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An approach through big cells to Clifford groups of low rank

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Abstract. So-called exceptional isomorphisms of low-dimensional spinor groups are directly worked out for the terminal two cases of rank five and six (in the sense of quadratic modules) at the level of special Clifford groups with norm characters as group schemes over a ring. Explicit formulas of identifications are available on big cells, restrictions to which being justified by scheme theory.

Introduction

It is one of the most interesting parts in the theory of Clifford algebras to have *spinor groups* of low rank, which the theory gives a method to construct generally, in turn identified with *classical groups*. While identifications are apparent at the level of Dynkin diagrams, actual constructions of group isomorphisms are rather complicated and indeed for the terminal cases of rank five and six somewhat other problems, mainly concerned with *classification*, usually take the place of realizing individual isomorphisms, see e.g. [9, Section V-5].

This paper deals exclusively with those cases which are essentially split (Section 1.3), and then gives a direct approach to identifications at the level of *special Clifford groups with norm characters*. The case of rank five (Theorem 1.4) is, contrary to [9], completely separated from that of rank six (Theorem 1.5) and the latter survives more generally (Remark 1.6). Our method rests on a principle of recovering groups from their 'big cells' (Section 1.7), an idea which goes back to Weil and grew extensively in the Demazure–Grothendieck theory of group schemes [1]. Once restricted to big cells there exists a systematic way of describing points of special Clifford groups and their norms ([7] and [8]), which enables us to obtain explicit formulas of identifications (Section 2.7 and Proposition 3.6).

Special Clifford groups with norm characters are preferred to spinor groups from the viewpoint of concrete realizations; being by construction the kernels of norm characters, spinor groups have, with few exceptions, additional properties like (semi)-simplicity, but their significance would be rather in general problems like classification, than in working out particular examples. Furthermore, we notice that interest in special Clifford groups has been recently aroused by Shimura in a number-theoretical context [11]. Though being far from touching upon the theory, this note is hoped to be one of the frames of the underlying structures in pure algebra.

1. Notation and main results

1.1. General notation

All modules, algebras, and schemes are to be considered over an arbitrary commutative base ring k of scalars. Modules are usually supposed finitely generated and projective, and schemes are treated as set-valued covariant functors on the category of commutative k-algebras, cf. [5]. If $\mathfrak{E}$ is any finitely generated projective module, $\mathfrak{E}^*$ denotes the k-module $\mathrm{Hom}(\mathfrak{E}, k)$ dual to $\mathfrak{E}$ and $\langle x, f\rangle$ the value of $f \in \mathfrak{E}^*$ at $x \in \mathfrak{E}$. We shall allow ourselves to identify the second dual $\mathfrak{E}^{**}$ with $\mathfrak{E}$ using the canonical isomorphism $\mathfrak{E} \xrightarrow{\sim} \mathfrak{E}^{**}$ of constructing 'point-wise distributions'. Furthermore, the role of $\mathfrak{E}$ is in many cases replaceable by the associated *vector bundle* $\mathbf{W}(\mathfrak{E})$, the functor sending any k-algebra k' to the k'-module $\mathfrak{E} \otimes_k k'$ which is in Grothendieck's notation the *affine k-scheme* of the symmetric algebra $\mathrm{Sym}(\mathfrak{E}^*)$. If $\mathcal{A}$ is any k-algebra which is finitely generated and projective as a k-module, $\mathbf{U}(\mathcal{A})$ denotes the *multiplicative group* assigning those groups for each scalar extensions $\mathcal{A} \otimes_k k'$; important examples being $\mathbf{U}(k) = \mathbf{G}_{\mathbf{m}k}$ (the usual 'multiplicative group') and $\mathbf{U}(\mathrm{End}(\mathfrak{E})) = \mathbf{GL}(\mathfrak{E})$ (the general linear group).

1.2. Exterior powers of modules

Let M be any finitely generated projective module. We put the *νth exterior powers* $\bigwedge^\nu(M)$ and $\bigwedge^\nu(M^*)$, for each integer $\nu \geq 0$, in duality by the pairing such that

$$\langle x_1 \wedge \ldots \wedge x_\nu, f_1 \wedge \ldots \wedge f_\nu \rangle := (-1)^{\nu(\nu-1)/2} \det(\langle x_i, f_j \rangle) \tag{1.2.1}$$

for $x_i \in M$ and $f_j \in M^*$, understood as the multiplication $k \times k \to k$ in the case $\nu = 0$, and then, taking their direct sum, extended to the full exterior powers. The wedge product $z \wedge z'$ in $\bigwedge(M)$ is also denoted by $l_z \cdot z'$ and considered to be giving a left $\bigwedge(M)$-module structure on $\bigwedge(M)$. Moreover, a left $\bigwedge(M^*)$-module structure is also considered by the (left) *interior product* $z^* \lrcorner z' = d_{z^*} \cdot z'$, for which we use the following two characterizations (see [3, Section 11] and [7, Section 1.2] for details): *under the pairing above,* $d_{z^*}\in\mathrm{End}(\bigwedge(M))$ *is dual to the right wedge product by* z^* in $\bigwedge(M^*)$, on the one hand, and d_f *for* $f\in M^*$ *is the unique anti-derivation extending* $f\colon \bigwedge^1(M)\to\bigwedge^0(M)$, on the other hand. The *exponential map* exp in an exterior algebra (e.g., in $\bigwedge(M)$) is understood, following Chevalley [4], to be the unique homomorphism $\bigwedge^2(M)\to\bigwedge(M)$ from the additive to the multiplicative group such that $\exp(x\wedge y)=1+x\wedge y$. By naturality it is in fact a homomorphism $\mathbf{W}(\bigwedge^2(M))\to\mathbf{U}(\bigwedge(M))$ of k-group schemes, and moreover in the target, $\bigwedge(M)$ may well be shrunk to the *even* part $\bigwedge^+(M):=\bigoplus_{p\geq 0}\bigwedge^{2p}(M)$.

1.3. Setups for special Clifford groups

The pairing $(x,f)\mapsto\langle x,f\rangle$ viewed as a quadratic form on the direct sum $M\oplus M^*=:H$ is by definition the *hyperbolic module* $\mathbf{H}(M)$, and has the Clifford algebra realized as $\mathrm{End}(\bigwedge(M))$ equipped with the map $(x,f)\mapsto l_x+d_f$ from H and with the 'checker-board grading' relative to the parity decomposition $\bigwedge(M)=\bigwedge^+(M)\oplus\bigwedge^-(M)$, cf. [9, Section IV.2.1]. Accordingly, we shall consider the *special Clifford group* $\mathrm{S}\Gamma(\mathbf{H}(M))$ to be the normalizer taken in $\mathrm{GL}(\bigwedge^+(M))\times\mathrm{GL}(\bigwedge^-(M))$ of the embedded $H\subset\mathrm{End}(\bigwedge(M))$, and employ bold-faced notation $\mathbf{S\Gamma}(\mathbf{H}(M))$ in the case considered scheme-theoretically. As for the odd rank case $\mathbf{H}(M)\perp\langle 1\rangle$, let $\mathfrak{e}\in\mathrm{End}(\bigwedge(M))$ denote the diagonal element $\mathrm{diag}(1,-1)$ relative to the parity decomposition above. Apart from the precise identification of the Clifford algebra, for which we refer to [8, Sections 1.1–1.3], $\bigwedge(M)$ turns out to be again the space of spinors, in the sense that one may identify the special Clifford group $\mathrm{S}\Gamma(\mathbf{H}(M)\perp\langle 1\rangle)$ with the full normalizer in $\mathrm{GL}(\bigwedge(M))$ of $H\oplus k\subset\mathrm{End}(\bigwedge(M))$ embedded by $(x,f,t)\mapsto l_x+d_f+t\mathfrak{e}$. For both cases, the special Clifford group has a character μ called the *norm* [9, Chapter IV, Lemma 6.1.1], the kernel of which being the so-called *spinor group.* One would like to thus make special Clifford groups and their norm characters explicit. For example, the case of $M=L$, an invertible module, soon yields $\mathbf{S\Gamma}(\mathbf{H}(L)\perp\langle 1\rangle)=\mathbf{GL}(\bigwedge(L))$ with $\mu=\det$ (the rank three case). Furthermore, it is well-known and easy to verify that if $M=P$ is of rank two then $\mu\colon \mathbf{S\Gamma}(\mathbf{H}(P))\to\mathbf{G}_{\mathbf{m}k}$, considered as an object over $\mathbf{G}_{\mathbf{m}k}$, equals the *fiber product* of $\mathbf{GL}(\bigwedge^+(P))$ and $\mathbf{GL}(\bigwedge^-(P))$, both considered over $\mathbf{G}_{\mathbf{m}k}$ by det (the rank four case). It is our purpose to obtain analogous descriptions for the next two cases of ranks five and six.

Theorem 1.4. (The rank five case) *Let P be any projective module of rank two. We consider the quadratic module $\mathbf{H}(P)\perp\langle 1\rangle$ of rank five and regard $\bigwedge(P)$ as its space of spinors. Furthermore, letting L denote the invertible module $\bigwedge^2(P)$, we* *construct an L-valued alternating form Ψ on $\bigwedge(P)$ by*

$$\Psi(Z, Z') := Z_0 Z_2' - Z_0' Z_2 + Z_1 \wedge Z_1', \tag{1.4.1}$$

where $Z, Z' \in \bigwedge(P)$ with subscripts indicating the degrees of components, e.g. $Z_0 \in k$, $Z_1 \in P$, $Z_2 \in \bigwedge^2(P) = L$, etc. Then the special Clifford group $\mathbf{S\Gamma}(\mathbf{H}(P) \perp \langle 1 \rangle)$ coincides with the similitude group $\mathbf{GSp}(\Psi)$, and so does the norm character with the similitude character.

A point is that an alternating form has appeared intrinsically; necessary rudiments for those forms with values in invertible modules are given in Section 2, which exposes some generalities culminating in a particular case of proving the theorem, cf. Section 2.7.

Theorem 1.5. (The rank six case) *Let N be any projective module of rank three. We consider the quadratic module $\mathbf{H}(N)$ of rank six and regard $\bigwedge(N) = \bigwedge^+(N) \oplus \bigwedge^-(N)$ as its space of spinors, decomposed into those of half-spinors. Then the diagram*

$$\begin{tikzcd} \mathbf{S\Gamma}(\mathbf{H}(N)) \arrow[r, "\text{norm}"] \arrow[d, "\text{half-spin representation}"'] & \mathbf{G}_{\mathbf{m}k} \arrow[d, "\text{square}"] \\ \mathbf{GL}(\bigwedge^-(N)) \arrow[r, "\det"'] & \mathbf{G}_{\mathbf{m}k} \end{tikzcd} \tag{1.5.1}$$

is commutative and Cartesian.

That (1.5.1) is commutative has been noticed in [9, Chapter V, (5.6.2)] in a more general setting, but the Cartesian property seems unnoticed. If $\mathbf{GL}^{(\mathbf{2})}(\bigwedge^-(N))$ denotes the actual fiber product of the lower right of the diagram, this amounts to saying that the homomorphism

$$\varkappa \colon \mathbf{S\Gamma}(\mathbf{H}(N)) \longrightarrow \mathbf{GL}^{(\mathbf{2})}(\bigwedge^-(N)) \tag{1.5.2}$$

induced by the commutativity is in fact an *isomorphism*; we shall make the *inverse* $\varkappa^{-1}$ explicit in Section 3.4.

Remark 1.6. By faithfully flat descent, Theorem 1.5 survives in more general cases of rank six, as far as the quadratic form q in question admits (half-)spin representation(s), e.g. when both Witt and Arf invariants are trivial (cf. [9, Section IV.8 p. 242]). Once a space $\mathfrak{S}^+$ of half-spinors is fixed we have $\mathbf{S\Gamma}(q)$ naturally identified with $\mathbf{GL}^{(\mathbf{2})}(\mathfrak{S}^+)$, by the homomorphism with the norm μ and the fixed half-spin representation $\mathbf{S\Gamma}(q) \to \mathbf{GL}(\mathfrak{S}^+)$ as components.

1.7. Big cells

Let $\mathbf{G}$ be a smooth separated finitely presented k-group scheme with connected fibers, e.g. $\mathbf{S\Gamma}(\mathbf{H}(M))$ or $\mathbf{S\Gamma}(\mathbf{H}(M)\perp\langle 1\rangle)$ for any finitely generated projective module M ([7], [8]). By a *big cell* of $\mathbf{G}$, we shall understand an open subscheme Ω which is universally scheme-theoretically dense in $\mathbf{G}$. Since in this case the multiplication map $\Omega\times\Omega\to\mathbf{G}$ becomes an fppf (fidélement plat et presentation fini, i.e. faithfully flat finitely presented) epimorphism (the same argument as employed in [10, remark on p. 31]), once a big cell Ω has been obtained working only with Ω, instead of considering the most general points, becomes sufficient in many cases; and this will be indeed our case when proving Theorems 1.4 and 1.5. It is thus appropriate to recall here the big cells constructed in [7] and [8]. Furthermore, by the same reason, it is theoretically important for our method to check the *smooth with connected fibers* property for other groups which are obviously affine finitely presented (cf. Propositions 2.3 and 3.2).

Now turning attention again to the exterior power $\bigwedge(M)$, cf. Section 1.2, let us consider the homomorphisms

$$\mathbf{W}(\textstyle\bigwedge^2(M)) \xrightarrow{Y_+} \mathbf{GL}(\bigwedge(M)) \xrightarrow{Y_-} \mathbf{W}(\bigwedge^2(M^*)), \tag{1.7.1}$$
$$Y_+(u):=l_{\exp(u)},\quad Y_-(v):=d_{\exp(v)},$$

$$Y_0\colon \mathbf{G_m}_k\times\mathbf{GL}(M)\longrightarrow\mathbf{GL}(\textstyle\bigwedge(M)),\quad Y_0(t,h):=t\det(h)^{-1}\bigwedge(h), \tag{1.7.2}$$

of k-group schemes. Here and in the sequel in describing scheme morphisms, expressions like u, v and (t,h) are to be understood for all scalar extensions too, e.g. $u\in\bigwedge^2(M)\otimes_k k'$ for any k-algebra k', etc. We recall [7, Proposition 3.7] that $Y_\pm$ and Y_0 are all factoring through $\mathbf{S\Gamma}(\mathbf{H}(M))$ with the product of sources being in fact embedded as an open subscheme Ω by the multiplication map $(v,(t,h),u)\mapsto Y_-(v)Y_0(t,h)Y_+(u)$; this Ω, together with a kind of coordinate system (Y_-,Y_0,Y_+), being our big cell. Furthermore, we recall [7, Section 3.6] that $Y_\pm$ factor through the spinor group, while Y_0 composed with the norm character yields $(t,h)\mapsto t^2\det(h)^{-1}$. In order to obtain a similar result for the odd rank case $\mathbf{H}(M)\perp\langle 1\rangle$, we have only to extend $Y_\pm$ to

$$\mathbf{W}(\textstyle\bigwedge^2(M)\oplus M) \xrightarrow{Y_+} \mathbf{GL}(\bigwedge(M)) \xrightarrow{Y_-} \mathbf{W}(\bigwedge^2(M^*)\oplus M^*), \tag{1.7.3}$$
$$Y_+(u,y):=Y_+(u)(1+l_y\mathfrak{e})=(1+l_y\mathfrak{e})Y_+(u),$$
$$Y_-(v,g):=Y_-(v)(1+d_g\mathfrak{e})=(1+d_g\mathfrak{e})Y_-(v),$$

where the sources are now to be understood equipped with new multiplications $(u,y)\bullet(u',y'):=(u+u'-y\wedge y',y+y')$, etc., so as to keep $Y_\pm$ homomorphic; with this being done, the triplet (Y_-,Y_0,Y_+) coordinatizes again a big cell of $\mathbf{S\Gamma}(\mathbf{H}(M)\perp\langle 1\rangle)$

and behaves similarly to the previous even rank case under the norm character (cf. [8, Sections 1.8–1.9]).

2. Alternating forms with values in invertible modules

2.1. Non-singularity, the inverse form

Let P be any finitely generated projective module and L be an invertible module. To any bilinear form Φ on P with values in L, there are two associated linear maps s_Φ and d_Φ of type $P\to\mathrm{Hom}(P,L)$ given by the formulas

$$s_\Phi(x)(y) := \Phi(x,y) =: d_\Phi(y)(x). \tag{2.1.1}$$

It is easy to see that d_Φ is recovered as the composition $\mathrm{Hom}(s_\Phi,1)\circ\rho$ of the map $\rho\colon P\to\mathrm{Hom}(\mathrm{Hom}(P,L),L)$, constructing 'point-wise distributions', with the map $\mathrm{Hom}(s_\Phi,1)$ sending $\delta\in\mathrm{Hom}(\mathrm{Hom}(P,L),L)$ to $\delta\circ s_\Phi\in\mathrm{Hom}(P,L)$. Similarly one has $\mathrm{Hom}(d_\Phi,1)\circ\rho=s_\Phi$, and since ρ is bijective, L being of rank one, it follows that s_Φ is bijective if and only if d_Φ is bijective; in this case, we say that Φ is *non-singular.* Supposing Φ being non-singular and setting $\mathrm{Hom}(P,L)\cong L\otimes_k P^*$ and $\mathrm{Hom}(P^*,L^*)\cong L^*\otimes_k P$ in natural duality, we regard both transpose-inverses $(s_\Phi^*)^{-1}$ and $(d_\Phi^*)^{-1}$ as maps of type $P^*\to\mathrm{Hom}(P^*,L^*)$ and make the following remark, verification of which being straightforward: *For any $f\in P^*$ the map $L\to\mathrm{Hom}(P,L)$ given by $\omega\mapsto\omega\otimes f$ is dual to the distribution $\delta_f\colon \mathrm{Hom}(P^*,L^*)\to L^*$ at f, and hence one has*

$$(d_\Phi^*)^{-1}(\delta_f\circ(s_\Phi^*)^{-1}) = (s_\Phi^*)^{-1}(\delta_f\circ(d_\Phi^*)^{-1}) = f. \tag{2.1.2}$$

The formula (2.1.2) is read as $\mathrm{Hom}((s_\Phi^*)^{-1},1)=(d_\Phi^*)^{-1}$ and $\mathrm{Hom}((d_\Phi^*)^{-1},1)=(s_\Phi^*)^{-1}$, where, contrary to the previous ρ, we have treated the map $f\mapsto\delta_f$ as an identification. Hence there exists one and only one L^*-valued bilinear form Φ^* on P^* defined by the two conditions

$$s_{\Phi^*} = (d_\Phi^*)^{-1} \quad\text{and}\quad d_{\Phi^*} = (s_\Phi^*)^{-1}, \tag{2.1.3}$$

which are equivalent. We call Φ^* the *inverse form* of Φ, a terminology consisting with [2, Section 1.7] in the case where $L=k$. By construction Φ^* is also non-singular and recovers Φ after again taking inverse. Furthermore, if $x\in P$ and $f\in P^*$ then (2.1.3) makes the two composite endomorphisms $d_\Phi(x)\circ s_{\Phi^*}(f)^*$ and $s_\Phi(x)\circ f_{\Phi^*}(f)^*$ of L to be equal to the scalar multiplication by $\langle x,f\rangle$. We shall record this fact, being acted upon $\omega\in L$, as the identities

$$\Phi(s_{\Phi^*}(f)^*\omega, x) = \Phi(x, d_{\Phi^*}(f)^*\omega) = \langle x,f\rangle\omega, \tag{2.1.4}$$

which actually anticipates part of the calculation needed later (cf. Section 2.5). As yet another application of (2.1.4), considering those x of the form $s_{\Phi^*}(g)^*\omega'$ we find the bilinear composition $\Phi\circ(s_{\Phi^*}(f)^*\times s_{\Phi^*}(g)^*)$ in L to be equal to $(\omega,\omega')\mapsto \langle\omega',\Phi^*(g,f)\rangle\omega$. Since there exists no non-zero alternating map on L, it follows in particular that *if* Φ *is alternating then so is* Φ^*.

2.2. Similitude groups

In the following, P is also supposed to be faithfully projective. We observe that *a surjective linear map* $u\colon P\to P'$ *onto a faithfully projective module* P' *is a unimodular element of the* k*-module* $\mathrm{Hom}(P,P')\cong P'\otimes_k P^*$. Indeed, there exist finitely many elements $x_i\in P$, $x'_i\in P'$ and $f'_i\in(P')^*$ such that $u(x_i)=x'_i$ and $\sum_i\langle x'_i,f'_i\rangle=1$, and putting $\gamma:=\sum_i x_i\otimes f'_i$ gives a linear form γ on $\mathrm{Hom}(P,P')$ with the wanted property $\gamma(u)=1$. Applying this to isomorphisms onto $P'=\mathrm{Hom}(P,L)$ we see that in the totality of all bilinear forms $P\times P\to L$ a *non-singular* one Φ is unimodular. Therefore, its *similitude group*, which is by formalistic definition the subgroup of $\mathrm{GL}(P)\times k^\times$ consisting of those pairs (g,t) such that

$$\Phi\circ(g\times g)=t\Phi, \tag{2.2.1}$$

may well be embedded into $\mathrm{GL}(P)$ by regarding $(g,t)\mapsto g$ as the inclusion, and thus equipped with a character $(g,t)\mapsto t$, the so-called *similitude character*. Furthermore, the criterion (2.2.1) may well be released from supposing g invertible in advance. Namely, (2.2.1) makes sense for general (g,t) in $\mathrm{End}(P)\times k$ and read as, under the identification $\mathrm{Hom}(P,L)\cong L\otimes_k P^*$,

$$(1\otimes g^*)\circ s_\Phi\circ g=ts_\Phi. \tag{2.2.2}$$

In particular $\det(g)^2\varphi=t^n\varphi$ for $\varphi:=\bigwedge^n(s_\Phi)$, $n:=\mathrm{rk}(P)$, and since φ is unimodular, again by the observation above, it follows that

$$\det(g)^2=t^n \tag{2.2.3}$$

for all $(g,t)\in\mathrm{End}(P)\times k$ satisfying (2.2.1). This will be used later (cf. Proposition 2.6) to assure the invertibility of g from that of t. Furthermore, it seems appropriate to record here another effect of rewriting (2.2.1) as (2.2.2), which is

$$\Phi^*\circ(g^{*-1}\times g^{*-1})=t^{-1}\Phi^* \tag{2.2.4}$$

for $(g,t)\in\mathrm{GK}(P)\times k^\times$ satisfying (2.2.1). This follows by taking transpose-inverses in (2.2.2) with (2.1.3) in mind. Our main interest is, actually, the case of *non-singular alternating forms*. In this case P is of *even* rank, as verified at once by localization, and abusing the standard notation $\mathrm{GSp}(\Phi)$, or $\mathbf{GSp}(\Phi)$ in the case considered scheme-theoretically, seems natural for the similitude group. There exists neither difficulty nor speciality of the case in proving that $\mathbf{GSp}(\Phi)$ is an affine finitely presented k-group scheme. We shall proceed to details in order to prove the following result.

Proposition 2.3. *The k-group $\mathbf{GSp}(\Phi)$ is smooth with connected fibers.*

Connectedness follows by an adaptation of the classical result [6, Proposition 4, p. 10] on generators. Namely, k being harmlessly supposed to be an algebraically closed field, the problem reduces to finding an irreducible subset $X \subset \mathrm{GSp}(\Phi)$ of generators containing the unit element, and this is done by taking all non-zero scalar multiples of all symplectic transvections. In order to proceed further, it suffices to find a smooth open neighborhood of the unit section, to be ultimately called the *big cell* (the same argument as employed in [7, Section 3.5]). The case of rank two and Φ given by the exterior product is trivial, because the formula $\bigwedge^2(g) = \det(g)$ shows at once that $\mathbf{GSp}(\Phi)$ equals the whole $\mathbf{GL}(P)$ with det the similitude character. Incidentally, we notice another example given by $P := k \oplus L$ equipped with the form $\Phi_L(\xi \oplus \omega, \xi' \oplus \omega') := \xi\omega' - \xi'\omega$, to be called the *fundamental form* as being naturally identified with the wedge product $P \times P \to \bigwedge^2(P)$.

As for the general case, changing notation we let $\Psi\colon Q \times Q \to L$ denote any non-singular alternating form on a projective module of rank ≥ 4. Moreover, as far as smoothness of $\mathbf{GSp}(\Psi)$ is concerned, it is actually harmless by descent to make an additional supposition that *Q has a direct factor isomorphic to L.* In this case, we claim that *Q has in fact a direct factor isomorphic to $k \oplus L$ on which is induced the fundamental form Φ_L from Ψ.* Indeed, let $e_1\colon L \to Q$ be an injection admitting a retraction $u\colon Q \to L$ and let $e_2 \in Q$ be an element such that $u = s_\Psi(e_2)$, which exists since Ψ is non-singular. From the surjectivity of u, together with the observation at the beginning of Section 2.2, it follows that e_2 is unimodular. Hence $k \cdot e_2$ is a direct factor, contained in $\ker(u)$, Ψ being alternating, while by construction $e_1(L) \cong L$ is supplemental to $\ker(u)$. So the map $\xi \oplus \omega \mapsto \xi e_2 + e_1(\omega)$ gives rise to a direct factor $k \oplus L \subset Q$, which clearly has the wanted property. This being an adaptation of the proof employed in [9, Chapter I, Theorem 4.1.1] for the usual case of scalar-valued forms, it is in fact possible along the same line as [9, pp. 16–17] to conclude furthermore that *Q decomposes into the direct sum of $k \oplus L$ with its Ψ-orthogonal supplement, say P, and that Ψ induces a non-singular form $\Phi\colon P \times P \to L$ on P.*

In order to complete the proof of Proposition 2.3, therefore, we may apply an induction which reduces the problem to constructing a smooth open neighborhood of the unit section of $\mathbf{GSp}(\Psi)$ with the aid of $\mathbf{GSp}(\Phi)$, the latter being harmlessly supposed smooth. This will be done in the following way.

2.4. The setup

Always L denotes an invertible module, and P a projective module of even rank with a non-singular alternating form $\Phi\colon P\times P\to L$. We denote by Q the direct sum

$$Q:=k\oplus P\oplus L, \tag{2.4.1}$$

and by Ψ the form $Q\times Q\to L$ identified, under the obvious switch, with the orthogonal sum $\Phi_L\perp\Phi$, that is,

$$\Psi(\xi\oplus x\oplus\omega,\xi'\oplus x'\oplus\omega'):=\xi\omega'-\xi'\omega+\Phi(x,x'). \tag{2.4.2}$$

Since both factors Φ_L and Φ are non-singular and alternating, so is Ψ. According to the decomposition (2.4.1), elements of $\mathrm{End}(Q)$ are to be represented as 3×3 matrices of type

$$\begin{pmatrix} k & P^* & L^* \\ P & \mathrm{End}(P) & \mathrm{Hom}(L,P) \\ L & \mathrm{Hom}(P,L) & k \end{pmatrix} \tag{2.4.3}$$

acting from the left. In fact, $\mathrm{Hom}(P,L)$-entries are well described by P through the isomorphisms s_Φ and d_Φ associated with Φ, and so are $\mathrm{Hom}(L,P)$-entries by P^* through those s_{Φ^*} and d_{Φ^*} associated with the inverse from Φ^*, provided we use the transposition isomorphism $\mathrm{Hom}(P^*,L^*)\cong\mathrm{Hom}(L,P)$. Among the composition rules for the entries, we call attention here to the relations

$$f\circ s_{\Phi^*}(f')^*=\Phi^*(f',f)=-\Phi^*(f,f'), \tag{2.4.4}$$

$$s_\Phi(x)\circ s_{\Phi^*}(f)^*=-\langle x,f\rangle \tag{2.4.5}$$

in f, $f\in P^*$, $x\in P$, which are proved by using the alternating property. Indeed, from this together with the relation $f\circ v^*=v(f)\in L^*$ in $v\in\mathrm{Hom}(P^*,L^*)$ follows (2.4.4), and since $s_\Phi(x)\circ s_{\Phi^*}(f)^*\cdot\omega$ may be equated to $-\Phi(s_{\Phi^*}(f)^*\omega,x)$, for any $\omega\in L$, (2.4.5) is a consequence of (2.1.4).

2.5. Embedded subgroups

The bilinear form $\Phi\colon P\times P\to L$ gives rise to an extension of P by L as additive groups, which is $P\oplus L$ with the multiplication $(x\oplus\omega)\bullet(x'\oplus\omega'):=(x+x')\oplus(\omega+\omega'-\Phi(x,x'))$, and similarly for Φ^* endowing $P^*\oplus L^*$ with $(f\oplus\tau)\bullet(f'\oplus\tau'):=(f+f')\oplus(\tau+\tau'-\Phi^*(f,f'))$, and these constructions carry naturally over to vector bundles so that both $\mathbf{W}(P\oplus L)$ and $\mathbf{W}(P^*\oplus L^*)$ are now treated as *unipotent* k-group schemes. This is to be explained by the fact, soon verified using (2.1.1) and (2.4.4), that as such they are embedded into $\mathbf{GL}(Q)$ by the homomorphisms

$$\mathbf{W}(P\oplus L) \xrightarrow{X_+} \mathbf{GL}(Q) \xrightarrow{X_-} \mathbf{W}(P^*\oplus L^*), \tag{2.5.1}$$

$$X_+(x\oplus\omega) := \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ \omega & -s_\Phi(x) & 1 \end{pmatrix}, \quad X_-(f\oplus\tau) := \begin{pmatrix} 1 & f & \tau \\ 0 & 1 & s_{\Phi^*}(f)^* \\ 0 & 0 & 1 \end{pmatrix}.$$

Furthermore, there is no difficulty in proving that $X_+(x\oplus\omega)$ stabilizes Ψ, and actually so does $X_-(f\oplus\tau)$ provided one has (2.1.4) at hand with the fact that $\bigwedge^2(L)=0$ in mind. Thus $X_\pm$ are in fact factoring through the symplectic group of Φ, *a fortiori*, through $\mathbf{GSp}(\Psi)$. On the other hand, letting λ (resp. μ) denote the similitude character of $\mathbf{GSp}(\Phi)$ (resp. $\mathbf{GSp}(\Psi)$), we embed $\mathbf{G_m}_k\times\mathbf{GSp}(\Phi)$ into $\mathbf{GSp}(\Psi)$ by

$$X_0\colon \mathbf{G_m}_k\times\mathbf{GSp}(\Phi) \longrightarrow \mathbf{GL}(Q), \quad X_0(t,h) := \begin{pmatrix} t & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & t^{-1}\lambda(h) \end{pmatrix}, \tag{2.5.2}$$

and observe that $\mu\circ X_0=\lambda\circ\mathrm{pr}_2$. Moreover, so embedded $\mathbf{G_m}_k\times\mathbf{GSp}(\Phi)$ clearly contains the center $\mathbf{G_m}_k\subset\mathbf{GL}(Q)$, and once the commutation relations $s_\Phi(x)\circ h^{-1}=\lambda(h)^{-1}s_\Phi(h\cdot x)$ and $h\circ s_{\Phi^*}(f)^*=\lambda(h)s_{\Phi^*}(h^{*-1}\cdot f)$ have been noticed from (2.2.1) and (2.2.4) it becomes straightforward to see that X_0 normalizes $X_\pm$ with the actual left actions

$$\begin{aligned} \mathrm{Int}(X_0(t,h))\cdot X_+(x\oplus\omega) &= X_+((t^{-1}h\cdot x)\oplus(t^{-2}\lambda(h)\omega)), \\ \mathrm{Int}(X_0(t,h))\cdot X_-(f\oplus\tau) &= X_-((th^{*-1}\cdot f)\oplus(t^2\lambda(h)^{-1}\tau)). \end{aligned} \tag{2.5.3}$$

It is our purpose to prove that the triplet (X_-,X_0,X_+) gives a coordinate system of a big cell of $\mathbf{GSp}(\Psi)$; a precise statement being Proposition 2.6 below, which will imply the smoothness of $\mathbf{GSp}(\Psi)$ from that of $\mathbf{GSp}(\Phi)$ and thus complete the induction employed in (2.3).

Proposition 2.6. *The morphism*

$$\begin{aligned} &\xi\colon \mathbf{W}(P^*\oplus L^*)\times(\mathbf{G_m}_k\times\mathbf{GSp}(\Phi))\times\mathbf{W}(P\oplus L) \longrightarrow \mathbf{GSp}(\Psi), \\ &\xi(f\oplus\tau,(t,h),x\oplus\omega) := X_-(f\oplus\tau)X_0(t,h)X_+(x\oplus\omega), \end{aligned} \tag{2.6.1}$$

is an open embedding with the principal open subscheme $\Omega\subset\mathbf{GSp}(\Psi)$ defined by the condition that the $(3,3)$*-entry of the matrix* (2.4.3) *be invertible as image.*

Proof. The actual calculation of the product $X_-(f\oplus\tau)X_0(t,h)X_+(x\oplus\omega)$ shows at once that it has the invertible scalar $t^{-1}\lambda(h)$ as $(3,3)$-entry, and is of the form $X_0(t',h')$ if and only if f, τ, x and ω are all zero, in which case $t'=t$ and $h'=h$. Thus

ξ is a monomorphism with the image $\mathrm{im}(\xi)$ contained in Ω, and our problem reduces to proving that *any point g of Ω belongs to* $\mathrm{im}(\xi)$. Without loss of generality we may suppose g to be a k-valued point and, moreover, to have 1 as the $(3,3)$-entry, since X_0 admits scaling and normalizes $X_\pm$, cf. (2.5.3). So we let g be described as

$$g =: \begin{pmatrix} * & f' & \tau \\ * & h & s_{\Phi^*}(f)^* \\ * & -s_\Phi(x) & 1 \end{pmatrix} \tag{2.6.2}$$

with $x \in P$, $f, f' \in P^*$, $h \in \mathrm{End}(P)$ and $\tau \in L^*$. Easy computation using (2.4.4) shows that $X_-(f \oplus \tau)^{-1} g$ has $(0,0,1) \in L^* \oplus \mathrm{Hom}(L,P) \oplus k$ as the third column, and $f'' := f' - f \circ h + \tau \circ s_\Phi(x) \in P^*$ as the $(1,2)$-entry; we shall prove that $f''=0$. Indeed, for any $x_0 \in P$ and $\omega_0 \in L$ with $\tilde{x}_0 := 0 \oplus x_0 \oplus 0$ and $\widetilde{\omega}_0 := 0 \oplus 0 \oplus \omega_0$ designating the elements of Q extended naturally, the pairing $\Psi(\tilde{x}_0, \widetilde{\omega}_0)$ is zero by (2.4.2), and hence so is $\Psi(g \cdot \tilde{x}_0, g \cdot \widetilde{\omega}_0)$. However

$$g \cdot \tilde{x}_0 = \langle x_0, f' \rangle \oplus h \cdot x_0 \oplus (-\Phi(x, x_0)) \quad \text{and} \quad g \cdot \widetilde{\omega}_0 = \langle \omega_0, \tau \rangle \oplus s_{\Phi^*}(f)^* \omega \oplus \omega_0$$

by (2.6.2), which combined with (2.4.2), (2.1.4), and with the fact that

$$\langle \omega_0, \tau \rangle \Phi(x, x_0) = \langle \Phi(x, x_0), \tau \rangle \omega_0 = \langle x_0, \tau \circ s_\Phi(x) \rangle \omega_0$$

as L is of rank one, soon proves that $\Psi(g \cdot \tilde{x}_0, g \cdot \widetilde{\omega}_0) = \langle x_0, f'' \rangle \omega_0$; thus $f''=0$, as expected. Now replacing g by $X_-(f \oplus \tau)^{-1} g$, which renders f, f' and τ in (2.6.2) all zero, we may and shall reset g in the form

$$g = \begin{pmatrix} t & 0 & 0 \\ x' & h & 0 \\ \omega & -s_\Phi(x) & 1 \end{pmatrix}, \tag{2.6.3}$$

where $t \in k$, $x, x' \in P$, $h \in \mathrm{End}(P)$ and $\omega \in L$. This having been done, we shall prove that *h belongs to* $\mathrm{GSp}(\Psi)$ *with the multiplier $t = \lambda(h)$ and $x' = h \cdot x$ being the image of x*. This will convert g (the anterior $X_-(f \oplus \tau)^{-1} g$) to the form $X_0(t, h) X_+(x \oplus \omega)$ and thus complete our proof. Previous notation like $\tilde{x}_0$ and $\widetilde{\omega}_0$, which designated indefinite elements of Q through those of components, being again employed and abused now for $\tilde{1} := 1 \oplus 0 \oplus 0$, we let the relation $\Psi \circ (g \times g) = \mu(g) \Psi$ act on $(\tilde{1}, \widetilde{\omega}_0)$; on account of (2.6.3) and (2.4.2), the result is $t\omega_0 = \mu(g)\omega_0$. Hence t is *invertible* and equal to $\mu(g)$. Then the same relation, rewritten now as $\Psi \circ (g \times g) = t\Psi$ and acted upon $(\tilde{x}_0, \tilde{y}_0)$ (where $y_0 \in P$ is also arbitrary), yields $\Phi \circ (h \times h) = t\Phi$ which, by the remark referred to as (2.2.3), amounts to $h \in \mathrm{GSp}(\Phi)$, $t = \lambda(h)$. This assures in particular that $t\Phi(x, x_0) = \Phi(h \cdot x, h \cdot x_0)$, and since $t\Phi(x, x_0) = \Phi(x', h \cdot x_0)$, as follows from the obvious nullity of $\Psi(g \cdot \tilde{1}, g \cdot \tilde{x}_0)$, we get the remaining relation $x' = h \cdot x$. $\square$

2.7. Proof of Theorem 1.4

We are now in position to consider the special case where P *is of rank two with* $L{:=}\bigwedge^2(P)$ *and* $\Phi\colon P\times P\to L$ *is given by the wedge product* $(x,y)\mapsto x\wedge y$. This being the case mentioned in Proposition 2.3, we have $\mathbf{GSp}(\Phi)=\mathbf{GL}(P)$, $Q=\bigwedge(P)$, and Ψ being the form defined in (1.4.1). It is also appropriate to observe here that if L^* is identified with $\bigwedge^2(P^*)$ by the pairing (1.2.1) then the inverse form $\Phi^*\colon P^*\times P^*\to L^*$ also becomes the wedge product $(f,g)\mapsto f\wedge g$. To see this, it suffices to use the formula

$$s_{\Phi^*}(f)^*\omega = -f\lrcorner\,\omega \tag{2.7.1}$$

since $\omega\mapsto f\lrcorner\,\omega$ is dual to $g\mapsto g\wedge f$, cf. Section 1.2, and this follows from (2.1.4), rewritten in the form $\Phi(s_{\Phi^*}(f)^*\omega,x)=\Phi(x,f\lrcorner\,\omega)$ by using $x\wedge(f\lrcorner\,\omega)=\langle x,f\rangle\omega$. This being said, put $\mathbf{H}{:=}\mathbf{G}_{\mathbf{m}k}\times\mathbf{GL}(P)$ and $\mathbf{U}^{\pm}{:=}\mathbf{W}(\bigwedge^2(P^{\pm})\oplus P^{\pm})$ with $P^+{:=}P$ and $P^-{:=}P^*$ for short; we shall prove that

$$Y_+(\omega,x)=X_+(x\oplus\omega),\quad Y_-(\tau,f)=X_+((-f)\oplus\tau), \tag{2.7.2}$$

$$Y_0(t,h)=X_0(t\det(h)^{-1},t\det(h)^{-1}h) \tag{2.7.3}$$

for all points $((\tau,f),(t,h),(\omega,x))$ of the k-scheme $\mathbf{U}^-\times\mathbf{H}\times\mathbf{U}^+$. Note that, in addition to the obvious isomorphisms $\mathbf{U}^{\pm}\cong\mathbf{W}(P^{\pm}\oplus L^{\pm})$ involved in (2.7.2), the map $(t,h)\mapsto(t_1,h_1)$ which we have from (2.7.3) rewritten as $Y_0(t,h)=X_0(t_1,h_1)$ is also an actual automorphism $\mathbf{H}\xrightarrow{\sim}\mathbf{G}_{\mathbf{m}k}\times\mathbf{GSp}(\Phi)$ inverse to $(t_1,h_1)\mapsto(t_1^{-1}\det(h_1),t_1^{-1}h_1)$. Therefore, (2.7.2) and (2.7.3) will prove in particular that the two big cells defined by (Y_-,Y_0,Y_+) and by (X_-,X_0,X_+) are *equal*, and hence generate the *same* group sheaves: $\mathbf{S\Gamma}(\mathbf{H}(P)\perp\langle 1\rangle)=\mathbf{GSp}(\Psi)$.

In order to prove (2.7.2) and (2.7.3), it is harmless to work with k-points and, in particular, from $x\wedge y=s_\Phi(x)(y)$ and (2.7.1) we see at once that

$$l_x=\begin{pmatrix}0&0&0\\x&0&0\\0&s_\Phi(x)&0\end{pmatrix}\quad\text{and}\quad d_f=\begin{pmatrix}0&f&0\\0&0&-s_{\Phi^*}(f)^*\\0&0&0\end{pmatrix} \tag{2.7.4}$$

in the matrix notation (2.4.3). Other matrices, e.g. $\mathfrak{e}=\mathrm{diag}(1,-1,1)$, $l_\omega=[\omega]_{31}$, the extension by zeros of ω inserted in the $(3,1)$-entry, and $d_\tau=[\tau]_{13}$ being more apparent, it is now a straightforward calculation to get (2.7.2) from the definitions $Y_+(\omega,x)=(1+l_\omega)(1+l_x\mathfrak{e})$, $Y_-(\tau,f)=(1+d_\tau)(1+d_f\mathfrak{e})$, cf. (1.7.3). Similarly (1.7.2) with $\bigwedge(h)=\mathrm{diag}(1,h,\det(h))$ makes (2.7.3) apparent, and the latter proves also that the norm character of $\mathbf{S\Gamma}(\mathbf{H}(P)\perp\langle 1\rangle)$ coincides with the similitude character of $\mathbf{GSp}(\Psi)$.

3. The rank six case

3.1. The group $\mathbf{GL^{(n)}}(P)$

Again let P be any faithfully projective module. For each integer $n \geq 1$, we denote by

$$\mathbf{GL^{(n)}}(P) := \mathbf{GL}(P) \times_{(\det, n)} \mathbf{G_m}_k \tag{3.1.1}$$

the *fiber product* relative to the determinant and the nth power, in other words, the closed subgroup scheme of $\mathbf{GL}(P) \times \mathbf{G_m}_k$ consisting of those points (g, t) such that $\det(g) = t^n$. This is a central extension of $\mathbf{GL}(P)$, in the sense that the first projection $\mathbf{GL^{(n)}}(P) \to \mathbf{GL}(P)$ is an fppf epimorphism with kernel μ_n, the group of nth roots of unity, inserted by $\zeta \mapsto (1, \zeta)$ which is central. As before, the role of big cells is central to our approach and we begin with the following result.

Proposition 3.2. *The k-group $\mathbf{GL^{(n)}}(P)$ is smooth with connected fibers.*

The proof follows the lines of that of Proposition 2.3 but requires less effort. Connectedness only needs to be checked for the set $\mathrm{GL}^{(n)}(P)$ of k-points, where k is supposed to be an algebraically closed field, and in this case, since P is a non-zero k-vector space, there exists a map $\delta\colon k^\times \to \mathrm{GL}(P)$ such that $\det(\delta(t)) = t^n$, which renders $\mathrm{GL}^{(n)}(P)$ the image of $\mathrm{SL}(P) \times k^\times$ under the surjection $\pi\colon (h, t) \mapsto (\delta(t)h, t)$; hence it is irreducible, *a fortiori*, connected. The next step is to find a smooth open neighborhood of the unit section, to be ultimately called the big cell, and so far as the smoothness is concerned we may localize to consider the following situation.

3.3. A big cell

Suppose P given as the direct sum

$$P := N \oplus L \tag{3.3.1}$$

of an arbitrary finitely generated projective module N and an arbitrary invertible module L. In this case, elements of $\mathrm{End}(P)$ being expressed as 2×2 matrices of type

$$\begin{pmatrix} \mathrm{End}(N) & \mathrm{Hom}(L, N) \\ \mathrm{Hom}(N, L) & k \end{pmatrix} \tag{3.3.2}$$

acting from the left and similarly for all scalar extensions, there are associated the following homomorphisms of k-group schemes:

$$\mathbf{W}(P\oplus\mathrm{Hom}(N,L))\xrightarrow{T_+}\mathbf{GL}^{(\mathbf{n})}(P)\xrightarrow{T_-}\mathbf{W}(\mathrm{Hom}(L,N)),\tag{3.3.3}$$

$$T_+(u):=\left(\begin{pmatrix}1&0\\u&1\end{pmatrix},1\right),\quad T_-(v):=\left(\begin{pmatrix}1&v\\0&1\end{pmatrix},1\right),$$

$$T_0\colon \mathbf{GL}(N)\times\mathbf{G}_{\mathbf{m}k}\longrightarrow\mathbf{GL}^{(\mathbf{n})}(P),\quad T_0(h,t):=\left(\begin{pmatrix}h&0\\0&t^n\det(h)^{-1}\end{pmatrix},t\right).\tag{3.3.4}$$

Apparently they are all monomorphic. In fact, it is also clear that T_0 identifies the closed subgroup scheme of $\mathbf{GL}^{(\mathbf{n})}(P)$ consisting of points (g,t) with the 'matrix part' g being *diagonal*, and this combined with the calculation of the product $T_-(v)T_0(h,t)T_+(u)=:\tau(v,(h,t),u)$ shows at once that the so defined τ is an *open embedding* with the principal open subscheme of $\mathbf{GL}^{(\mathbf{n})}(P)$ defined by the invertibility of the $(2,2)$-entry of the matrix part as image. Therefore, $\mathbf{GL}^{(\mathbf{n})}(P)$ is smooth with $\mathrm{im}(\tau)$ being a big cell isomorphic to the product of sources of the triplet (T_-,T_0,T_+). This completes the proof of Proposition 3.2, and we are now in position to consider the original situation of Theorem 1.5.

3.4. The construction of the inverse $\varkappa^{-1}$

In the following, N is supposed to be of *rank three* and L denotes the invertible module $\bigwedge^3(N)$. Notice that the previous $P=N\oplus L$ is then the space $\bigwedge^-(N)$ of 'odd spinors' for the hyperbolic module $\mathbf{H}(N)$. In order to describe the even space $\bigwedge^+(N)=k\oplus\bigwedge^2(N)$ in terms of $\bigwedge^-(N)$, we shall make use of the linear map

$$u\longmapsto\hat{u}\colon\ \mathrm{Hom}(N,L)\cong L\otimes_k N^*\longrightarrow\textstyle\bigwedge^2(N),\quad \widehat{(\omega\otimes f)}:=f\lrcorner\,\omega,\tag{3.4.1}$$

which, apart from local-global transition, is in fact a special case of well-known isomorphisms attached to exterior algebras (cf. [3, Section 11.11]). Actually, tensoring L to the direct sum $L^*\oplus N^*$ considered obviously dual to $\bigwedge^-(N)$, we set the *negative* of (3.4.1), together with the natural $L\otimes_k L^*\xrightarrow{\sim}k$, constituting an isomorphism

$$L\otimes_k(\textstyle\bigwedge^-(N))^*\xrightarrow{\ \sim\ }\textstyle\bigwedge^+(N)\tag{3.4.2}$$

to be treated as the identification; the sign minus being needed for balancing some calculations, cf. Section 3.8. Accordingly, the assignment $(g,t)\mapsto t\otimes g^{*-1}$ is considered to be a homomorphism from $\mathrm{GL}(\bigwedge^-(N))\times k^\times$ to $\mathrm{GL}(\bigwedge^+(N))$. In fact, considering functorially and singling the component g out, we shall pay attention to the homomorphism

$$\iota\colon \mathbf{GL}(\textstyle\bigwedge^-(N))\times\mathbf{G_m}_k \longrightarrow \mathbf{GL}(\textstyle\bigwedge^+(N))\times\mathbf{GL}(\textstyle\bigwedge^-(N)), \qquad \iota(g,t):=(t\otimes g^{*-1},g), \tag{3.4.3}$$

of k-group schemes, which is clearly monomorphic and has $\mathbf{GL^{(2)}}(\bigwedge^-(N))$ (resp. $\mathbf{S\Gamma}(\mathbf{H}(N))$) as a subgroup of the source (resp. the target). Our ultimate goal is to prove that *between these subgroups* ι *induces an isomorphism* $\mathbf{GL^{(2)}}(\bigwedge^-(N))\xrightarrow{\sim}\mathbf{S\Gamma}(\mathbf{H}(N))$ *whose inverse is what Theorem* 1.5 *states to exist* (Proposition 3.6). This will be done by working with big cells, and at present we continue to fix some more isomorphisms.

3.5. The 'hat' isomorphisms

Following (1.2.1) we consider $\bigwedge^3(N^*)$ dual to $\bigwedge^3(N)=L$ and settle an analogous isomorphism

$$v\longmapsto\hat{v}\colon\ \mathrm{Hom}(L,N)\cong N\otimes_k L^*\longrightarrow\textstyle\bigwedge^2(N^*),\quad \widehat{(x\otimes\tau)}:=x\lrcorner\,\tau, \tag{3.5.1}$$

to (3.4.1). Both (3.4.1) and (3.5.1) are now to be used as identifications of sources of the homomorphisms $T_\pm$ (3.3.3) and $Y_\pm$ (1.7.1). As for T_0 and Y_0, we distinguish the source $\mathbf{GL}(N)\times\mathbf{G_m}_k$ of T_0 from the source $\mathbf{G_m}_k\times\mathbf{GL}(N)$ of Y_0, cf. (3.3.4) and (1.7.2), and call attention to the homomorphism

$$\hat{\ }\colon\ \mathbf{GL}(N)\times\mathbf{G_m}_k\longrightarrow\mathbf{G_m}_k\times\mathbf{GL}(N),\qquad \widehat{(h,t)}:=(t^2\det(h)^{-1},t\det(h)^{-1}h), \tag{3.5.2}$$

which, N being of rank three, is in fact an 'involution' in the sense that actually the same formula $(t_1,h_1)\mapsto(t_1\det(h_1)^{-1}h_1,t_1^2\det(h_1)^{-1})$ gives the inverse. It is our purpose to have these isomorphisms all subsumed under ι. Namely, we shall prove the following result.

Proposition 3.6. *One has*

$$\iota(T_+(u))=Y_+(\hat{u})\quad\text{and}\quad \iota(T_-(v))=Y_-(\hat{v}), \tag{3.6.1}$$

$$\iota(T_0(h,t))=Y_0(\widehat{(h,t)}). \tag{3.6.2}$$

In particular, every point $s=(s_+,s_-)$ *of* $\mathbf{S\Gamma}(\mathbf{H}(N))$ *has components, relative to the half-spin representations in* $\bigwedge^\pm(N)$, *related by* $s_+=\mu(s)\otimes(s_-)^{*-1}$ *with* $\mu(s)$ *being the norm. Moreover one has* $\det(s_+)=\det(s_-)=\mu(s)^2$.

Note that the last two statements on $s = (s_+, s_-)$ are immediate consequences of (3.6.1) and (3.6.2), since their right-hand sides generate $\mathbf{S\Gamma}(\mathbf{H}(N))$ as an fppf group sheaf. Also since the second of them, $\det(s_+) = \det(s_-) = \mu(s)^2$, amounts to the commutativity of the diagram (1.5.1) and the first to $\iota \circ \varkappa = \mathrm{Id}$ for the $\varkappa$ introduced in (1.5.2), they are actually sufficient for proving the statement at the end of Section 3.4, *a fortori*, Theorem 1.5. Therefore, we have only to prove (3.6.1) and (3.6.2). Furthermore, to do so it is harmless to restrict ourselves to k-points. We notice also that the identification (3.4.2) is to be now involved intimately. To avoid confusion, besides (3.3.2) we shall employ for elements of $\mathrm{End}(\bigwedge^+(N))$ the matrix notation of type

$$\begin{bmatrix} k & \bigwedge^2(N^*) \\ \bigwedge^2(N) & \mathrm{End}(\bigwedge^2(N)) \end{bmatrix} \tag{3.6.3}$$

acting on $\bigwedge^+(N) = k \oplus \bigwedge^2(N)$ from the left. Though the outline is the same, we begin with the proof of (3.6.2) which is more straightforward.

Easy computation using the definitions (1.7.2) and (3.5.2) describes $Y_0(\widehat{(h,t)})$ as the pair (α, β) of matrices

$$\alpha := \begin{bmatrix} t^{-1}\det(h) & 0 \\ 0 & t\det(h)^{-1}\bigwedge^2(h) \end{bmatrix} \quad \text{and} \quad \beta := \begin{pmatrix} h & 0 \\ 0 & t^2\det(h)^{-1} \end{pmatrix}, \tag{3.6.4}$$

which compared to (3.4.3), together with (3.3.4), reduces the proof of (3.6.2) to that of $\alpha = t \otimes \beta^{*-1}$. Now taking $\omega_0 \in L$, $f_0 \in N^*$ and $\tau_0 \in L^*$ arbitrary we let

$$Z_0 := \omega_0 \otimes (f_0, \tau_0) \in L \otimes_k (\textstyle\bigwedge^-(N))^* = L \otimes_k (N^* \oplus L^*) \tag{3.6.5}$$

which, under (3.4.2), corresponds to

$$Z_0^+ := \langle \omega_0, \tau_0 \rangle \oplus (-f_0 \lrcorner\, \omega_0) \in \textstyle\bigwedge^+(N) = k \oplus \bigwedge^2(N). \tag{3.6.6}$$

The problem being then to have $(t \otimes \beta^{*-1}) \cdot Z_0$ corresponding to $\alpha \cdot Z_0^+$, we are indeed done by deducing $(t \otimes \beta^{*-1}) \cdot Z_0 = t\omega_0 \otimes (h^* \cdot f_0, t^{-2}\det(h)\tau_0)$ from (3.6.4) and by bearing the obvious formula $(h^{*-1} \cdot f_0) \lrcorner\, \omega_0 = \det(h)^{-1} \bigwedge^2(h) \cdot (f_0 \lrcorner\, \omega_0)$ in mind. In order to proceed along the same lines for proving (3.6.1), we need some addenda to the constructions (3.4.1) and (3.5.1).

3.7. Basic identities

Let $x \in N$, $f \in N^*$, $\omega \in L$ and $\tau \in L^*$ be arbitrary. The formula $x \wedge (f \lrcorner\, \omega) = \langle x, f \rangle \omega$ read in the notation of (3.4.1) implies that any $u \in \mathrm{Hom}(N, L)$ takes value $x \wedge \hat{u}$ at x, namely

$$\hat{u} \wedge x = u(x). \tag{3.7.1}$$

The similar formula $f\wedge(x\lrcorner\tau)=\langle x,f\rangle\tau$ acted upon the coupling with ω equates $\langle(x\lrcorner\tau)\lrcorner\omega,f\rangle$ to $\langle(x\otimes\tau)(\omega),f\rangle$, in which $x\otimes\tau$ is clearly replaceable by a general $v\in\mathrm{Hom}(L,N)$. Thus, in the notation of (3.5.1),

$$\hat{v}\lrcorner\omega=v(\omega). \tag{3.7.2}$$

Note that (3.7.2) again coupled with f gives $\langle\omega,f\wedge\hat{v}\rangle=\langle v(\omega),f\rangle$ also, which we shall record in the form

$$\langle f\lrcorner\omega,\hat{v}\rangle=\langle\omega,v^*(f)\rangle. \tag{3.7.3}$$

As a similar formula, relating u^* with $\hat{u}$, we shall prove

$$u^*(\tau)\lrcorner\omega=\langle\omega,\tau\rangle\hat{u}. \tag{3.7.4}$$

Indeed, introducing another $\omega'\in L$ we may harmlessly suppose that $u=\omega'\otimes f$, in which case $u^*(\tau)\lrcorner\omega$ equals $\langle\omega',\tau\rangle f\lrcorner\omega$, while since $\langle\omega',\tau\rangle\omega=\langle\omega,\tau\rangle\omega'$, L being of rank one, this amounts to the wanted relation, cf. (3.4.1).

3.8. Proof of (3.6.1)

Note that $Y_+(\hat{u})$ and $Y_-(\hat{v})$ are now the transformations $Z\mapsto Z+\hat{u}\wedge Z$ and $Z\mapsto Z+\hat{v}\lrcorner Z$, respectively. Both actions are apparent on $\bigwedge^+(N)$ and as for those on $\bigwedge^-(N)$, we use (3.7.1) and (3.7.2) to conclude that $Y_+(\hat{u})=(\alpha_1,\beta_1)$, $Y_-(\hat{v})=(\alpha_2,\beta_2)$, where

$$\alpha_1:=\begin{bmatrix}1 & 0\\ \hat{u} & 1\end{bmatrix},\quad \beta_1:=\begin{pmatrix}1 & 0\\ u & 1\end{pmatrix},\quad \alpha_2:=\begin{bmatrix}1 & \hat{v}\\ 0 & 1\end{bmatrix}\quad\text{and}\quad \beta_2:=\begin{pmatrix}1 & v\\ 0 & 1\end{pmatrix}.$$

Hence, by (3.4.3) and (3.3.3), we are reduced to proving that $\alpha_i=1\otimes\beta_i^*$ for $i=1,2$. Now, in the notations of (3.6.5) and (3.6.6), we have

$$\begin{cases}(1\otimes\beta_1^{*-1})\cdot Z_0=\omega_0\otimes(f_0-u^*(\tau_0),\tau_0),\\ \qquad\qquad \alpha_1\cdot Z_0'=\langle\omega_0,\tau_0\rangle\oplus(\langle\omega_0,\tau_0\rangle\hat{u}-f_0\lrcorner\omega_0),\end{cases}$$
$$\begin{cases}(1\otimes\beta_2^{*-1})\cdot Z_0=\omega_0\otimes(f_0,-v^*(f_0)+\tau_0),\\ \qquad\qquad \alpha_2\cdot Z_0'=(\langle\omega_0,\tau_0\rangle-\langle f_0\lrcorner\omega_0,\hat{v}\rangle)\oplus(-f_0\lrcorner\omega_0),\end{cases}$$

and due to the sign convention made in (3.4.2), as well as to the previous identities (3.7.3) and (3.7.4), both pairs are indeed identified, q.e.d.

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