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Indefinite higher Riesz transforms

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Abstract. Stein's higher Riesz transforms are translation invariant operators on $L^2(\mathbf{R}^n)$ built from multipliers whose restrictions to the unit sphere are eigenfunctions of the Laplace–Beltrami operators. In this article, generalizing Stein's higher Riesz transforms, we construct a family of translation invariant operators by using discrete series representations for hyperboloids associated to the indefinite quadratic form of signature (p,q). We prove that these operators extend to L^r-bounded operators for $1<r<\infty$ if the parameter of the discrete series representations is generic.

1. Introduction and statement of main results

For a measurable bounded function m on $\mathbf{R}^n$, the *multiplier operator* $T_m\colon f\mapsto\mathcal{F}^{-1}(m\mathcal{F}f)$ defines a continuous, translation invariant operator on $L^2(\mathbf{R}^n)$, where $\mathcal{F}$ is the Fourier transform.

Classic examples are the Riesz transforms

$$(R_jf)(x):=\lim_{\varepsilon\to 0}\frac{\Gamma((n+2)/2)}{\pi^{(n+2)/2}}\int_{|y|>\varepsilon}\frac{y_j}{|y|^{n+1}}f(x-y)\,dy,\quad 1\le j\le n,$$

which are associated with the multipliers $m_j(\xi):=|\xi|^{-1}\xi_j$. One of the important properties of the Riesz transforms is that R_j extends to a continuous operator on the Banach space $L^r(\mathbf{R}^n)$ for any $1<r<\infty$. From a group theoretic view point, the space $\mathcal{M}_2(\mathbf{R}^n)$ of all continuous, translation invariant operators on $L^2(\mathbf{R}^n)$ is naturally a representation space of the general linear group $\mathrm{GL}(n,\mathbf{R})$ by

$$\mathcal{M}_2(\mathbf{R}^n)\longrightarrow\mathcal{M}_2(\mathbf{R}^n),\quad T\longmapsto L_g\circ T\circ L_g^{-1},\quad g\in\mathrm{GL}(n,\mathbf{R}),$$

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where $(L_g f)(x) := f(g^{-1}x)$. Then, it is noteworthy that the Riesz transforms R_j, $1 \le j \le n$, span the simplest non-trivial representation (an irreducible n-dimensional representation) of the orthogonal group $\mathrm{O}(n)$.

More generally, E. M. Stein introduced a family of bounded translation invariant operators T_m (*higher Riesz transforms*) associated with multipliers m having the following two properties:

$$m \text{ is a homogeneous function of degree } 0, \tag{1.1}$$

$$m|_{S^{n-1}} \in \mathcal{H}^k(\mathbf{R}^n). \tag{1.2}$$

The condition (1.1) is equivalent to the fact that m is constant on rays emanating from the origin. Thus, m is completely determined by its restriction to the unit sphere. The condition (1.2) concerns this restriction. Here, $\mathcal{H}^k(\mathbf{R}^n)$ denotes the space of spherical harmonics of degree k defined by

$$\mathcal{H}^k(\mathbf{R}^n) := \{f \in C^\infty(S^{n-1}) : \Delta_{S^{n-1}} f = -k(k+n-2)f\}, \tag{1.3}$$

where $\Delta_{S^{n-1}}$ is the Laplace–Beltrami operator on the unit sphere S^{n-1}. Then, we have the following consequence.

Fact 1.1. (See [6, Chapter II, Theorem 3]) Suppose $k \in \mathbf{N}$. Then, for any m satisfying (1.1) and (1.2), T_m extends to a continuous operator on the Banach space $L^r(\mathbf{R}^n)$ for any $1 < r < \infty$.

We note that T_m corresponds to the identity operator for $k=0$, and to the Riesz transforms for $k=1$. For general k, $\{T_m : m \text{ satisfies (1.1) and (1.2)}\}$ forms an irreducible $\mathrm{O}(n)$ submodule in $\mathcal{M}_2(\mathbf{R}^n)$.

In the previous paper [4], we analyzed translation invariant operators from group theoretic view points, and found the following phenomenon: L^2-bounded translation invariant operators with 'large symmetries' are mostly unbounded on $L^r(\mathbf{R}^n)$, $r \ne 2$, except for the cases when they are built from higher Riesz transforms, as far as 'large symmetries' are defined by *finite*-dimensional representations of affine subgroups (see e.g. [4, Theorem 9]).

The aim of this paper is to construct a family of L^r-bounded translation invariant operators, $1 < r < \infty$, with 'large symmetries' by using *infinite*-dimensional representations.

To be more precise, we take a quadratic form

$$Q(\xi) := \xi_1^2 + \dots + \xi_p^2 - \xi_{p+1}^2 - \dots - \xi_{p+q}^2$$

of a general signature (p, q), $p > 1$, and work on the hyperboloid

$$X_{p,q} := \{\xi \in \mathbf{R}^{p+q} : Q(\xi) = 1\},$$

which is a (non-singular) submanifold in the open domain $\mathbf{R}_+^{p,q}:=\{\xi\in\mathbf{R}^{p+q}: Q(\xi)>0\}$ of $\mathbf{R}^{p+q}$. We endow $X_{p,q}$ with the standard pseudo-Riemannian structure of signature $(p-1,q)$, and introduce a dense subspace of L^2-eigenfunctions of the Laplace–Beltrami operator $\Delta\equiv\Delta_{X_{p,q}}$ as

$$\mathcal{H}^k(\mathbf{R}^{p,q}):=\{f\in L^2(X_{p,q}):\Delta f=-k(k+p+q-2)f\}_{K\text{-finite}}.$$

See Section 2 for more details. In the case $(p,q)=(n,0)$, we note $X_{p,q}=S^{n-1}$, $\mathbf{R}_+^{p,q}=\mathbf{R}^n$, and $\mathcal{H}^k(\mathbf{R}^{p,q})=\mathcal{H}^k(\mathbf{R}^n)$.

In our setting for general p and q, we replace (1.2) with

$$m\,|_{X_{p,q}}\in\mathcal{H}^k(\mathbf{R}^{p,q})\quad\text{and}\quad\operatorname{supp} m\subset\overline{\mathbf{R}_+^{p,q}}. \tag{1.4}$$

Then, the following subspace of $\mathcal{M}_2(\mathbf{R}^{p+q})$:

$$\{T_m : m \text{ satisfies (1.1) and (1.4)}\}$$

forms a dense subspace of an irreducible (infinite-dimensional) unitary representation of the indefinite orthogonal group $\mathrm{O}(p,q)$ if $p>1$ and $q>0$. Our L^r-boundedness theorem is now stated as follows:

Theorem 1. *Suppose $k>4$, if $p+q$ is even, or $k>3$, if $p+q$ is odd. Then, for any m satisfying* (1.1) *and* (1.4), *the multiplier operator T_m extends to a continuous operator on $L^r(\mathbf{R}^{p+q})$ for any $1<r<\infty$.*

Remark 1.2. In place of $X_{p,q}\subset\mathbf{R}_+^{p,q}$, we can also consider the open domain $\mathbf{R}_-^{p,q}:=\{\xi\in\mathbf{R}^{p+q}:Q(\xi)<0\}$ and L^2-eigenfunctions on another hyperboloid $X'_{p,q}:=\{\xi\in\mathbf{R}^{p+q}:Q(\xi)=-1\}$. Then, for m supported on $\overline{\mathbf{R}_-^{p,q}}$ an analogous result also holds by swapping p and q because $X'_{p,q}\simeq X_{q,p}$ and $\mathbf{R}_-^{p,q}\simeq\mathbf{R}_+^{p,q}$.

The operators T_m with m satisfying (1.1) and (1.4) may be regarded as a generalization of Stein's higher Riesz transforms in the following sense:

$$\begin{aligned}\text{spherical harmonics on } S^{n-1} &\Longrightarrow \text{discrete series for } X_{p,q},\\ \mathrm{O}(n) &\Longrightarrow \text{indefinite orthogonal group } \mathrm{O}(p,q).\end{aligned}$$

We shall call T_m *indefinite Riesz transforms.* Then, Theorem 1 for indefinite Riesz transforms is a generalization of Fact 1.1.

A distinguishing feature of our generalization is that the restriction of the multiplier m to the unit sphere is no more infinitely differentiable. Our multiplier m has the following property:

$$m(\xi)=0\quad\text{if } \xi_1^2+\ldots+\xi_p^2\le\xi_{p+1}^2+\ldots+\xi_{p+q}^2.$$

Unlike Fefferman's ball multiplier theorem [2] and its descendants [4] for multiplier operators with 'large symmetries', the indefinite Riesz transforms T_m remain L^r-bounded for any r, $1<r<\infty$.

The crucial point of the proof of Theorem 1 is the asymptotic estimate of the multiplier $m(\xi)$ together with its differentials as ξ approaches the boundary of $\mathbf{R}_+^{p,q}$. This estimate is carried out by using techniques of infinite-dimensional representation theory of $\mathrm{O}(p,q)$ and non-commutative harmonic analysis (see e.g. [1], [5] and [7]).

Notation. $\mathbf{R}_+:=\{x\in\mathbf{R}:x>0\}$, $\mathbf{N}=\{0,1,2,...\}$, and $\mathbf{N}_+:=\{1,2,...\}$.

2. Basic properties of discrete series for $X_{p,q}$

In this section, after a quick review of some of the fundamental facts concerning discrete series representations for hyperboloids $X_{p,q}$, we introduce the linear vector space $\mathcal{V}_k^\infty$ consisting of smooth functions on an open domain $\mathbf{R}_+^{p,q}$ satisfying a certain decay condition. This space $\mathcal{V}_k^\infty$ will bridge discrete series for $X_{p,q}$ and 'indefinite higher multipliers'.

In what follows, we shall use the notation

$$\begin{aligned} \xi &= (\xi',\xi'')\in\mathbf{R}^{p+q},\\ Q(\xi) &= |\xi'|^2-|\xi''|^2=\xi_1^2+...+\xi_p^2-\xi_{p+1}^2-...-\xi_{p+q}^2,\\ |\xi|^2 &= |\xi'|^2+|\xi''|^2=\xi_1^2+...+\xi_p^2+\xi_{p+1}^2+...+\xi_{p+q}^2. \end{aligned}$$

Then, the indefinite orthogonal group

$$\mathrm{O}(p,q):=\{g\in\mathrm{GL}(p+q;\mathbf{R}):Q(g\xi)=Q(\xi)\text{ for any }\xi\in\mathbf{R}^{p+q}\}$$

is non-compact if $p,q>0$. Throughout this paper, we shall write $G:=\mathrm{O}(p,q)$, and denote by $\mathfrak{g}$ the Lie algebra $\mathfrak{o}(p,q)$ of G. The group G contains

$$K:=\left\{\begin{pmatrix}A & 0\\ 0 & B\end{pmatrix}:A\in\mathrm{O}(p)\text{ and }B\in\mathrm{O}(q)\right\}\simeq\mathrm{O}(p)\times\mathrm{O}(q)$$

as a maximal compact subgroup. We note that in the case $q=0$, $G=K$ is nothing but the orthogonal group $\mathrm{O}(p)$.

We denote by $\mathbf{R}^{p,q}$ the Euclidean space $\mathbf{R}^{p+q}$ equipped with the flat pseudo-Riemannian structure $ds^2=d\xi_1^2+...+d\xi_p^2-d\xi_{p+1}^2-...-d\xi_{p+q}^2$. Then, ds^2 is non-degenerate when restricted to the submanifold $X_{p,q}$, and defines a pseudo-Riemannian structure $g_{X_{p,q}}$ of signature $(p-1,q)$ on $X_{p,q}$. Obviously, the group $\mathrm{O}(p,q)$ acts on $\mathbf{R}^{p,q}$ and $X_{p,q}$, respectively, as isometries.

The action of $\mathrm{O}(p,q)$ on $X_{p,q}$ is transitive, and the isotropy subgroup at ${}^t(1,0,...,0)$ is identified with $\mathrm{O}(p-1,q)$. Thus, $X_{p,q}$ is realized as a homogeneous space:

$$\mathrm{O}(p,q)/\,\mathrm{O}(p-1,q) \simeq X_{p,q}.$$

The group G acts on the space of functions on $\mathbf{R}^{p,q}$, $\mathbf{R}^{p,q}_+$ and also on $X_{p,q}$ by translations:

$$\pi(g)\colon f \longmapsto f(g^{-1}\cdot\,).$$

In particular, G acts unitarily on the Hilbert space $L^2(X_{p,q})$ consisting of square integrable functions on $X_{p,q}$ with respect to the measure induced by $g_{X_{p,q}}$.

The differential of π, denoted by $d\pi$, is formally defined by

$$d\pi(Y)f := \frac{d}{dt}\bigg|_{t=0} f(e^{-tY}\cdot\,) \quad \text{for } Y \in \mathfrak{g}.$$

Next, we consider L^2-eigenfunctions of the Laplace–Beltrami operator $\Delta \equiv \Delta_{X_{p,q}}$ on $X_{p,q}$:

$$L^2_k(X_{p,q}) := \{f \in L^2(X_{p,q}) : \Delta f = -k(k+p+q-2)f\}.$$

Here, the differential equation is interpreted as that of distributions. Then, $L^2_k(X_{p,q})$ is a closed subspace of the Hilbert space $L^2(X_{p,q})$ (possibly, equal to zero). Since Δ commutes with the G-action, $L^2_k(X_{p,q})$ is a G-invariant subspace.

Suppose $f \in L^2_k(X_{p,q})$. We say that f is *K-finite* if $\mathbf{C}$-span$\{\pi(k)f : k\in K\}$ is finite-dimensional. We set the vector space consisting of K-finite vectors as

$$\mathcal{H}^k(\mathbf{R}^{p,q}) := L^2_k(X_{p,q})_{K\text{-finite}}. \tag{2.1}$$

We note that any function of $\mathcal{H}^k(\mathbf{R}^{p,q})$ is real-analytic although the differential operator Δ is not elliptic. We also note that if $q=0$ then $G=K$ and $\mathcal{H}^k(\mathbf{R}^{p,0}) = L^2_k(S^{p-1})_{K\text{-finite}} = L^2_k(S^{p-1})$.

We set

$$\rho := \frac{p+q-2}{2} \quad \text{and} \quad \Lambda_+(p,q) := \begin{cases} \{k\in\mathbf{Z} : k > -\rho\}, & q\neq 0, \\ \{k\in\mathbf{Z} : k\geq 0\}, & q=0. \end{cases}$$

We collect below some known results on discrete series representations for hyperboloids $X_{p,q}$. See [1] and [7] for the pioneering work. See also [5, Fact 5.4] for a survey on algebraic, geometric, and analytic aspects of these representations from modern representation theory.

Fact 2.1. Suppose $p>1$.

(1) If $k\in\Lambda_+(p,q)$ then $L^2_k(X_{p,q})$ is non-zero. It is irreducible as a representation of G. Conversely, any (non-zero) irreducible closed subspace of $L^2(X_{p,q})$ is of the form $L^2_k(X_{p,q})$ for some $k\in\Lambda_+(p,q)$.

(2) $\mathcal{H}^k(\mathbf{R}^{p,q})$ is a K-invariant dense subspace of $L^2_k(X_{p,q})$. As a representation of K,

$$\mathcal{H}^k(\mathbf{R}^{p,q})\simeq\begin{cases}\displaystyle\bigoplus_{\substack{a,b\in\mathbf{N}\\ a-b\geq k+q\\ a-b\equiv k+q \bmod 2}}\mathcal{H}^a(\mathbf{R}^p)\otimes\mathcal{H}^b(\mathbf{R}^q), & q>0,\\ \mathcal{H}^k(\mathbf{R}^p), & q=0.\end{cases}$$

(3) Suppose further that $q>0$. If $f\in\mathcal{H}^k(\mathbf{R}^{p,q})$, then there exists a function $a(\omega,\eta)\in C^\infty(S^{p-1}\times S^{q-1})$ such that

$$f(\omega\cosh t,\eta\sinh t)=a(\omega,\eta)e^{-(k+2\rho)t}(1+e^{-2t}\,\mathrm{O}(t)),\quad \text{as } t\to\infty.$$

(4) $d\pi(Y)\mathcal{H}^k(\mathbf{R}^{p,q})\subset\mathcal{H}^k(\mathbf{R}^{p,q})$ for any $Y\in\mathfrak{g}$.

The irreducible unitary representation of G realized on a closed subspace of $L^2(X_{p,q})$ is called a *discrete series representation* for the hyperboloid $X_{p,q}$. Discrete series representations for $X_{p,q}$ exist if $p>1$. By Fact 2.1 (1), $\Lambda_+(p,q)$ is the parameter space of discrete series representations for $X_{p,q}$.

Remark 2.2. (1) In the literature, the normalization of the parameter is often taken to be

$$\lambda:=k+\rho\left(=k+\frac{p+q-2}{2}\right).$$

Then, for $p>1$ and $q>0$, $k\in\Lambda_+(p,q)$ if and only if

$$\lambda>0\quad\text{and}\quad\lambda\in\mathbf{Z}+\frac{p+q}{2}.$$

(2) If $f\in\mathcal{H}^k(\mathbf{R}^{p,q})$ belongs to the K-type $\mathcal{H}^a(\mathbf{R}^p)\otimes\mathcal{H}^b(\mathbf{R}^q)$ (see Fact 2.1 (2)), then we have an explicit formula of f as follows:

$$f(\omega\cosh t,\eta\sinh t)=h_a(\omega)h_b(\eta)(\cosh t)^a(\sinh t)^b\varphi_{i\lambda}^{(b+q/2-1,a+p/2-1)}(t), \tag{2.2}$$

for some functions $h_a\in\mathcal{H}^a(\mathbf{R}^p)$ and $h_b\in\mathcal{H}^b(\mathbf{R}^q)$. Here $\lambda=k+\rho$, and $\varphi_{i\lambda}^{(\lambda'',\lambda')}(t)$, $\lambda''\neq-1,-2,...$, is the Jacobi function which is the unique solution to the following differential equation:

$$(L+(\lambda'+\lambda''+1)^2-\lambda^2)\varphi=0,\quad \varphi(0)=1,$$

if we set $L:=d^2/dt^2+((2\lambda'+1)\tanh t+(2\lambda''+1)\coth t)\,d/dt$. Equivalently, in terms of the hypergeometric function ${}_2F_1$, we have

$$\varphi_{i\lambda}^{(\lambda'',\lambda')}(t)={}_2F_1\left(\frac{\lambda'+\lambda''+1-\lambda}{2},\frac{\lambda'+\lambda''+1+\lambda}{2};\lambda''+1;-\sinh^2 t\right).$$

With these preparations, let us investigate the asymptotic behavior of the multiplier m near the boundary of $\mathbf{R}_+^{p,q}$. For this, we fix $\varkappa\in\mathbf{R}_+$ and let

$$\mathcal{V}_\varkappa:=\left\{f\in C^\infty(\mathbf{R}_+^{p,q}):(1)\ f \text{ is a homogeneous function of degree } 0,\right.$$

$$\left.(2)\ \sup_{\xi\in\mathbf{R}_+^{p,q}}|f(\xi)|\left(\frac{Q(\xi)}{|\xi|^2}\right)^{-\varkappa}<\infty\right\}. \tag{2.3}$$

Remark 2.3. Obviously, $\mathcal{V}_\varkappa\subset\mathcal{V}_{\varkappa'}$ if $\varkappa>\varkappa'$.

For any $g\in G$, we set $c:=\max(\|g\|,\|g^{-1}\|)$, where $\|g\|$ denotes the operator norm of g. This means that

$$c^{-1}|\xi|\le|g\xi|\le c|\xi|,\quad \xi\in\mathbf{R}^{p+q}.$$

Further, $Q(g\xi)=Q(\xi)$. Hence, $\mathcal{V}_\varkappa$ is a G-invariant subspace of $C^\infty(\mathbf{R}_+^{p,q})$. We define

$$\mathcal{V}_\varkappa^\infty:=\{f\in\mathcal{V}_\varkappa:d\pi(X_1)\circ...\circ d\pi(X_l)f\in\mathcal{V}_\varkappa \text{ for any } l=0,1,... \text{ and } X_1,...,X_l\in\mathfrak{g}\}.$$

Lemma 2.4. *Let m be as in Theorem* 1. *Then* $m|_{\mathbf{R}_+^{p,q}}\in\mathcal{V}_{(k/2)+\rho}^\infty$.

Proof. Suppose $\xi\in\mathbf{R}_+^{p,q}$. Then, $Q(\xi)>0$ and $Q(\xi)^{-1/2}\xi\in X_{p,q}$. Hence, we can find $\omega\in S^{p-1}$, $\eta\in S^{q-1}$ and $t\in\mathbf{R}$ such that

$$Q(\xi)^{-1/2}\xi=(\omega\cosh t,\eta\sinh t).$$

This means that

$$Q(\xi)^{-1}|\xi|^2=\cosh^2 t+\sinh^2 t=\cosh 2t.$$

If m satisfies (1.1), then $m(\xi)=m(Q(\xi)^{-1/2}\xi)=m(\omega\cosh t,\eta\sinh t)$. Therefore, we have

$$\sup_{\xi\in\mathbf{R}_+^{p,q}}\left(\frac{|\xi|^2}{Q(\xi)}\right)^{k/2+\rho}m(\xi)<\infty$$

by Fact 2.1 (3). Hence $m|_{\mathbf{R}_+^{p,q}}\in\mathcal{V}_{k/2+\rho}$. Hence, the lemma follows by iterating Fact 2.1 (4). $\square$

3. Proof of L^p-boundedness

For an open subset V in $\mathbf{R}^n$, we write $C^k(V)$ for the space of functions on V with continuous derivatives up to order k.

We recall from [6, Section IV, Theorem 3] the Hörmander–Mikhlin condition for L^r-multipliers:

Fact 3.1. Suppose $m \in C^{[n/2]+1}(\mathbf{R}^n \setminus \{0\})$ satisfies

$$\sup_{\xi \in \mathbf{R}^n \setminus \{0\}} |\xi|^{|\alpha|} \left| \frac{\partial^\alpha m(\xi)}{\partial \xi^\alpha} \right| < \infty \tag{3.1}$$

for all multi-indices α such that $|\alpha| \le [n/2]+1$. Then, the multiplier operator T_m extends to a continuous operator on $L^r(\mathbf{R}^n)$ for any r, $1<r<\infty$.

In Section 5, we shall show the following result.

Proposition 3.2. *If $f \in \mathcal{V}_\varkappa^\infty$, then*

$$\sup_{\xi \in \mathbf{R}_+^{p,q}} |\xi|^{|\alpha|} \left| \frac{\partial^\alpha f}{\partial \xi^\alpha} \right| < \infty \tag{3.2}$$

for any multi-index $\alpha \in \mathbf{N}^{p+q}$ with $|\alpha| \le \varkappa$.

Proposition 3.3. *For $f \in \mathcal{V}_\varkappa^\infty$ let F be the extension by zero of f to all of $\mathbf{R}^n$. Let N be any non-negative integer such that $N<\varkappa$. Then $F \in C^N(\mathbf{R}^n \setminus \{0\})$. In particular, F satisfies* (3.1) *for any α with $|\alpha|<\varkappa$.*

Admitting Propositions 3.2 and 3.3 for a while, let us complete the proof of Theorem 1.

Proof of Theorem 1. Suppose m is as in Theorem 1. Then $m|_{\mathbf{R}_+^{p,q}} \in \mathcal{V}_{k/2+\rho}^\infty$ by Lemma 2.4. Hence, m satisfies (3.1) for any multi-index α with $|\alpha|<k/2+\rho$ by Proposition 3.3.

Since the assumption $k>4$, if $p+q$ is even, or $k>3$, if $p+q$ is odd, implies that

$$\frac{k}{2}+\rho > \left[\frac{p+q}{2}\right]+1$$

the Hörmander–Mikhlin condition for m is fulfilled. Therefore, the operator is bounded on $L^r(\mathbf{R}^n)$ by Fact 3.1. □

4. Differential operators along $\mathrm{O}(p,q)$-orbits

The vector space $\mathcal{V}_\varkappa^\infty$ in which our multiplier lives (see Lemma 2.4) is stable under the action of the Lie algebra $\mathfrak{g}$ and the Euler operator $E=\sum_{i=1}^{p+q} \xi_i \, \partial/\partial \xi_i$. In this section, we shall give a formula for the standard derivatives $\partial/\partial\xi_i$, $1\le i\le p+q$, by means of $d\pi(Y)$, $Y\in\mathfrak{g}$, and E. The main result of this section is Proposition 4.3, and we shall study the space H_1 of coefficients (or more generally H_N; see (4.11)) in Section 5.

In the polar coordinate for the first p-factor

$$\mathbf{R}_+\times S^{p-1}\times\mathbf{R}^q\longrightarrow\mathbf{R}^{p+q},\quad (r,\omega,\xi'')\longmapsto(r\omega,\xi''), \tag{4.1}$$

an easy computation shows that

$$\frac{\partial}{\partial\xi_i}=a_i(\omega)\frac{\partial}{\partial r}+\frac{1}{r}Y_i(\omega),\quad 1\le i\le p, \tag{4.2}$$

where $a_i(\omega)\in C^\infty(S^{p-1})$ and Y_i is a smooth vector field on S^{p-1}.

In order to rewrite (4.2) by using the Lie algebra action $d\pi$, we note that $\mathfrak{g}=\mathfrak{o}(p,q)$ is given by matrices as

$$\begin{aligned}\mathfrak{g}&\simeq\left\{\begin{pmatrix}A & B\\ {}^tB & C\end{pmatrix}: {}^tA=-A,\ {}^tC=-C \text{ and } B\in M(p,q;\mathbf{R})\right\}\\ &=(\mathfrak{o}(p)+\mathfrak{o}(q))+\mathfrak{p}\quad\text{(Cartan decomposition)},\end{aligned}$$

where we set

$$\mathfrak{p}:=\left\{\begin{pmatrix}0 & B\\ {}^tB & 0\end{pmatrix}: B\in M(p,q;\mathbf{R})\right\}.$$

Let $\mathfrak{X}(S^{p-1})$ be the vector space consisting of smooth vector fields on S^{p-1}. Since $\mathrm{O}(p)$ acts transitively on S^{p-1}, the map

$$C^\infty(S^{p-1})\otimes\mathfrak{o}(p)\longrightarrow\mathfrak{X}(S^{p-1}),\quad (b,X)\longmapsto b\,d\pi(X),$$

is surjective. Let $\left\{K_h:1\le h\le\frac{1}{2}p(p-1)\right\}$ be a basis of the Lie algebra $\mathfrak{o}(p)$. Then, we can find $b_i^h\in C^\infty(S^{p-1})$ such that

$$Y_i(\omega)=\sum_h b_i^h(\omega)\,d\pi(K_h). \tag{4.3}$$

Next, we set

$$Y_{ij}:=E_{i,p+j}+E_{p+j,i},\quad 1\le i\le p,\ 1\le j\le q.$$

Here, E_{ij} are matrix units in $M(p+q,\mathbf{R})$. By definition, Y_{ij} spans $\mathfrak{p}$ and $d\pi(Y_{ij})$ is the vector field on $\mathbf{R}^{p+q}$ given as

$$d\pi(Y_{ij})=\xi_{p+j}\,\frac{\partial}{\partial\xi_i}+\xi_i\,\frac{\partial}{\partial\xi_{p+j}}. \tag{4.4}$$

Lemma 4.1. *For* $1 \le i \le p$ *we have*

$$\frac{\partial}{\partial \xi_i} = \frac{a_i(\omega)}{rQ(\xi)}\left(r^2E - \sum_{k=1}^{p}\sum_{j=1}^{q} \xi_k\xi_{p+j}\, d\pi(Y_{kj})\right) + \frac{1}{r}\sum_h b_i^h(\omega)\, d\pi(K_h), \tag{4.5}$$

and for $1 \le j \le q$

$$\frac{\partial}{\partial \xi_{p+j}} = \frac{1}{r^2}\left(\sum_{i=1}^{p} \xi_i\, d\pi(Y_{ij}) - \frac{\xi_{p+j}}{Q(\xi)}\left(r^2E - \sum_{i=1}^{p}\sum_{k=1}^{q} \xi_i\xi_{p+k}\, d\pi(Y_{ik})\right)\right). \tag{4.6}$$

Proof. By multiplying (4.4) by ξ_i and summing over i, $1 \le i \le p$, we get

$$\frac{\partial}{\partial \xi_{p+j}} = \frac{1}{r^2}\left(\sum_{i=1}^{p} \xi_i\, d\pi(Y_{ij}) - \xi_{p+j} r\frac{\partial}{\partial r}\right), \tag{4.7}$$

where we have used that $r\,\partial/\partial r = \sum_{i=1}^{p} \xi_i\, \partial/\partial \xi_i$.

Next, we multiply (4.7) by ξ_{p+j} and sum over j, $1 \le j \le q$, we obtain the identity for the Euler operator $E_{\xi''} = \sum_{j=1}^{q} \xi_{p+j}\, \partial/\partial \xi_{p+j}$:

$$E_{\xi''} = \frac{1}{r^2}\sum_{i=1}^{p}\sum_{j=1}^{q} \xi_i\xi_{p+j}\, d\pi(Y_{ij}) - \frac{|\xi''|^2}{r^2} r\frac{\partial}{\partial r}.$$

Combining with the identity

$$E_{\xi''} + r\frac{\partial}{\partial r} = E$$

we get

$$r\frac{\partial}{\partial r} = \frac{1}{Q(\xi)}\left(r^2E - \sum_{i=1}^{p}\sum_{j=1}^{q} \xi_i\xi_{p+j}\, d\pi(Y_{ij})\right). \tag{4.8}$$

By (4.7), this proves (4.6).

To prove (4.5) we insert into (4.2) the expressions for $Y_i(\omega)$ and $\partial/\partial r$ obtained in (4.3) and (4.8), respectively. □

To handle the coefficients of (4.5) and (4.6), we introduce the subspace, denoted by $H_{a,b,c}$, of $C^\infty(\mathbf{R}_+^{p,q})$ for $(a,b,c) \in \mathbf{N}^3$ that consists of finite linear combinations of functions of the form

$$\frac{A(\omega)P_a(\xi'')}{r^{b-c}Q(\xi)^c} = \frac{A(\omega)P_a(\xi'')}{r^{b-c}(r^2 - |\xi''|^2)^c}, \tag{4.9}$$

where P_a is a homogeneous polynomial of $\xi''=(\xi_{p+1},...,\xi_{p+q})\in\mathbf{R}^q$ of degree a and $A\in C^\infty(S^{p-1})$. If $f\in H_{a,b,c}$ and $g\in H_{a',b',c'}$ then $fg\in H_{a+a',b+b',c+c'}$, and likewise for finite linear combinations of such terms. We state this as

$$H_{a,b,c}H_{a',b',c'}\subset H_{a+a',b+b',c+c'}. \tag{4.10}$$

We also define the space

$$H_N:=\bigoplus_{\substack{a,b,c\in\mathbf{N}\\ a\le 2N,\ c\le N\\ b-a+c=N}} H_{a,b,c}. \tag{4.11}$$

The following lemma is an immediate consequence of (4.10).

Lemma 4.2. $H_N H_{N'}\subset H_{N+N'}$.

We write $H_N\,d\pi(\mathfrak{g})$ for the vector space consisting of differential operators on $\mathbf{R}^{p,q}_+$ which are of the form $\sum_j f_j\,d\pi(X_j)$ (a finite sum) for some $f_j\in H_N$ and $X_j\in\mathfrak{g}$. The point of the definition of H_N is that we have the following result.

Proposition 4.3. *On* $\mathbf{R}^{p,q}_+$,

$$\frac{\partial}{\partial\xi_i}\in H_1\,d\pi(\mathfrak{g})+C^\infty(\mathbf{R}^{p,q}_+)E,\quad 1\le i\le p+q.$$

Proof. In light of the formulas (4.5) and (4.6), it is sufficient to show that the coefficients

$$\frac{a_i(\omega)\xi_l\xi_{p+j}}{rQ(\xi)},\ \frac{b_i^h(\omega)}{r},\ \frac{\xi_i}{r^2},\ \frac{\xi_i\xi_{p+j}\xi_{p+k}}{r^2Q(\xi)}\in H_1$$

for any $1\le i,l\le p$ and $1\le j,k\le q$. In fact, these coefficients belong to

$$H_{1,1,1},\quad H_{0,1,0},\quad H_{0,1,0},\quad H_{2,2,1},$$

respectively, by definition. □

5. Proof of Propositions 3.2 and 3.3

Lemma 5.1. *For* $1\le i\le p+q$,

$$\frac{\partial}{\partial\xi_i}H_N\subset H_{N+1}.$$

Proof. Since $H_{a,b,c}$ is spanned by functions of the form (4.9), we get

$$a_i(\omega)\frac{\partial}{\partial r}(H_{a,b,c}) \subset H_{a,b+1,c} \oplus H_{a,b,c+1},$$
$$\frac{1}{r}Y_i(\omega)H_{a,b,c} \subset H_{a,b+1,c}.$$

Thus, by using (4.2) we have

$$\frac{\partial}{\partial \xi_i}H_{a,b,c} \subset H_{a,b+1,c} \oplus H_{a,b,c+1}, \quad 1 \le i \le p.$$

For the variables $\xi'' = (\xi_{p+1}, \dots, \xi_{p+q})$, we obtain directly

$$\frac{\partial}{\partial \xi_{j+p}}H_{a,b,c} \subset H_{a-1,b,c} \oplus H_{a+1,b+1,c+1}, \quad 1 \le j \le q.$$

The lemma now follows from the definition (4.11) of H_N. □

We denote by $H_N \cdot \mathcal{V}_\varkappa^\infty$ the subspace of $C^\infty(\mathbf{R}_+^{p,q})$ consisting of finite linear combinations of products of elements from H_N and $\mathcal{V}_\varkappa^\infty$. We then have the following result.

Lemma 5.2.

$$\frac{\partial}{\partial \xi_i}\mathcal{V}_\varkappa^\infty \subset H_1 \cdot \mathcal{V}_\varkappa^\infty, \quad 1 \le i \le p+q.$$

Proof. Since the Euler operator E acts on $\mathcal{V}_\varkappa^\infty$ by zero and $d\pi(X)\mathcal{V}_\varkappa^\infty \subset \mathcal{V}_\varkappa^\infty$, $X \in \mathfrak{g}$, the lemma follows from Proposition 4.3. □

Proposition 5.3. *For any multi-index* $\alpha \in \mathbf{N}^{p+q}$,

$$\frac{\partial^\alpha}{\partial \xi^\alpha}\mathcal{V}_\varkappa^\infty \subset H_{|\alpha|} \cdot \mathcal{V}_\varkappa^\infty. \tag{5.1}$$

Proof. We have already proved (5.1) for $|\alpha| = 1$ in Lemma 5.2. Suppose we have proved (5.1) for $|\alpha| \le N$. Then,

$$\begin{aligned}\frac{\partial}{\partial \xi_i}\frac{\partial^\alpha}{\partial \xi^\alpha}\mathcal{V}_\varkappa^\infty &\subset \frac{\partial}{\partial \xi_i}(H_{|\alpha|} \cdot \mathcal{V}_\varkappa^\infty) \\ &\subset \left(\frac{\partial}{\partial \xi_i}H_{|\alpha|}\right) \cdot \mathcal{V}_\varkappa^\infty + H_{|\alpha|} \cdot \left(\frac{\partial}{\partial \xi_i}\mathcal{V}_\varkappa^\infty\right) \\ &\subset H_{|\alpha|+1} \cdot \mathcal{V}_\varkappa^\infty + H_{|\alpha|}(H_1 \cdot \mathcal{V}_\varkappa^\infty),\end{aligned}$$

by Lemmas 5.1 and 5.2. Since $H_{|\alpha|}H_1 \subset H_{|\alpha|+1}$ by Lemma 4.2, (5.1) holds for $|\alpha| = N+1$. Hence, Proposition 5.3 is proved by induction on $|\alpha|$. □

Lemma 5.4. *Let* $f\in\mathcal{V}_\varkappa$ *and* $g\in H_{a,b,c}$. *Then*

$$|f(\xi)g(\xi)|\le\frac{C}{|\xi|^{b-a+c}}\left(\frac{Q(\xi)}{|\xi|^2}\right)^{\varkappa-c},\quad \xi\in\mathbf{R}_+^{p,q}.$$

In particular, if $f\in\mathcal{V}_\varkappa$ *and* $g\in H_N$ *are such that* $N\le\varkappa$, *then we have*

$$|f(\xi)g(\xi)|\le C|\xi|^{-N},\quad \xi\in\mathbf{R}_+^{p,q}.$$

Proof. By the definition (2.3) of $\mathcal{V}_\varkappa$, f satisfies

$$|f(\xi)|\le C_1\left(\frac{Q(\xi)}{|\xi|^2}\right)^{\varkappa}\quad\text{for } \xi\in\mathbf{R}_+^{p,q},$$

for some constant $C_1>0$. Hence, in view of (4.9), there exists $C'>0$ such that

$$|f(\xi)g(\xi)|\le C'\frac{1}{r^{b-c}}\frac{|\xi''|^a}{Q(\xi)^c}\left(\frac{Q(\xi)}{|\xi|^2}\right)^{\varkappa}. \tag{5.2}$$

We note that for $\xi\in\mathbf{R}_+^{p,q}$, we have $r>|\xi''|$ and therefore $|\xi|=(r^2+|\xi''|^2)^{1/2}$ satisfies

$$r<|\xi|<\sqrt{2}r.$$

Hence the first factor of (5.2) is bounded by

$$\frac{1}{r^{b-c}}\le\frac{C''}{|\xi|^{b-c}}.$$

The last two factors of (5.2) are estimated as

$$\frac{|\xi''|^a}{Q(\xi)^c}\left(\frac{Q(\xi)}{|\xi|^2}\right)^{\varkappa}\le\frac{1}{|\xi|^{2c-a}}\left(\frac{Q(\xi)}{|\xi|^2}\right)^{\varkappa-c}.$$

Combining these estimates, we have proved that

$$|f(\xi)g(\xi)|\le\frac{C'C''}{|\xi|^{b-a+c}}\left(\frac{Q(\xi)}{|\xi|^2}\right)^{\varkappa-c}. \quad\square$$

Proof of Proposition 3.2. Suppose $f\in\mathcal{V}_\varkappa^\infty$. Then $\partial^\alpha f/\partial\xi^\alpha\in H_{|\alpha|}\cdot\mathcal{V}_\varkappa^\infty$ by Proposition 5.3. If $|\alpha|\le\varkappa$ then $|\partial^\alpha f/\partial\xi^\alpha|\le C|\xi|^{-|\alpha|}$ for $\xi\in\mathbf{R}_+^{p,q}$ by Lemma 5.4. Hence, Proposition 3.2 is proved. $\square$

Proof of Proposition 3.3. Let $f\in\mathcal{V}_\varkappa^\infty$. It is sufficient to prove that if $|\alpha|<\varkappa$ then

$$\frac{\partial^\alpha f}{\partial\xi^\alpha}(\xi)\to 0$$

as $\xi\in\mathbf{R}_+^{p,q}$ approaches the boundary of $\mathbf{R}_+^{p,q}$ in $\mathbf{R}^{p+q}\setminus\{0\}$, namely, the isotropic cone $\{\xi\in\mathbf{R}^{p+q}\setminus\{0\}:Q(\xi)=0\}$. This follows again from Lemma 5.4. Hence, Proposition 3.3 is also proved. □

Thus, the proof of Theorem 1 is completed.

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