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The chamber basis of the Orlik–Solomon algebra and Aomoto complex

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Abstract. We introduce a basis of the Orlik–Solomon algebra labeled by chambers, the so called chamber basis. We consider structure constants of the Orlik–Solomon algebra with respect to the chamber basis and prove that these structure constants recover D. Cohen's minimal complex from the Aomoto complex.

1. Introduction

Let $\mathcal{A}=\{H_1,...,H_n\}$ be an affine hyperplane arrangement in the real vector space $\mathbb{R}^l$. Choose for each $H\in\mathcal{A}$ an affine linear form α_H with $H=\alpha_H^{-1}(0)$. Denote by $\mathsf{ch}(\mathcal{A})$ the set of all chambers and by $\mathsf{M}(\mathcal{A})=\mathbb{C}^l\setminus\bigcup_{H\in\mathcal{A}}H\otimes\mathbb{C}$ the complement to the complexified hyperplanes.

The set $\mathsf{ch}(\mathcal{A})$ of chambers has been known to carry information about the topology of $\mathsf{M}(\mathcal{A})$. For example $|\mathsf{ch}(\mathcal{A})|=\sum_{j=0}^{l} b_j(\mathsf{M}(\mathcal{A}))$, [22], the homotopy type of $\mathsf{M}(\mathcal{A})$ can be obtained from the face poset [15], and [9] uses bounded chambers to construct a basis of the local system cohomology group. The relation between the Orlik–Solomon algebra and the ring $\mathbb{Z}[\mathsf{ch}(\mathcal{A})]$ of $\mathbb{Z}$-valued functions over the set of chambers was studied in [19] (see also [4], [5] and [8]). References [21] and [16] considered the relation between structures of chambers and minimal CW-decomposition. We will pursue these topological interpretations of chambers in the context of rank-one local system cohomology groups.

Let $\lambda_H\in\mathbb{C}$ be complex weights. A rank-one local system $\mathcal{L}$ on $\mathsf{M}(\mathcal{A})$ is defined with monodromy $\exp(2\pi i\lambda_H)$ around the hyperplane H. Let $\omega_H=d\alpha_H/2\pi i\alpha_H$, $A^\bullet=H^\bullet(\mathsf{M}(\mathcal{A}),\mathbb{C})$ and

$$\omega_\lambda=\sum_{H\in\mathcal{A}}\lambda_H\omega_H.$$

Under some genericity conditions on the weights λ_H, Esnault–Schechtman–Viehweg [7] (and [17]) proved that the Aomoto complex $(A^\bullet, 2\pi i\omega_\lambda\wedge)$ is quasi-isomorphic to the de Rham complex with coefficients in $\mathcal{L}$. In particular, if the weights λ_H are sufficiently small, e.g. $|\lambda_H|<1/2(n+1)$ for all $H\in\mathcal{A}$, then

$$H^p(\mathsf{M}(\mathcal{A}),\mathcal{L})\cong H^p(A^\bullet, 2\pi i\omega_\lambda\wedge).$$

Thus the local system cohomology group $H^\bullet(\mathsf{M}(\mathcal{A}),\mathcal{L})$ can be calculated from the Aomoto complex if $\mathcal{L}$ is close to the trivial one (the tangent-cone theorem). However, in general, these two cohomology groups have different dimensions [3] and [18]. The Aomoto complex does not compute $H^\bullet(\mathsf{M}(\mathcal{A}),\mathcal{L})$ directly. This suggests the problem whether the Aomoto complex $(A^\bullet, 2\pi i\omega_\lambda\wedge)$ can recover the local system cohomology group $H^\bullet(\mathsf{M}(\mathcal{A}),\mathcal{L})$.

Another interpretation of the Aomoto complex is related to the minimality of $\mathsf{M}(\mathcal{A})$, [1], [6], [12] and [14]. In [1], Cohen constructed a complex $(K^\bullet(\mathcal{A}),\Delta^\bullet(\lambda))$ which computes $H^\bullet(\mathsf{M}(\mathcal{A}),\mathcal{L})$ and the terms of this complex have Betti numbers as their dimensions, that is, satisfying the minimality: $\dim K^p=b_p(\mathsf{M}(\mathcal{A}))$. When $\mathcal{L}$ is trivial, the minimality implies that all the coboundary maps vanish, $\Delta^p(0)=0$. However the coboundary $\Delta(\lambda)$ of the minimal complex is difficult to compute for $\lambda\neq 0$. Cohen–Orlik [2] determined the first order approximation of $\Delta(\lambda)$. They proved that the linearization of the minimal complex is chain equivalent to the Aomoto complex,

$$\left(K^\bullet(\mathcal{A}), \left.\frac{d}{dt}\right|_{t=0}\Delta^\bullet(t\lambda)\right)\cong (A^\bullet, 2\pi i\omega_\lambda\wedge).$$

The purpose of this paper is to study an "integration" of the Aomoto complex for obtaining the minimal complex for a real arrangement $\mathcal{A}$. To do this, we introduce a basis of the Orlik–Solomon algebra $A^\bullet$, the so called "chamber basis", which is depending on a fixed generic flag with orientations. Then we have a matrix expression of the linear map $2\pi i\omega_\lambda\wedge\colon A^\bullet\to A^{\bullet+1}$. We will prove that the minimal complex can be recovered from these matrix entries. Roughly speaking, it is done just by replacing each matrix entry with its value of the hyperbolic sine function.

The proof is based on the constructions in [21]. In the previous paper [21], we explicitly constructed the attaching maps of cells arising from the Lefschetz hyperplane section theorem for a real arrangement $\mathcal{A}$. Moreover we also obtained a description of the minimal complex in terms of chambers. However the formula of the boundary map in [21] contains an integer $\deg(C',C)$ which is difficult to compute (see also Remark 4.2). The situation is changed in this paper. We will prove that the integer $\deg(C',C)$ appears as a "structure constant" of the Orlik–Solomon algebra with respect to the chamber basis. Moreover we also give an algorithm relating the chamber basis and the classical generator of the Orlik–Solomon algebra, which is a more combinatorics friendly object than the original definition of $\deg(C', C)$.

The paper is organized as follows. In Section 2, we recall some basic facts on the topology of $\mathsf{M}(\mathcal{A})$ and constructions from [21]. In particular, using a generic flag $\mathcal{F}$, we divide the set of chambers into a disjoint union $\mathsf{ch}(\mathcal{A})=\bigsqcup_{q=0}^{l} \mathsf{ch}^q_{\mathcal{F}}(\mathcal{A})$ such that $|\mathsf{ch}^q_{\mathcal{F}}(\mathcal{A})|=b_q(\mathsf{M}(\mathcal{A}))$ and construct an isomorphism $\nu^q\colon \mathbb{Z}[\mathsf{ch}^q_{\mathcal{F}}(\mathcal{A})]\xrightarrow{\cong} H^q(\mathsf{M}(\mathcal{A}),\mathbb{Z})$. This leads us to introduce the notion of chamber basis $\{\nu^q(C)\,|\,C\in\mathsf{ch}^q(\mathcal{A})\}$ of A^q. In Section 3, we will construct the inverse map $\xi^q=(\nu^q)^{-1}\colon H^q(\mathsf{M},\mathbb{Z})\to\mathbb{Z}[\mathsf{ch}^q_{\mathcal{F}}(\mathcal{A})]$ which enables us to express $\nu^q(C)\in H^q(\mathsf{M}(\mathcal{A}))$ in terms of the differential forms ω_H. The wedge product $\omega_\lambda\wedge\nu^q(C)$ can be uniquely expressed as

$$\sum_{C'\in\mathsf{ch}^{q+1}} \Gamma_{C,C'}(\lambda)\nu^{q+1}(C')$$

for some coefficients $\Gamma_{C,C'}(\lambda)\in\mathbb{C}$. In Section 4, two main results are stated and proved. First we assert that the coefficient $\Gamma_{C,C'}(\lambda)$ has a decomposition as a product of a linear form of weights λ_H and an integer. Furthermore the linear factor of the weights is explicitly described by using the notion of separating hyperplanes. The other integral factor is essentially equivalent to $\deg(C', C)$ mentioned above. The second result is recovering the minimal complex from these coefficients using the hyperbolic sine function. In the appendix, Section 5, a generalized version of the linearization theorem for a minimal CW-complex is proved.

2. Preliminary

2.1. Basic constructions

Let V be an l-dimensional vector space. A finite set of affine hyperplanes $\mathcal{A}=\{H_1, ..., H_n\}$ is called a *hyperplane arrangement*. For each hyperplane H_j we fix a defining equation α_j such that $H_j=\alpha_j^{-1}(0)$. Let $L(\mathcal{A})$ be the set of nonempty intersections of elements of $\mathcal{A}$. Define a partial order on $L(\mathcal{A})$ by $X\le Y\Leftrightarrow Y\subseteq X$ for $X, Y\in L(\mathcal{A})$. Note that this is reverse inclusion.

Define a *rank function* on $L(\mathcal{A})$ by $r(X)=\operatorname{codim} X$. Write $L^p(\mathcal{A})=\{X\in L(\mathcal{A})\,|\, r(X)=p\}$. We call $\mathcal{A}$ *essential* if $L^l(\mathcal{A})\neq\varnothing$.

Let $\mu\colon L(\mathcal{A})\to\mathbb{Z}$ be the *Möbius function* of $L(\mathcal{A})$ defined by

$$\mu(X)=\begin{cases} 1 & \text{for } X=V, \\ -\sum_{Y<X}\mu(Y) & \text{for } X>V. \end{cases}$$

The *Poincaré polynomial* of $\mathcal{A}$ is $\pi(\mathcal{A},t)=\sum_{X\in L(\mathcal{A})}\mu(X)(-t)^{r(X)}$ and we also define the numbers $b_j(\mathcal{A})$ by the formula

$$\pi(\mathcal{A},t)=\sum_{j=0}^{l} b_j(\mathcal{A})t^j.$$

We also define the *β-invariant* $\beta(\mathcal{A})$ by

$$\beta(\mathcal{A})=|\pi(\mathcal{A},-1)|,$$

if $\mathcal{A}$ is an essential arrangement, the sign can be precisely enumerated as $\beta(\mathcal{A})=(-1)^l\pi(\mathcal{A},-1)$. See [11] for details.

2.2. Classical results

Let $\mathcal{A}$ be an arrangement in a real vector space $V_{\mathbb{R}}$. Then the following relations between the set $\mathsf{ch}(\mathcal{A})$ of chambers and the complexified complement $\mathsf{M}(\mathcal{A})$ are known.

Theorem 2.1. ([10] and [22]) (i) *Let $\mathcal{A}$ be an essential real l-arrangement. Then $|\mathsf{ch}(\mathcal{A})|=\pi(\mathcal{A},1)$, and $|\mathsf{bch}(\mathcal{A})|=(-1)^l\pi(\mathcal{A},-1)=\beta(\mathcal{A})$, where $\mathsf{bch}(\mathcal{A})$ is the set of all bounded chambers.*

(ii) *Let $\mathcal{A}$ be a complex arrangement. Then $b_j(\mathcal{A})$ is equal to the topological Betti number $b_j(\mathsf{M}(\mathcal{A}))$, that is,*

$$\mathrm{Poin}(\mathsf{M}(\mathcal{A}),t)=\pi(\mathcal{A},t).$$

In particular, the absolute value of the topological Euler characteristic $|\chi(\mathsf{M}(\mathcal{A}))|$ of the complement is equal to $\beta(\mathcal{A})$.

2.3. Generic flags and topology of $\mathsf{M}(\mathcal{A})$

Let $\mathcal{A}$ be an l-arrangement. A q-dimensional affine subspace $\mathcal{F}^q\subset V$ is called *generic* or *transversal* to $\mathcal{A}$ if $\dim\mathcal{F}^q\cap X=q-r(X)$ for $X\in L(\mathcal{A})$. A *generic flag* $\mathcal{F}$ is defined to be a complete flag (of affine subspaces) in V,

$$\mathcal{F}:\ \varnothing=\mathcal{F}^{-1}\subset\mathcal{F}^0\subset\mathcal{F}^1\subset\dots\subset\mathcal{F}^l=V,$$

where each $\mathcal{F}^q$ is a generic q-dimensional affine subspace.

For a generic subspace $\mathcal{F}^q$ we have an arrangement in $\mathcal{F}^q$,

$$\mathcal{A}\cap\mathcal{F}^q:=\{H\cap\mathcal{F}^q\mid H\in\mathcal{A}\}.$$

The genericity provides an isomorphism of posets

$$L(\mathcal{A}\cap\mathcal{F}^q)\cong L^{\le q}(\mathcal{A}):=\bigcup_{j\le q}L^j(\mathcal{A}). \tag{1}$$

In [10], Orlik and Solomon gave a presentation of the cohomology ring $H^*(\mathsf{M}(\mathcal{A}),\mathbb{Z})$ in terms of the poset $L(\mathcal{A})$ for a complex arrangement $\mathcal{A}$. The next proposition follows from (1).

Proposition 2.2. *Let $\mathcal{A}$ be a complex arrangement and $\mathcal{F}^q$ be a q-dimensional generic subspace. Then the natural inclusion $i\colon \mathsf{M}(\mathcal{A})\cap\mathcal{F}^q\hookrightarrow\mathsf{M}(\mathcal{A})$ induces isomorphisms*

$$i_k\colon H_k(\mathsf{M}(\mathcal{A})\cap\mathcal{F}^q,\mathbb{Z})\xrightarrow{\cong}H_k(\mathsf{M}(\mathcal{A}),\mathbb{Z})$$

for $k=0,1,...,q$.

In particular, the Poincaré polynomial of $\mathcal{A}\cap\mathcal{F}^q$ is given by

$$\pi(\mathcal{A}\cap\mathcal{F}^q,t)=\pi(\mathcal{A},t)^{\leq q}, \tag{2}$$

where $(\sum_{j\geq 0}a_jt^j)^{\leq q}=\sum_{j=0}^{q}a_jt^j$ is the truncated polynomial. From these formulas and Theorem 2.1, we have the following proposition. (For the proof see [21, Proposition 2.3.2] for example.)

Proposition 2.3. *Let $\mathcal{A}$ be a real l-arrangement and $\mathcal{F}$ a generic flag. Define*

$$\mathsf{ch}^q_{\mathcal{F}}(\mathcal{A})=\{C\in\mathsf{ch}(\mathcal{A})\mid C\cap\mathcal{F}^q\neq\varnothing \text{ and } C\cap\mathcal{F}^{q-1}=\varnothing\}$$

for each $q=0,1,...,l$. Then $|\mathsf{ch}^q_{\mathcal{F}}(\mathcal{A})|=b_q(\mathsf{M}(\mathcal{A}))$.

In particular, the number of chambers which does not intersect with a generic hyperplane $\mathcal{F}^{l-1}$ satisfies

$$|\mathsf{ch}^l_{\mathcal{F}}(\mathcal{A})|=b_l(\mathsf{M}(\mathcal{A})).$$

This formula has a topological meaning. Let us recall the construction in [21] briefly. First to fix an orientation, we fix a basis $(v_1,...,v_l)$ of V such that

$$\mathcal{F}^q=\mathcal{F}^0+\sum_{j=1}^{q}\mathbb{R}v_j.$$

The orientation of $\mathcal{F}^q$ is determined by the ordered basis $(v_1,...,v_q)$. Also define positive and negative half spaces, $\mathcal{F}^q_+$ and $\mathcal{F}^q_-$, by

$$\mathcal{F}^q_+=\mathcal{F}^{q-1}+\mathbb{R}_{>0}v_q,$$
$$\mathcal{F}^q_-=\mathcal{F}^{q-1}+\mathbb{R}_{<0}v_q,$$

respectively.

Definition 2.4. The map sgn: $\mathsf{ch}^q_{\mathcal{F}}(\mathcal{A})\to\{\pm 1\}$ is defined by

$$\operatorname{sgn}(C)=\begin{cases} 1, & \text{if } \mathcal{F}^q\cap C\subset\mathcal{F}^q_+,\\ -1, & \text{if } \mathcal{F}^q\cap C\subset\mathcal{F}^q_-.\end{cases}$$

Let $\mathcal{F}^q\subset\mathbb{R}^l$. Denote by $\mathcal{F}^q_{\mathbb{C}}=\mathcal{F}^q\otimes\mathbb{C}$ the complexification of $\mathcal{F}^q$ and put $\mathsf{M}^q:=\mathcal{F}^q_{\mathbb{C}}\cap\mathsf{M}(\mathcal{A})$. Note that $\mathsf{M}^l=\mathsf{M}(\mathcal{A})$. We fix the orientation of $\mathcal{F}^q_{\mathbb{C}}$ by the ordered basis

$$(v_1,\dots,v_q,iv_1,\dots,iv_q).$$

Note that this orientation is different by $(-1)^{q(q-1)/2}$ from the canonical orientation of a complex vector space.

For each $C\in\mathsf{ch}^l_{\mathcal{F}}(\mathcal{A})$, we can explicitly construct a continuous map

$$\sigma_C\colon (D^l,\partial D^l)\longrightarrow(\mathsf{M}^l,\mathsf{M}^{l-1}),$$

such that, [21, Section 5.2],

$$\begin{aligned} &\text{(Transversality)} && \sigma_C(0)\in C \quad\text{and}\quad \sigma_C(D^l)\pitchfork C=\{\sigma_C(0)\},\\ &\text{(Non-intersecting)} && \sigma_C(D^l)\cap C'=\varnothing \quad \text{for } C'\in\mathsf{ch}^l_{\mathcal{F}}(\mathcal{A})\setminus\{C\}.\end{aligned}\tag{3}$$

These properties guarantee the following homotopy equivalence [21, Theorem 4.3.1]:

$$\mathsf{M}^l\simeq\mathsf{M}^{l-1}\cup_{(\partial\sigma_C)}\Bigg(\bigsqcup_{C\in\mathsf{ch}^l_{\mathcal{F}}(\mathcal{A})}D^l\Bigg),\tag{4}$$

where the right-hand side is obtained by attaching l-dimensional disks to M^{l-1} along $\partial\sigma_C\colon\partial D^l\to\mathsf{M}^{l-1}$ for $C\in\mathsf{ch}^l_{\mathcal{F}}(\mathcal{A})$. Since the natural inclusion $\mathsf{M}^{l-1}\hookrightarrow\mathsf{M}^l$ induces $H_{l-1}(\mathsf{M}^{l-1},\mathbb{Z})\cong H_{l-1}(\mathsf{M}^l,\mathbb{Z})$ and $H_l(\mathsf{M}^l,\mathbb{Z})\cong H_l(\mathsf{M}^l,\mathsf{M}^{l-1},\mathbb{Z})$, $\sigma_C(D^l)$ can be considered as an element of $H_l(\mathsf{M}^l,\mathbb{Z})$. Furthermore, $\{\sigma_C(D^l)\}_{C\in\mathsf{ch}^l_{\mathcal{F}}}$ form a basis of $H_l(\mathsf{M}^l,\mathbb{Z})$. We choose an orientation of σ_C so that the intersection number satisfies

$$[C]\cdot[\sigma_C]=1.$$

Then chambers $\{[C]\}_{C\in\mathsf{ch}^l_{\mathcal{F}}}$ form the dual basis of the locally finite homology group $H^{\mathrm{lf}}_l(\mathsf{M}^l,\mathbb{Z})$, which is isomorphic to $H^l(\mathsf{M}^l,\mathbb{Z})$. Thus we have the isomorphism

$$\begin{array}{ccccccc} \mathbb{Z}[\mathsf{ch}^q_{\mathcal{F}}(\mathcal{A})] & \xrightarrow{\ \cong\ } & H^{\mathrm{lf}}_q(\mathsf{M}^q,\mathbb{Z}) & \xrightarrow{\ \cong\ } & H^q(\mathsf{M}^q,\mathbb{Z}) & \xrightarrow{\ \cong\ } & H^q(\mathsf{M}^l,\mathbb{Z})\\ C & & & & & & \nu^q(C).\end{array}$$

Denote by ν^q the composite map

$$\nu^q\colon\mathbb{Z}[\mathsf{ch}^q_{\mathcal{F}}(\mathcal{A})]\xrightarrow{\cong}H^q(\mathsf{M}^l,\mathbb{Z}).\tag{5}$$

Definition 2.5. The set $\{\nu^q(C)|C\in\mathsf{ch}^q_{\mathcal{F}}(\mathcal{A})\}$ is called the *chamber basis* of $H^q(\mathsf{M},\mathbb{Z})$ with respect to a flag $\mathcal{F}$.

Remark 2.6. In [19], Varchenko and Gel′fand constructed a filtration $0\subset P^0\subset\ldots\subset P^l=\mathbb{Z}[\mathsf{ch}(\mathcal{A})]$ and an isomorphism $P^q/P^{q-1}\cong H^{\mathrm{lf}}_{2l-q}(\mathsf{M}(\mathcal{A}),\mathbb{Z})$. Our subspace $\mathbb{Z}[\mathsf{ch}^l_{\mathcal{F}}(\mathcal{A})]$ gives a section of the quotient map $P^l\to P^l/P^{l-1}$. Using a generic section $\mathcal{F}^q$, we can construct an isomorphism $\mathbb{Z}[\mathsf{ch}^q_{\mathcal{F}}(\mathcal{A})]\cong P^q/P^{q-1}$. The map ν^q is equivalent to Varchenko–Gel′fand's isomorphism under the identification $H^{\mathrm{lf}}_{2l-q}(\mathsf{M}(\mathcal{A}),\mathbb{Z})\cong H^q(\mathsf{M}(\mathcal{A}),\mathbb{Z})$ up to sign.

3. An algorithm relating chambers and differential forms

From the result of Brieskorn, Orlik and Solomon, the cohomology ring $A^*=H^*(\mathsf{M}(\mathcal{A}),\mathbb{Z})$ is generated by $\omega_j=d\alpha_j/2\pi i\alpha_j$, $j=1,\ldots,n$. In this section, we express $\nu^q(C)$ in terms of generators ω_H.

Let $J=\{j_1,\ldots,j_l\}\subset\{1,\ldots,n\}$ be an ordered subset of l indices, $\mathcal{A}(J):=\{H_{j_1},\ldots,H_{j_l}\}$ be a subarrangement consisting of l hyperplanes. Suppose $H_{j_1},\ldots,H_{j_l}$ are independent, that is, $d\alpha_{j_1}\wedge\ldots\wedge d\alpha_{j_l}\neq 0$. Obviously $\mathsf{ch}(\mathcal{A}(J))$ consists of 2^l chambers and there exists a unique chamber $C_0(J)\in\mathsf{ch}(\mathcal{A}(J))$ with $C_0(J)\cap\mathcal{F}^{l-1}=\varnothing$. Choose a normal vector $w_{j_k}\perp H_{j_k}$ for each H_{j_k} such that $C_0(J)$ is contained in the half space $H_{j_k}+\mathbb{R}_{>0}\cdot w_{j_k}$.

Definition 3.1. For an ordered l-tuple $J=(j_1,\ldots,j_l)\subset\{1,\ldots,n\}$, define $\varepsilon(J)$ by

$$\varepsilon(J)=\begin{cases}0, & \text{if } H_{j_1},\ldots,H_{j_l} \text{ are dependent,}\\ 1, & \text{if } (w_{j_1},\ldots,w_{j_l}) \text{ is a positive basis,}\\ -1, & \text{if } (w_{j_1},\ldots,w_{j_l}) \text{ is a negative basis.}\end{cases}$$

For an ordered l-tuple $J=(j_1,\ldots,j_l)$, set $\omega_J=\omega_{j_1}\wedge\ldots\wedge\omega_{j_l}$. Let us define

$$\xi^l(\omega_J)=\varepsilon(J)[C_0(J)]\in\mathbb{Z}[\mathsf{ch}^l_{\mathcal{F}}(\mathcal{A})].$$

Theorem 3.2. *The map* ξ^l *induces the isomorphism*

$$\xi^l\colon H^l(\mathsf{M}(\mathcal{A}),\mathbb{Z})\xrightarrow{\cong}\mathbb{Z}[\mathsf{ch}^l_{\mathcal{F}}(\mathcal{A})],$$

and $\xi^l=(\nu^l)^{-1}$.

Proof. We prove that $\nu^l(\xi^l(\omega_J))=\omega_J$. It is enough to show that

$$\int_{\sigma_C}\omega_J=\varepsilon(J)[C_0(J)]\cdot\sigma_C$$

for $C \in \mathsf{ch}^l_{\mathcal{F}}(\mathcal{A})$. Note that the inclusion $\iota\colon \mathsf{M}(\mathcal{A}) \hookrightarrow \mathsf{M}(\mathcal{A}(J))$ induces a surjective map $\iota_*\colon H_l(\mathsf{M}(\mathcal{A}),\mathbb{Z}) \to H_l(\mathsf{M}(\mathcal{A}(J)),\mathbb{Z})$. Since $\mathsf{M}(\mathcal{A}(J))$ is homeomorphic to $(\mathbb{C}^*)^l$, the top homology $H_l(\mathsf{M}(\mathcal{A}(J)),\mathbb{Z})$ is generated by the cycle

$$T := \{(\alpha_{j_1},...,\alpha_{j_l}) \mid |\alpha_{j_1}| = |\alpha_{j_2}| = ... = |\alpha_{j_l}| = 1\}.$$

Fix the orientation of T so that $[C_0(J)]\cdot T = 1$. Then since $[C_0]$ is the dual basis to T, from (3), we have,

$$\iota_*(\sigma_C) = \begin{cases} T, & \text{if } C \subseteq C_0(J), \\ 0, & \text{if } C \cap C_0(J) = \varnothing, \end{cases}$$

for $C \in \mathsf{ch}^l_{\mathcal{F}}(\mathcal{A})$. Now we have $\int_{\sigma_C} \omega_J = \int_{\iota_*(\sigma_C)} \omega_J$ and $\int_T \omega_J = \varepsilon(J)$ completes the proof. □

Similarly, we can define the isomorphism $\xi^q\colon H^q(\mathsf{M}(\mathcal{A}),\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}[\mathsf{ch}^q_{\mathcal{F}}(\mathcal{A})]$ by using that $H^q(\mathsf{M}(\mathcal{A}),\mathbb{Z}) \cong H^q(\mathsf{M}(\mathcal{A}) \cap \mathcal{F}^q,\mathbb{Z})$ for $0 \le q \le l-1$. Theorem 3.2 enables us to express the chamber basis $\nu^q(C)$ in terms of the generators ω_H.

Example 3.3. Let $\mathcal{A} = \{H_1,...,H_4\}$ be the arrangement of 4-lines as in Figure 1 with flag $\mathcal{F}^\bullet$ defined by v_1 and v_2. Then $\mathsf{ch}^0_{\mathcal{F}}(\mathcal{A}) = \{A\}$, $\mathsf{ch}^1_{\mathcal{F}}(\mathcal{A}) = \{B_1,B_2,B_3,B_4\}$, and $\mathsf{ch}^2_{\mathcal{F}}(\mathcal{A}) = \{C_1,C_2,C_3,C_4,C_5\}$. From Theorem 3.2, $\xi(1) = [A]$, $\nu(A) = 1$ and

$$\begin{aligned}
\xi(\omega_1) &= -[B_1], & \nu(B_1) &= -\omega_1, \\
\xi(\omega_2) &= [B_2]+[B_3]+[B_4], & \nu(B_2) &= \omega_2-\omega_3, \\
\xi(\omega_3) &= [B_3]+[B_4], & \nu(B_3) &= \omega_3-\omega_4, \\
\xi(\omega_4) &= [B_4], & \nu(B_4) &= \omega_4, \\
\xi(\omega_{12}) &= -[C_1]-[C_3], & \nu(C_1) &= -\omega_{12}+\omega_{14}-\omega_{24}, \\
\xi(\omega_{13}) &= -[C_1]-[C_2]-[C_3]-[C_4], & \nu(C_2) &= \omega_{12}-\omega_{13}+\omega_{24}-\omega_{34}, \\
\xi(\omega_{14}) &= -[C_3]-[C_4]-[C_5], & \nu(C_3) &= -\omega_{14}+\omega_{24}, \\
\xi(\omega_{24}) &= -[C_4]-[C_5], & \nu(C_4) &= -\omega_{24}+\omega_{34}, \\
\xi(\omega_{34}) &= -[C_5], & \nu(C_5) &= -\omega_{34},
\end{aligned}$$

$$\begin{aligned}
\omega_\lambda \wedge \nu(A) &= -\lambda_1\nu(B_1) + \lambda_2\nu(B_2) + \lambda_{23}\nu(B_3) + \lambda_{234}\nu(B_4), \\
\omega_\lambda \wedge \nu(B_1) &= -\lambda_{23}\nu(C_1) - \lambda_3\nu(C_2) - \lambda_{234}\nu(C_3) - \lambda_{34}\nu(C_4) - \lambda_4\nu(C_5), \\
\omega_\lambda \wedge \nu(B_2) &= +\lambda_{123}\nu(C_2) + \lambda_{1234}\nu(C_4), \\
\omega_\lambda \wedge \nu(B_3) &= -\lambda_1\nu(C_1) - \lambda_{12}\nu(C_2) + \lambda_{1234}\nu(C_5), \\
\omega_\lambda \wedge \nu(B_4) &= -\lambda_1\nu(C_3) - \lambda_{12}\nu(C_4) - \lambda_{123}\nu(C_5),
\end{aligned}$$

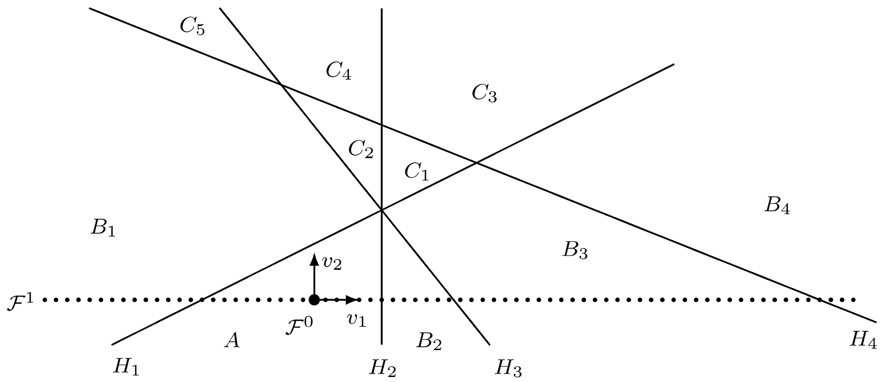

Figure 1. Example.

4. Aomoto complex via chamber basis

4.1. Main result

Let $(\lambda_1, ..., \lambda_n) \in \mathbb{C}^n$ and put $\omega_\lambda = \sum_{j=1}^n \lambda_j \omega_j$. Since $\omega_\lambda \wedge \omega_\lambda = 0$, we have a cochain complex $(A^\bullet, 2\pi i \omega_\lambda \wedge)$, which is called the *Aomoto complex.* We shall study this complex using the chamber basis $\{\nu^q(C)\}_{C \in \mathsf{ch}^q_{\mathcal{F}}(\mathcal{A})}$ of A^*. For a chamber $C \in \mathsf{ch}^q_{\mathcal{F}}(\mathcal{A})$, $\omega_\lambda \wedge \nu^q(C)$ is uniquely expressed as

$$\omega_\lambda \wedge \nu^q(C) = \sum_{C' \in \mathsf{ch}^{q+1}_{\mathcal{F}}(\mathcal{A})} \Gamma_{C,C'}(\lambda) \nu^{q+1}(C'),$$

for some complex numbers $\Gamma_{C,C'}(\lambda) \in \mathbb{C}$. We may consider the coefficients $\{\Gamma_{C,C'}(\lambda)\}$ as structure constants of the cohomology ring with respect to the chamber basis.

Let $\mathcal{L}_\lambda$ be a rank-one local system on $\mathsf{M}(\mathcal{A})$ determined by monodromies $q_j = e^{2\pi i \lambda_j} \in \mathbb{C}^*$ around the hyperplane H_j. For two given chambers $C, C' \in \mathsf{ch}(\mathcal{A})$, denote by $S(C, C')$ the set

$$S(C, C') = \{H \in \mathcal{A} \mid H \text{ separates } C \text{ and } C'\},$$

of hyperplanes separating C and C', and set $\lambda_{S(C,C')} = \sum_{H \in S(C,C')} \lambda_H$.

The main result is the following.

Theorem 4.1. (a) *The coefficient* $\Gamma_{C,C'}(\lambda)$ *has the decomposition*

$$\Gamma_{C,C'}(\lambda) = N_{C,C'} \lambda_{S(C,C')},$$

where $N_{C,C'} \in \mathbb{Z}$.

(b) *Let us define a linear map* $\widetilde{\nabla}_\lambda\colon \mathbb{C}[\mathsf{ch}^q_{\mathcal{F}}(\mathcal{A})]\to\mathbb{C}[\mathsf{ch}^{q+1}_{\mathcal{F}}(\mathcal{A})]$ *by*

$$\widetilde{\nabla}_\lambda([C]) = -\sum_{C'\in\mathsf{ch}^{q+1}_{\mathcal{F}}(\mathcal{A})} 2N_{C,C'}\sinh(\pi i\lambda_{S(C,C')})[C'].$$

Then $(\mathsf{ch}^\bullet_{\mathcal{F}}(\mathcal{A}),\widetilde{\nabla}_\lambda)$ *is a cochain complex and*

$$H^p(\mathsf{ch}^\bullet_{\mathcal{F}}(\mathcal{A}),\widetilde{\nabla}_\lambda)\cong H^p(\mathsf{M}(\mathcal{A}),\mathcal{L}_\lambda).$$

Thus $(\mathsf{ch}^\bullet_{\mathcal{F}}(\mathcal{A}),\widetilde{\nabla}_\lambda)$ is the minimal complex which is obtained as an integration of the Aomoto complex.

4.2. Proof

First we recall some more notation from [21]. We defined the degree map, [21, Section 6.3],

$$\deg\colon \mathsf{ch}^{p+1}_{\mathcal{F}}(\mathcal{A})\times\mathsf{ch}^{p}_{\mathcal{F}}(\mathcal{A})\longrightarrow\mathbb{Z}.$$

Furthermore the notion of $\deg(C',C)$ enables us to express the twisted cellular coboundary map $\nabla_{\mathcal{L}_\lambda}\colon \mathbb{C}[\mathsf{ch}^p_{\mathcal{F}}(\mathcal{A})]\to\mathbb{C}[\mathsf{ch}^{p+1}_{\mathcal{F}}(\mathcal{A})]$ as follows [21, Theorem 6.4.1]:

$$\nabla_{\mathcal{L}_\lambda}(C) = -\sum_{C'\in\mathsf{ch}^{p+1}_{\mathcal{F}}(\mathcal{A})}\operatorname{sgn}(C')\deg(C',C)\cdot 2\sinh(\pi i\lambda_{S(C',C)})[C'].$$

In particular, $(\mathbb{C}[\mathsf{ch}^\bullet_{\mathcal{F}}],\nabla_{\mathcal{L}_\lambda})$ is a cochain complex, and the cohomology group is isomorphic to cohomology with coefficients in $\mathcal{L}_\lambda$: $H^p(\mathbb{C}[\mathsf{ch}^\bullet_{\mathcal{F}}],\nabla_{\mathcal{L}_\lambda})\cong H^p(\mathsf{M}(\mathcal{A}),\mathcal{L}_\lambda)$.

We now apply the linearization theorem by Cohen–Orlik [2]. Consider the local system $\mathcal{L}_{t\lambda}$ with $t\in\mathbb{C}$. Since

$$\left.\frac{d}{dt}\right|_{t=0} 2\sinh\pi it\lambda_{S(C',C)} = 2\pi i\lambda_{S(C,C')}, \tag{6}$$

we have

$$\left.\frac{d}{dt}\right|_{t=0}\nabla_{\mathcal{L}_{t\lambda}}(C) = -2\pi i\sum_{C'\in\mathsf{ch}^{p+1}_{\mathcal{F}}}\operatorname{sgn}(C')\deg(C',C)\lambda_{S(C,C')}[C'].$$

From the construction in [21, Section 6.4], $[C]$ can be identified with $\nu(C)$ here. Thus we have $\Gamma_{CC'}(\lambda) = -\operatorname{sgn}(C')\deg(C',C)\lambda_{S(C,C')}$. The map $\widetilde{\nabla}_\lambda$ of (b) is clearly equivalent to $\nabla_{\mathcal{L}_\lambda}$.

Remark 4.2. In [21], a minimal CW-decomposition such that each k-cell is labeled by a chamber $C\in\mathsf{ch}^k_{\mathcal{F}}(\mathcal{A})$ is constructed. From the minimality, the incidence numbers vanish, $[C':C]=0$. On the other hand, a minimal CW-decomposition of $\mathsf{M}(\mathcal{A})$ induces a $\mathbb{Z}^{b_1}$-equivariant CW-decomposition of the homology covering $\widetilde{\mathsf{M}}$. The degree above can be considered as the incidence number at the level of homology covering.

5. Appendix: The linearization theorem for minimal CW-complexes

In this section, we give a proof of the linearization theorem by Cohen–Orlik [2] in a generalized setting.

Let X be a connected minimal CW-complex, that is, a finite CW-complex with exactly as many k-cells as the kth Betti number, for all k. Denote by $\mathcal{S}_k$ the set of k-cells and by X_k the k-skeleton of X. Suppose $|\mathcal{S}_1|=n$, then $H_1(X,\mathbb{Z})\cong\mathbb{Z}^n$ and $H^1(X,\mathbb{Z})=\mathbb{Z}^n$. An example of a minimal CW-complex is the n-torus $T^n=(S^1)^{\times n}$. The n-torus T^n admits the canonical minimal cell decomposition as follows. Let $e_1,...,e_n$ be the standard basis of $\mathbb{R}^n$. The torus T^n can be identified with the quotient space $(\bigoplus_{j=1}^n \mathbb{R}e_j)/(\bigoplus_{j=1}^n \mathbb{Z}e_j)$. For any subset $\Phi=\{p_1,...,p_k\}\subset[1,n]:=\{1,...,n\}$, denote by K_Φ the k-cube

$$K_\Phi=\{t_1e_{p_1}+...+t_ke_{p_k}\,|\,0\le t_j\le 1,\ j=1,...,k\}\subset\mathbb{R}^n$$

and by $e_\Phi=p(K_\Phi)$ the image of the cube by the quotient map $p\colon\mathbb{R}^n\to\mathbb{R}^n/\mathbb{Z}^n$. This gives a cell decomposition of $T^n=\bigcup_\Phi e_\Phi$. Note that $\dim e_\Phi=|\Phi|$. Obviously the quotient map p gives the universal covering of T^n. Let us denote by x_j the multiplicative generator of the deck transformation group corresponding to e_j. The deck transformation group is identified with the multiplicative group of Laurent monomials $\{x^\alpha=x_1^{\alpha_1}x_2^{\alpha_2}...x_n^{\alpha_n}\,|\,\alpha=(\alpha_1,...,\alpha_n)\in\mathbb{Z}^n\}$. The covering space $\widetilde{T}^n\cong\mathbb{R}^n$ has the following cell decomposition,

$$\widetilde{T}^n=\bigcup_{\Phi\subset[1,n]}\bigcup_{\alpha\in H_1}x^\alpha\cdot K_\Phi.$$

Since T^n is the $K(\mathbb{Z}^n,1)$-space, the abelianization map $\pi_1(X)\to H_1(X,\mathbb{Z})\cong\mathbb{Z}^n$ determines, uniquely up to homotopy, a continuous map $f\colon X\to T^n$ such that $f_*\colon H_1(X,\mathbb{Z})\xrightarrow{\cong}H_1(T^n,\mathbb{Z})$. By the cellular approximation theorem, we may assume that f is cellular, i.e., preserving skeletons $f(X_k)\subset(T^n)_k$. A k-cell $\sigma\in\mathcal{S}_k$ is expressed as a characteristic map

$$\sigma\colon(D^k,\partial D^k)\longrightarrow(X_k,X_{k-1}),$$

from the k-disk to the k-skeleton. For simplicity, we assume that the base point $p_\sigma\in D^k$ is mapped to X_0 by σ.

As is the case of T^n, the $H_1(X,\mathbb{Z})$-covering $\widetilde{X}$ of X has the structure of a $\mathbb{Z}^n$-equivariant CW-complex. Indeed from the minimality, fixing a base point

$p_0 \in \widetilde{X}$ over X_0, each cell $\sigma\colon D^k \to X$ can be lifted uniquely to $\tilde{\sigma}\colon D^k \to \widetilde{X}$ such that $\tilde{\sigma}(p_\sigma)=p_0$. Then $\widetilde{X}$ is decomposed as

$$\widetilde{X} = \bigcup_{\sigma\in\mathcal{S}} \bigcup_{\alpha\in H_1} x^\alpha \cdot \tilde{\sigma},$$

where $\mathcal{S}=\bigcup_j \mathcal{S}_j$ is the set of all cells. The boundary of the cellular chain complex can be expressed as

$$\partial(x^\alpha \cdot \tilde{\sigma}) = \sum_{\tau\in\mathcal{S}_{k-1}} \sum_{\beta\in H_1} [\tilde{\sigma} : (x^\beta \cdot \tilde{\tau})] x^{\alpha+\beta} \cdot \tilde{\tau} \in H_k(\widetilde{X}_k, \widetilde{X}_{k-1}; \mathbb{Z}), \tag{7}$$

where $[\tilde{\sigma} : (x^\beta \cdot \tilde{\tau})] \in \mathbb{Z}$ is the incidence number.

Since f induces the isomorphism $H_1(X,\mathbb{Z}) \cong H_1(T^n,\mathbb{Z})$, any complex rank-one local system on X can be obtained as a pull-back by f. Recall that a local system on T^n is determined by a homomorphism $\rho\colon H_1(T^n,\mathbb{Z}) \to \mathbb{C}^*$. Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$, and define a local system $\mathcal{L}_\lambda$ on T^n by $\rho(e_{\{j\}}) = q_j = e^{2\pi i \lambda_j} \in \mathbb{C}^*$. We also set $\rho(x^\alpha) = q^\alpha = \prod_{j=1}^n q_j^{\alpha_j}$. Now we describe the boundary map of an $\mathcal{L}_\lambda$-coefficients cellular complex. Recall that, [20, Section VI.2], a chain with coefficients in a local system $\mathcal{L}_\lambda$ is a pair of a cell σ and a section $c(\sigma) \in (\mathcal{L}_\lambda)_{\sigma(p_\sigma)}$ at the base point. Let us fix a trivialization $(\mathcal{L}_\lambda)_{X_0} \cong \mathbb{C}$ at the base point. We identify the cell σ with a chain $[\sigma]$ with $\mathcal{L}_\lambda$-coefficient which takes the value $c=1 \in \mathbb{C} \cong (\mathcal{L}_\lambda)_{X_0}$. Then, for a k-cell $\sigma \in \mathcal{S}_k$, the boundary with $\mathcal{L}_\lambda$-coefficients is expressed as

$$\partial_{\mathcal{L}_\lambda}[\sigma] = \sum_{\tau\in\mathcal{S}_{k-1}} \sum_{\alpha\in H_1} [\tilde{\sigma} : (x^\alpha \cdot \tau)] q^\alpha [\tau] \in H_k(X_k, X_{k-1}; \mathbb{C}). \tag{8}$$

Let us denote the dual basis of $e_{\{j\}}$ by $dt_j \in H^1(T^n,\mathbb{Z})$ and define

$$\omega_\lambda = 2\pi i \sum_{j=1}^n \lambda_j \, dt_j \in H^1(T^n, \mathbb{C}).$$

Clearly the minimality implies that $\lim_{t\to 0} \partial_{\mathcal{L}_{t\lambda}}[\sigma] = 0$. By differentiating (8), we have the following lemma.

Lemma 5.1. *Let X be a minimal CW-complex. Let $t \in \mathbb{C}$ and $\sigma \in \mathcal{S}_k$ be a k-cell of X. Then*

$$\left.\frac{d}{dt}\right|_{t=0} \partial_{\mathcal{L}_{t\lambda}}[\sigma] = \sum_{\tau\in\mathcal{S}_{k-1}} \sum_{\alpha\in H_1} [\tilde{\sigma} : (x^\alpha \cdot \tau)] \cdot \langle \omega_\lambda, \alpha \rangle \cdot [\tau]. \tag{9}$$

The next result asserts that, after-push-forward by f_*, it is obtained by the cap product on the torus.

Theorem 5.2.

$$f_*\left(\frac{d}{dt}\bigg|_{t=0} \partial_{\mathcal{L}_{t\lambda}}[\sigma]\right) = \omega_\lambda \cap f_*[\sigma]. \tag{10}$$

Proof. We prove this in two steps. The first step is commuting the operators f_* and $(d/dt)|_{t=0}\partial_{\mathcal{L}_{t\lambda}}$. This is essentially done by the fact that a cellular map induces a chain map between cellular chain complexes. Therefore it is enough to show that

$$\frac{d}{dt}\bigg|_{t=0} \partial_{\mathcal{L}_{t\lambda}}[f\circ\sigma] = \omega_\lambda \cap [f\circ\sigma]. \tag{11}$$

Let us denote the left-hand side of this formula by η. Note that the continuous map $f\colon X\to T^n$ can also be lifted to $\tilde{f}\colon \widetilde{X}\to \widetilde{T}^n$. From Lemma 5.1, we have

$$\eta = \sum_{|\Phi|=k-1} \sum_{\alpha\in H_1} [\tilde{f}\circ\tilde{\sigma} : (x^\alpha \cdot e_\Phi)]\langle \omega_\lambda, \alpha\rangle [e_\Phi].$$

Let $\Phi_0 \subset [1,n]$ with $|\Phi_0| = k-1$. Then (11) is equivalent to

$$\int_\eta dt_{\Phi_0} = \int_{[\tilde{f}\circ\tilde{\sigma}]} \omega_\lambda \wedge dt_{\Phi_0} \tag{12}$$

for all Φ_0. The left-hand side of (12) is equal to

$$\sum_\alpha [\tilde{f}\circ\tilde{\sigma} : (x^\alpha \cdot e_{\Phi_0})]\langle \omega_\lambda, \alpha\rangle.$$

We compute the right-hand side of (12) using $\omega_\lambda = d(2\pi i \sum_{j=1}^n \lambda_j t_j)$, by Stokes' theorem,

$$\begin{aligned}
\int_{[\tilde{f}\circ\tilde{\sigma}]} \omega_\lambda \wedge dt_{\Phi_0} &= \sum_{j=1}^n \int_{\partial[\tilde{f}\circ\tilde{\sigma}]} 2\pi i \lambda_j t_j \, dt_{\Phi_0} \\
&= 2\pi i \sum_{j=1}^n \sum_{|\Phi|=k-1} \sum_{\alpha\in H_1} \int_{x^\alpha e_\Phi} [(\tilde{f}\circ\tilde{\sigma}) : (x^\alpha e_\Phi)]\lambda_j t_j \, dt_{\Phi_0} \\
&= 2\pi i \sum_{j=1}^n \sum_{\alpha\in H_1} \int_{e_{\Phi_0}} [(\tilde{f}\circ\tilde{\sigma}) : (x^\alpha e_{\Phi_0})]\lambda_j \alpha_j \, dt_{\Phi_0} \\
&= \sum_{\alpha\in H_1} [(\tilde{f}\circ\tilde{\sigma}) : (x^\alpha e_{\Phi_0})]\langle \omega_\lambda, \alpha\rangle.
\end{aligned}$$

This completes the proof of (12). □

Corollary 5.3. *If X satisfies that $f_*: H_k(X,\mathbb{Z}) \to H_k(T^n,\mathbb{Z})$ is injective, then*

$$\left.\frac{d}{dt}\right|_{t=0} \partial_{\mathcal{L}_{t\lambda}}[\sigma] = f^*(\omega_\lambda) \cap \sigma \tag{13}$$

for every k-cell $\sigma \in \mathcal{S}_k$.

Remark 5.4. If X is the complement of a hyperplane arrangement, then f_* is injective for every k. Thus we have the linearization theorem by [2]. Recently Papadima and Suciu [13] proved that the the linearization theorem (13) holds for any minimal CW-complex X.

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