using conditions (2.1) and depending on if $t \in V_n$ or $t \notin V_n$, one of the two following possible cases holds

$$|(f_0(1-g_n) \pm h_n)(t)| = |h_n(t)|, \quad |(f_0(1-g_n) \pm h_n)(t)| = |f_0(1-g_n)|,$$

and so, both sequences $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ are in the unit ball of $\mathcal{C}(K)$. Finally, by using that the space $\mathcal{C}(K)$ has the Dunford–Pettis property and, by [Ry, Theorem 2.1], it has the polynomial Dunford–Pettis property, that is, polynomials on $\mathcal{C}(K)$ preserve weak convergence of sequences, we obtain that

$$P(u_n) \to P(f_0) \quad \text{and} \quad P(v_n) \to P(f_0).$$

Hence, for n large enough, u_n and v_n satisfy

$$\operatorname{Re} P(u_n) > \|P\| - \delta \quad \text{and} \quad \operatorname{Re} P(v_n) > \|P\| - \delta.$$

Now let us note that

$$\|u_n - v_n\| = \|2h_n\| = 2.$$

Since $\|f_0\| = 1$, assume for instance that $t \in K$ satisfies that $|f_0(t)| = 1$, since the open sets U_n are mutually disjoint, for n large enough, $t \notin U_n$ and so

$$\|u_n + v_n\| = \|2f_0(1-g_n)\| \geq 2|(f_0(1-g_n))(t)| = 2|f_0(t)| = 2. \quad \square$$

3. $L_1(\mu)$-spaces

Our aim now is to show similar results for $L_1(\mu)$ (real or complex case). As we already remarked, if μ has an atom of finite measure, then $B_{L_1(\mu)}$ has a strongly exposed point, and so the unit ball of $\bigotimes_{\pi,s}^N L_1(\mu)$ has slices of arbitrarily small diameter. Under reasonable assumptions on the measure, the next result shows that only two extreme cases happen: all slices of the unit ball of the above space have diameter two or there are arbitrarily small slices.

Theorem 3.1. *Let (Ω, Σ, μ) be a measure space and assume that μ is σ-finite and atomless. For a positive integer N, we write $X = \otimes_{\pi,s}^N L_1(\mu)$. Then every slice of B_X has diameter 2. Consequently, $\eta(\mathcal{P}^N(L_1(\mu)), P) = 2$ for every P in $S_{\mathcal{P}^N(L_1(\mu))}$.*

Proof. We use that X^* is identified with $\mathcal{P}^N(L_1(\mu))$. So, a slice is associated to a certain element $P \in S_{\mathcal{P}^N(L_1(\mu))}$ and a positive real number $\delta < 1$. We can clearly assume that $\|P\|=\sup_{f\in B_{L_1(\mu)}} \operatorname{Re} P(f)$. Since the subset of simple functions is dense in $L_1(\mu)$ and $S(B_X, \widehat{P}, \delta)$ is an open subset in the unit ball of X, we can choose a function $s=\sum_{i=1}^p \alpha_i\, \chi_{A_i}/\mu(A_i)$ satisfying $\operatorname{Re} P(s)>1-\delta$, for some positive integer p, some measurable sets $\{A_i : 1\leq i\leq p\}$ that are mutually disjoint and have positive measure, and some scalars satisfying $\sum_{i=1}^p |\alpha_i|=1$. By assumption, μ has no atoms, and so for each $1\leq i\leq p$, there is a sequence $\{r_{i,n}\}_{n=1}^\infty$ of Rademacher functions supported on A_i (see [AA, Definition 11.55]). The sequence $\{r_{i,n}/\mu(A_i)\}_{n=1}^\infty$ is contained in the unit sphere of $L_1(\mu)$ and converges weakly to zero for each i. We define $\{f_{i,n}\}_{n=1}^\infty$ and $\{g_{i,n}\}_{n=1}^\infty$ by

$$f_{i,n} := \frac{\chi_{A_i}+r_{i,n}}{\mu(A_i)} \quad \text{and} \quad g_{i,n} := \frac{\chi_{A_i}-r_{i,n}}{\mu(A_i)},$$

which are sequences of functions in $S_{L_1(\mu)}$ weakly converging to $\chi_{A_i}/\mu(A_i)$, satisfying

$$\|f_{i,n}-g_{i,n}\| = 2, \quad i\in\{1,...,p\}.$$

We write $f_n:=\sum_{i=1}^p \alpha_i f_{i,n}$ and $g_n:=\sum_{i=1}^p \alpha_i g_{i,n}$. Then $\{f_n\}_{n=1}^\infty$ and $\{g_n\}_{n=1}^\infty$ are sequences in $B_{L_1(\mu)}$ weakly converging to the simple function s. Since $L_1(\mu)$ has the Dunford–Pettis property, P is weakly sequentially continuous. Since $\operatorname{Re} P(s)>1-\delta$, $\operatorname{Re} P(f_{n_0}), \operatorname{Re} P(g_{n_0})>1-\delta$ for n_0 large enough. It follows that $f_{n_0}^N, g_{n_0}^N\in S(B_X, \widehat{P}, \delta)$ for suitable $n_0\in\mathbb{N}$. For $i\in\{1,...,p\}$, let λ_i be a scalar such that $\alpha_i=|\alpha_i|\lambda_i$ and $|\lambda_i|=1$, and μ_i be such that $\mu_i\lambda_i=1$. We consider the functionals $u_{n_0}^*:=\sum_{i=1}^p \mu_i\chi_{A_i}$ and $v_{n_0}^*:=\sum_{i=1}^p \mu_i r_{i,n_0}$ in $S_{L_1(\mu)^*}$. It is clear that $u_{n_0}^*(f_{i,n_0})=\mu_i=u_{n_0}^*(g_{i,n_0})$ and $v_{n_0}^*(f_{i,n_0})=\mu_i=-v_{n_0}^*(g_{i,n_0})$.

If N is odd, then the N-homogeneous polynomial given by $Q_{n_0}:=(v_{n_0}^*)^N$ satisfies $\|Q_{n_0}\|=1$ and

$$Q_{n_0}(f_{n_0})-Q_{n_0}(g_{n_0}) = 2.$$

If N is even, then we take the N-homogeneous polynomial $Q_{n_0}:=(v_{n_0}^*)^{N-1}u_{n_0}^*$, that satisfies $\|Q_{n_0}\|=1$ and

$$Q_{n_0}(f_{n_0})-Q_{n_0}(g_{n_0}) = 2.$$

In any case, we have that $\operatorname{diam} S(B_X, \widehat{P}, \delta)\geq \|f_{n_0}^N - g_{n_0}^N\|=2$. In view of [DGZ, Proposition I.1.11], we obtain that $\eta(\mathcal{P}^N(X), P)=2$. $\square$

From the above proof we also obtain the following consequence.

Corollary 3.2. *Let (Ω, Σ, μ) be a measure space, where μ is σ-finite and atomless. If N is a natural number and $P\in\mathcal{P}^N(L_1(\mu))$, then for every $\delta>0$, there are* *elements* $f, g \in S_{L_1(\mu)}$ *such that*

$$|\mathrm{Re}\, P(f)| \geq \|P\| - \delta, \quad |\mathrm{Re}\, P(g)| \geq \|P\| - \delta,$$
$$\mathrm{Re}\, P(f)\, \mathrm{Re}\, P(g) \geq 0, \quad \|f - g\| = \|f + g\| = 2.$$

To conclude this paper, let us state the analogous version of Theorem 3.1 for non-empty weakly open sets of the closed unit ball. In this case, the proof works only in some cases.

Theorem 3.3. *Let* (Ω, Σ, μ) *be a measure space, where* μ *is* σ*-finite and atomless. For an odd natural number* N*, take* $X := \bigotimes_{\pi,s}^N L_1(\mu)$*, where we consider only the real case. Then every non-empty open set of* (B_X, w) *has diameter two.*

Proof. Assume that W is a non-empty weakly open set in (B_X, w). Since μ has no atoms, $L_1(\mu)$ is infinite-dimensional and W contains an element in the unit sphere of X. By denseness of the simple functions in $L_1(\mu)$, W contains an element x that can be expressed as $x := \sum_{i=1}^k \alpha_i s_i^N$, where $\sum_{i=1}^k |\alpha_i| = 1$ and each s_i is a normalized simple function. Since N is odd, we can also assume that $\alpha_i > 0$ for every i. Now each simple function s_i can be written as $s_i = \sum_{j=1}^m \beta_{ij}\, \chi_{A_j}/\mu(A_j)$, where the subsets $A_1, ..., A_m$ are measurable and pairwise disjoint sets with $0 < \mu(A_i) < \infty$ and $\sum_{j=1}^m |\beta_{ij}| = 1$ for every i. For every fixed $1 \leq i \leq k$, consider the subset

$$P_i := \{j \in \{1, ..., m\} : \beta_{ij} \geq 0\}.$$

Since μ has no atoms, for each $1 \leq j \leq m$, we will denote by $\{r_{j,n}\}_{n=1}^\infty$ a sequence of Rademacher functions supported on A_j (see [AA, Definition 11.55]). We recall that $\{r_{j,n}/\mu(A_j)\}_{n=1}^\infty$ is a sequence of functions in the sphere of $L_1(\mu)$ that converges weakly to zero. Then we define the following sequences of functions

$$u_{i,n} := \sum_{j \in P_i} \beta_{ij} \frac{\chi_{A_j} + r_{j,n}}{\mu(A_j)} + \sum_{\substack{1 \leq j \leq m \\ j \notin P_i}} \beta_{ij} \frac{\chi_{A_j} - r_{j,n}}{\mu(A_j)}$$

and

$$v_{i,n} := \sum_{j \in P_i} \beta_{ij} \frac{\chi_{A_j} - r_{j,n}}{\mu(A_j)} + \sum_{\substack{1 \leq j \leq m \\ j \notin P_i}} \beta_{ij} \frac{\chi_{A_j} + r_{j,n}}{\mu(A_j)}.$$

The sequences $\{u_{i,n}\}_{n=1}^\infty$ and $\{v_{i,n}\}_{n=1}^\infty$ of functions in $S_{L_1(\mu)}$ are weakly converging to $\sum_{j=1}^m \beta_{ij}\, \chi_{A_j}/\mu(A_j)$. Now we consider the elements in the unit ball of X given by

$$u_n := \sum_{i=1}^k \alpha_i u_{i,n}^N \quad \text{and} \quad v_n := \sum_{i=1}^k \alpha_i v_{i,n}^N.$$

Since $L_1(\mu)$ has the Dunford–Pettis property, we have that every polynomial on $L_1(\mu)$ is weakly sequentially continuous. Thus for n large enough $u_n, v_n \in W$. Now we will check that the distance between u_n and v_n is two.

For every natural number n, let us choose a function $f \in L_\infty(\mu)$ such that

$$\|f\| = 1, \quad \int_{A_j} f \frac{r_{j,n}}{\mu(A_j)}\, d\mu = 1 \text{ and } \int_{A_j} f\, d\mu = 0 \quad \text{for all } j.$$

The polynomial Q given by

$$Q(h) = \left(\int_\Omega f h\, d\mu \right)^N, \quad h \in L_1(\mu)$$

is in the unit ball of the space of N-homogeneous polynomials on $L_1(\mu)$. By using that N is odd, we obtain that

$$Q(u_{i,n}) = \Bigg(\sum_{j \in P_i} \beta_{i,j} - \sum_{\substack{1 \le j \le m \\ j \notin P_i}} \beta_{ij} \Bigg)^N = 1$$

and

$$Q(v_{i,n}) = \Bigg(- \sum_{j \in P_i} \beta_{i,j} + \sum_{\substack{1 \le j \le m \\ j \notin P_i}} \beta_{ij} \Bigg)^N = -1.$$

Hence

$$\widehat{Q}(u_n - v_n) = 2 \sum_{i=1}^k \alpha_i = 2.$$

We deduce that $\operatorname{diam} W \ge \|u_n - v_n\| = 2$, as we wanted to show. □

The above theorem suggest the following open problem:

Open question. We do not know if the above result holds for the complex space $L_1(\mu)$, nor for the real case if N is even.

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