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A moment problem for pseudo-positive definite functionals

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Abstract. A moment problem is presented for a class of signed measures which are termed pseudo-positive. Our main result says that for every pseudo-positive definite functional (subject to some reasonable restrictions) there exists a representing pseudo-positive measure.

The second main result is a characterization of determinacy in the class of equivalent pseudo-positive representation measures. Finally the corresponding truncated moment problem is discussed.

1. Introduction

Let $\mathbb{C}[x_1, ..., x_d]$ denote the space of all polynomials in d variables with complex coefficients and let $T\colon \mathbb{C}[x_1, ..., x_d] \to \mathbb{C}$ be a linear functional. The *multivariate moment problem* asks for conditions on the functional T such that there exists a *non-negative* measure μ on $\mathbb{R}^d$ with

$$T(P) = \int_{\mathbb{R}^d} P(x)\, d\mu(x) \tag{1}$$

for all $P \in \mathbb{C}[x_1, ..., x_d]$. It is well known that *positive definiteness* of the functional T is a necessary condition which means that

$$T(P^* P) \geq 0 \quad \text{for all } P \in \mathbb{C}[x_1, ..., x_d];$$

here P^* is the polynomial whose coefficients are the complex conjugates of the coefficients of P. By a theorem of Haviland, a necessary and sufficient condition for the existence of a non-negative measure μ satisfying (1) is the *positivity* of the functional T, i.e. $P(x) \geq 0$ for all $x \in \mathbb{R}^d$ implies $T(P) \geq 0$ for all $P \in \mathbb{C}[x_1, ..., x_d]$,

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cf. [5, p. 111]. In the case $d=1$ it is a classical fact that a functional T is positive if and only if it is positive-definite, which is proved by using the representation of a non-negative polynomial as a sum of two squares of polynomials, cf. [1, Chapter 1, Section 1.1]. A counterexample of D. Hilbert shows that a representation of a multivariate non-negative polynomial as a finite sum of squares is in general not possible, cf. [14] and [6]. Many authors have tried to find additional assumptions on the functional T such that positive definiteness and positivity become equivalent, see [6], [12], [13, p. 47], [22], [24], [25] and [30].

In this paper we shall be concerned with a *modified* moment problem which arose in the investigation of a new cubature formula of Gauss–Jacobi type for measures μ in the multivariate setting, see [19], [20] and [21]. In contrast to the classical multivariate moment problem we allow the measures μ under consideration to be *signed* measures on $\mathbb{R}^d$. Our approach is based on the new notions of pseudo-positive definite functionals T and pseudo-positive signed measures μ, to be explained below.

A cornerstone of our approach is the *Gauss representation* of a polynomial which we provide below. First we recall some definitions and notation: Let $|x|=\sqrt{x_1^2+...+x_d^2}$ be the euclidean norm and $\mathbb{S}^{d-1}:=\{x\in\mathbb{R}^d:|x|=1\}$ be the unit sphere. We shall write $x\in\mathbb{R}^d$ in spherical coordinates $x=r\theta$ with $\theta\in\mathbb{S}^{d-1}$. Let $\mathcal{H}_k(\mathbb{R}^d)$ be the set of all harmonic homogeneous complex-valued polynomials of degree k. Then $f\in\mathcal{H}_k(\mathbb{R}^d)$ is called a *solid harmonic* and the restriction of f to $\mathbb{S}^{d-1}$ a *spherical harmonic* of degree k. Throughout the paper we shall assume that $Y_{k,l}\colon \mathbb{R}^d\to\mathbb{R}$, $l=1,...,a_k:=\dim\mathcal{H}_k(\mathbb{R}^d)$, is an orthonormal basis of $\mathcal{H}_k(\mathbb{R}^d)$ with respect to the scalar product $\langle f,g\rangle_{\mathbb{S}^{d-1}}:=\int_{\mathbb{S}^{d-1}} f(\theta)\overline{g(\theta)}\,d\theta$. We shall often use the trivial identity $Y_{k,l}(x)=r^kY_{kl}(\theta)$. The Gauss representation (cf. [3], [28] or [18, Theorem 10.2]) tells us that for every $P\in\mathbb{C}[x_1,...,x_d]$ there exist polynomials $p_{k,l}$ such that

$$P(x)=\sum_{k=0}^{\deg P}\sum_{l=1}^{a_k}p_{k,l}(r^2)r^kY_{k,l}(\theta)=\sum_{k=0}^{\deg P}\sum_{l=1}^{a_k}p_{k,l}(|x|^2)Y_{k,l}(x), \tag{2}$$

where $\deg P$ is the degree of the polynomial P. By this formula it is clear that the set of polynomials

$$\{|x|^{2j}Y_{k,l}(x): j\geq 0, k\geq 0 \text{ and } l=1,2,...,a_k\}$$

forms a basis for the space of all polynomials, hence this is an alternative basis to the standard basis $\{x^\alpha:\alpha\in\mathbb{Z}^d \text{ and } \alpha\geq 0\}$. The numbers

$$c_{j,k,l}:=\int_{\mathbb{R}^d}|x|^{2j}Y_{k,l}(x)\,d\mu(x) \tag{3}$$

are sometimes called the *distributed moments* of μ, cf. [8], [9], [15], [16] and [17]. Let us remark that for fixed k and l one may consider the correspondence $j\mapsto c_{j,k,l}$

as a univariate moment sequence in the variable $j\in\mathbb{N}_0$. The distributed moments can be expressed linearly by the classical *monomial moments*

$$\int_{\mathbb{R}^d} x^\alpha \, d\mu(x) \tag{4}$$

which are considered in the standard approach, and vice versa.

Now we will introduce our basic notions: A signed measure μ over $\mathbb{R}^d$ is *pseudo-positive with respect to the orthonormal basis* $Y_{k,l}$, $l=1,\dots,a_k$, $k\in\mathbb{N}_0$, if the inequality

$$\int_{\mathbb{R}^d} h(|x|) Y_{k,l}(x) \, d\mu(x) \geq 0 \tag{5}$$

holds for every non-negative continuous function $h\colon [0,\infty)\to[0,\infty)$ with compact support, and for all $k\in\mathbb{N}_0$ and $l=1,2,\dots,a_k$. Obviously, the *radially-symmetric measures* represent a subclass of the pseudo-positive measures.

Given a linear functional $T\colon \mathbb{C}[x_1,\dots,x_d]\to\mathbb{C}$ and $Y_{k,l}\in\mathcal{H}_k(\mathbb{R}^d)$ we define the "component functional" $T_{k,l}\colon \mathbb{C}[x_1]\to\mathbb{C}$ by putting

$$T_{k,l}(p) := T(p(|x|^2) Y_{k,l}(x)) \quad \text{for every } p\in\mathbb{C}[x_1]. \tag{6}$$

Note that in the notation (3), $T_{k,l}(p)=c_{j,k,l}$ for $p(t)=t^j$ with $j\in\mathbb{N}_0$. We say that the functional T is *pseudo-positive definite with respect to the orthonormal basis* $Y_{k,l}$, $l=1,\dots,a_k$, $k\in\mathbb{N}_0$, if

$$T_{k,l}(p^*(t)p(t)) \geq 0 \quad \text{and} \quad T_{k,l}(tp^*(t)p(t)) \geq 0$$

for every $p(t)\in\mathbb{C}[x_1]$, and for every $k\in\mathbb{N}_0$ and $l=1,\dots,a_k$.

Our main result in Section 2 provides a reasonable *sufficient criterion* guaranteeing that for a pseudo-positive definite functional $T\colon \mathbb{C}[x_1,\dots,x_d]\to\mathbb{C}$ there exists a pseudo-positive signed measure μ on $\mathbb{R}^d$ with

$$\int_{\mathbb{R}^d} P(x) \, d\mu = T(P) \quad \text{for all } P\in\mathbb{C}[x_1,\dots,x_d]. \tag{7}$$

This means that we give a solution to the *pseudo-positive moment problem*: this problem asks for conditions on the moments (3) which provide the existence of a pseudo-positive (signed) measure μ satisfying the equalities (3). The sufficient criterion is a *summability assumption* of the type

$$\sum_{k=0}^{\infty} \sum_{l=1}^{a_k} \int_0^\infty r^N r^{-k} \, d\sigma_{k,l}(r) < \infty \quad \text{for all } N\in\mathbb{N}_0, \tag{8}$$

where the measures $\sigma_{k,l}$ are representing measures of the component functionals $T_{k,l}$, cf. Proposition 2.2.

An essential advantage of our approach is that there exists a naturally defined *truncated moment problem* in the class of pseudo-positive definite functionals. In Section 3 we shall formulate and solve this problem which is important also from the practical point of view.

The second main result in Section 4 says that the pseudo-positive representing measure μ of a pseudo-positive definite functional $T\colon \mathbb{C}[x_1,\dots,x_d]\to\mathbb{C}$ is unique in the class of all pseudo-positive signed measures whenever each functional $T_{k,l}$ defined in (5) has a unique representing measure on $[0,\infty)$ in the sense of Stieltjes (for the precise definition see Section 4). And vice versa, if a pseudo-positive functional T is determinate in the class of all pseudo-positive signed measures and the summability condition (8) is satisfied, then each functional $T_{k,l}$ is determinate in the sense of Stieltjes. The proof is essentially based on the properties of the Nevanlinna extremal measures. In the last section we shall give examples and some further properties of pseudo-positive definite functionals.

Let us recall some terminology from measure theory: a signed measure on $\mathbb{R}^d$ is a set function on the Borel σ-algebra on $\mathbb{R}^d$ which takes real values and is σ-additive. For the standard terminology, as Radon measure, Borel σ-algebra, etc., we refer to [6]. By the *Jordan decomposition* [11, p. 125], a signed measure μ is the difference of two non-negative finite measures, say $\mu=\mu^+-\mu^-$ with the property that there exists a Borel set A such that $\mu^+(A)=0$ and $\mu^-(\mathbb{R}^n\setminus A)=0$. The *variation* of μ is defined as $|\mu|:=\mu^++\mu^-$. The signed measure μ is called *moment measure* if all polynomials are integrable with respect to μ^+ and μ^-, which is equivalent to integrability with respect to the total variation. The *support of a non-negative measure* μ on $\mathbb{R}^d$ is defined as the complement of the largest open set U such that $\mu(U)=0$. In particular, the *support of the zero measure* is the *empty set*. The *support of a signed measure* σ is defined as the support of the total variation $|\sigma|=\sigma_++\sigma_-$ (see [11, p. 226]). Recall that in general, the supports of σ_+ and σ_- are not disjoint (cf. Exercise 2 in [11, p. 231]). For a surjective measurable mapping $\varphi\colon X\to Y$ and a measure ν on X the *image measure* ν^φ on Y is defined by

$$\nu^\varphi(B):=\nu(\varphi^{-1}B) \tag{9}$$

for all Borel subsets B of Y. The equality $\int_X g(\varphi(x))\,d\nu(x)=\int_Y g(y)\,d\nu^\varphi(y)$ holds for all integrable functions g.

2. The moment problem for pseudo-positive definite functionals

Recall that for a continuous function $f\colon \mathbb{R}^d\to\mathbb{C}$ the *Laplace–Fourier coefficient* is defined by

$$f_{k,l}(r) = \int_{\mathbb{S}^{d-1}} f(r\theta) Y_{k,l}(\theta)\, d\theta. \tag{10}$$

The formal expansion

$$f(r\theta) = \sum_{k=0}^{\infty} \sum_{l=1}^{a_k} f_{k,l}(r) Y_{k,l}(\theta) \tag{11}$$

is the *Laplace–Fourier series.* The following result may be found e.g. in [4] or [27].

Proposition 2.1. *The Laplace–Fourier coefficient* $f_{k,l}$ *of a polynomial* f *given by* (10) *is of the form* $f_{k,l}(r) = r^k p_{k,l}(r^2)$, *where* $p_{k,l}$ *is a univariate polynomial. Hence, the Laplace–Fourier series* (11) *is equal to*

$$f(x) = \sum_{k=0}^{\deg f} \sum_{l=1}^{a_k} p_{k,l}(|x|^2) Y_{k,l}(x). \tag{12}$$

The next two propositions characterize pseudo-positive definite functionals:

Proposition 2.2. *Let* $T\colon \mathbb{C}[x_1, ..., x_d] \to \mathbb{C}$ *be a pseudo-positive definite functional. Then for each* $k \in \mathbb{N}_0$, *and* $l = 1, ..., a_k$, *there exist non-negative measures* $\sigma_{k,l}$ *with support in* $[0, \infty)$ *such that*

$$T(f) = \sum_{k=0}^{\deg f} \sum_{l=1}^{a_k} \int_0^{\infty} f_{k,l}(r) r^{-k}\, d\sigma_{k,l}(r) \tag{13}$$

holds for all $f \in \mathbb{C}[x_1, ..., x_d]$, *where* $f_{k,l}(r)$, $k \in \mathbb{N}_0$, $l = 1, ..., a_k$, *are the Laplace–Fourier coefficients of* f.

Proof. By the solution of the Stieltjes moment problem there exists a non-negative measure $\mu_{k,l}$ with support in $[0, \infty)$ representing the functional $T_{k,l}$, i.e. satisfying

$$T_{k,l}(p) = \int_0^{\infty} p(t)\, d\mu_{k,l}(t) \quad \text{for every } p \in \mathbb{C}[t]. \tag{14}$$

Let now $\varphi\colon [0, \infty) \to [0, \infty)$ be defined by $\varphi(t) = \sqrt{t}$. Then we put $\sigma_{k,l} := \mu_{k,l}^{\varphi}$, where $\mu_{k,l}^{\varphi}$ is the image measure defined in (9). We obtain

$$\int_0^{\infty} h(t)\, d\mu_{k,l}(t) = \int_0^{\infty} h(r^2)\, d\mu_{k,l}^{\varphi}(r). \tag{15}$$

Now use (12), the linearity of T and the definition of $T_{k,l}$ in (6), and the equations (14) and (15) to obtain

$$T(f)=\sum_{k=0}^{\deg f}\sum_{l=1}^{a_k} T_{k,l}(p_{k,l})=\sum_{k=0}^{\deg f}\sum_{l=1}^{a_k}\int_0^\infty p_{k,l}(r^2)\,d\mu_{k,l}^{\varphi}(r).$$

Since $p_{k,l}(r^2)=r^{-k}f_{k,l}(r)$ the claim (13) follows from the last equation, which ends the proof. □

The next result shows that the converse of Proposition 2.2 is also true; not less important, it is a natural way of defining pseudo-positive definite functionals.

Proposition 2.3. *Let $\sigma_{k,l}$, $k\in\mathbb{N}_0$, $l=1,\dots,a_k$, be non-negative moment measures with support in $[0,\infty)$. Then the functional $T\colon \mathbb{C}[x_1,\dots,x_d]\to\mathbb{C}$ defined by*

$$T(f):=\sum_{k=0}^{\deg f}\sum_{l=1}^{a_k}\int_0^\infty f_{k,l}(r)r^{-k}\,d\sigma_{k,l} \tag{16}$$

is pseudo-positive definite, where $f_{k,l}(r)$, $k\in\mathbb{N}_0$, $l=1,\dots,a_k$, are the Laplace–Fourier coefficients of f.

Proof. Let us compute $T_{k,l}(p)$ where p is a univariate polynomial: by definition, $T_{k,l}(p)=T(p(|x|^2)Y_{k,l}(x))$. The Laplace–Fourier series of the function $x\mapsto |x|^{2j}p(|x|^2)Y_{k,l}(x)$ is equal to $r^{2j}p(r^2)r^kY_{k,l}(\theta)$, hence

$$T_{k,l}(t^jp(t))=T(|x|^{2j}p(|x|^2)Y_{k,l}(x))=\int_0^\infty r^jp(r^2)\,d\sigma_{k,l}$$

for every natural number j. Taking $j=0$ and $j=1$ one concludes that $T_{k,l}(p^*(t)p(t))\geq 0$ and $T_{k,l}(tp^*(t)p(t))\geq 0$ for all univariate polynomials p, hence T is pseudo-positive definite. □

By $C(X)$ we denote the space of all continuous complex-valued functions on a topological space X while $C_c(X)$ is the set of all $f\in C(X)$ having compact support. Further $C_{\mathrm{pol}}(\mathbb{R}^d)$ is the space of all polynomially bounded, continuous functions, so for each $f\in C_{\mathrm{pol}}(\mathbb{R}^d)$ there exists $N\in\mathbb{N}_0$ such that $|f(x)|\leq C_N(1+|x|)^N$ for some constant C_N (depending on f) for all $x\in\mathbb{R}^d$. A useful space of test functions is

$$C^{\times}(\mathbb{R}^d):=\left\{\sum_{k=0}^{N}\sum_{l=1}^{a_k} f_{k,l}(|x|)Y_{k,l}(x): N\in\mathbb{N}_0 \text{ and } f_{k,l}\in C[0,\infty)\right\} \tag{17}$$

which can be rephrased as the set of all continuous functions with a finite Laplace–Fourier series.

Proposition 2.4. *Let μ be a pseudo-positive moment measure on $\mathbb{R}^d$. Then there exist unique moment measures $\mu_{k,l}$ defined on $[0,\infty)$ such that*

$$\int_0^\infty h(t)\,d\mu_{k,l}(t) = \int_{\mathbb{R}^d} h(|x|)Y_{k,l}(x)\,d\mu \tag{18}$$

holds for all $h\in C_{\mathrm{pol}}[0,\infty)$. Further for each $f\in C^\times(\mathbb{R}^d)\cap C_{\mathrm{pol}}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} f(x)\,d\mu = \sum_{k=0}^{\infty}\sum_{l=1}^{a_k}\int_0^\infty f_{k,l}(r)r^{-k}\,d\mu_{k,l}.$$

Proof. By definition of pseudo-positivity, $M_{k,l}(h):=\int_{\mathbb{R}^d} h(|x|)Y_{k,l}(x)\,d\mu$ defines a positive functional on $C_c([0,\infty))$. By the Riesz representation theorem there exists a unique non-negative measure $\mu_{k,l}$ such that $M_{k,l}(h)=\int_0^\infty h(t)\,d\mu_{k,l}$ for all $h\in C_c([0,\infty))$. We want to show that (18) holds for all $h\in C_{\mathrm{pol}}[0,\infty)$. For this, let $u_R\colon[0,\infty)\to[0,1]$ be a *cut-off function,* so u_R is continuous and decreasing such that

$$u_R(r)=1 \text{ for all } 0\le r\le R \quad\text{and}\quad u_R(r)=0 \text{ for all } r\ge R+1. \tag{19}$$

Let $h\in C_{\mathrm{pol}}[0,\infty)$. Then $u_Rh\in C_c([0,\infty))$ and

$$\int_0^\infty u_R(t)h(t)\,d\mu_{k,l} = \int_{\mathbb{R}^d} u_R(|x|)h(|x|)Y_{k,l}(x)\,d\mu. \tag{20}$$

Note that $|u_R(t)h(t)|\le|u_{R+1}(t)h(t)|$ for all $t\in[0,\infty)$. Hence by the monotone convergence theorem

$$\int_0^\infty |h(t)|\,d\mu_{k,l} = \lim_{R\to\infty}\int_0^\infty |u_R(t)h(t)|\,d\mu_{k,l}. \tag{21}$$

On the other hand, it is obvious that

$$\left|\int_{\mathbb{R}^d} u_R(|x|)\big|h(|x|)\big|Y_{k,l}(x)\,d\mu\right| \le \int_{\mathbb{R}^d}\big|h(|x|)Y_{k,l}(x)\big|\,d|\mu|. \tag{22}$$

The last expression is finite since μ is a moment measure. From (21), (20) applied to $|h|$ and (22) it follows that $|h|$ is integrable for $\mu_{k,l}$. Using (20) and Lebesgue's convergence theorem for μ it is easy to see that (18) holds. For the last statement recall that each $f\in C^\times(\mathbb{R}^d)$ has a finite Laplace–Fourier series, and it is easy to see that the Laplace–Fourier coefficients $f_{k,l}$ are in $C_{\mathrm{pol}}[0,\infty)$ if $f\in C_{\mathrm{pol}}(\mathbb{R}^d)$, see (25) below. $\square$

The next theorem is the main technical result of this section.

Theorem 2.5. *Let $\sigma_{k,l}$, $k\in\mathbb{N}_0$, $l=1,...,a_k$, be non-negative measures with support in $[0,\infty)$ such that for any $N\in\mathbb{N}_0$,*

$$C_N := \sum_{k=0}^{\infty}\sum_{l=1}^{a_k}\int_0^{\infty} r^N r^{-k}\, d\sigma_{k,l} < \infty. \tag{23}$$

Then for the functional $T\colon \mathbb{C}[x_1,...,x_d]\to\mathbb{C}$ defined by (16) there exists a pseudo-positive, signed moment measure σ such that

$$T(f) = \int_{\mathbb{R}^n} f\, d\sigma \quad \text{for all } f\in\mathbb{C}[x_1,...,x_d].$$

Remark 2.6. (1) If the measures $\sigma_{k,l}$ have supports in the compact interval $[\rho,R]$ for all $k\in\mathbb{N}_0$, $l=1,...,a_k$, then the measure σ in Theorem 2.5 has support in the annulus $\{x\in\mathbb{R}^d:\rho\le|x|\le R\}$.

(2) In the case of $R<\infty$, it obviously suffices to assume that $C_0<\infty$ instead of $C_N<\infty$ for all $N\in\mathbb{N}_0$.

(3) The proof of Theorem 2.5 shows that $\sigma_{k,l}$ is equal to the measure induced by σ with respect to the solid harmonic $Y_{k,l}(x)$, cf. (5).

Proof. (1) We show at first that T can be extended to a linear functional $\widetilde{T}$ defined on $C_{\text{pol}}(\mathbb{R}^d)$ by the formula

$$\widetilde{T}(f) := \sum_{k=0}^{\infty}\sum_{l=1}^{a_k}\int_0^{\infty} f_{k,l}(r) r^{-k}\, d\sigma_{k,l} \tag{24}$$

for $f\in C_{\text{pol}}(\mathbb{R}^d)$, where $f_{k,l}(r)$ are the Laplace–Fourier coefficients of f. Indeed, since $f\in C_{\text{pol}}(\mathbb{R}^d)$ is of polynomial growth there exist $C>0$ and $N\in\mathbb{N}$ such that $|f(x)|\le C(1+|x|^N)$. Let ω_{d-1} denote the surface area of the unit sphere. It follows from (10) that

$$|f_{k,l}(r)| \le C(1+r^N)\sqrt{\omega_{d-1}}\sqrt{\int_{\mathbb{S}^{d-1}} |Y_{k,l}(\theta)|^2\, d\theta} = C(1+r^N)\sqrt{\omega_{d-1}}, \tag{25}$$

where we used the Cauchy–Schwarz inequality and the fact that $Y_{k,l}$ is orthonormal. Hence,

$$\int_0^{\infty} |f_{k,l}(r)| r^{-k}\, d\sigma_{k,l} \le \sqrt{\omega_{d-1}}C\int_0^{\infty}(1+r^N)r^{-k}\, d\sigma_{k,l}.$$

By assumption (23) the latter integral exists, so $f_{k,l}(r)r^{-k}$ is integrable with respect to $\sigma_{k,l}$. By summing over all k and l we obtain by (23) that

$$\sum_{k=0}^{\infty}\sum_{l=1}^{a_k}\left|\int_0^{\infty} f_{k,l}(r)r^{-k}\, d\sigma_{k,l}\right| < \infty,$$

which implies the convergence of the series in (24). It follows that $\widetilde{T}$ is well-defined.

(2) Let T_0 be the restriction of the functional $\widetilde{T}$ to the space $C_c(\mathbb{R}^d)$. We will show that T_0 is continuous. Let $f \in C_c(\mathbb{R}^d)$ and suppose that f has support in the annulus $\{x \in \mathbb{R}^d : \rho \le |x| \le R\}$ (for the case $\rho = 0$ this is a ball). Then by a similar technique as above $|f_{k,l}(r)| \le \sqrt{\omega_{d-1}} \max_{\rho \le |x| \le R} |f(x)|$. Using (24) one arrives at

$$|T_0(f)| \le \max_{\rho \le |x| \le R} |f(x)| \sqrt{\omega_{d-1}} \sum_{k=0}^{\infty} \sum_{l=1}^{a_k} \int_{\rho}^{R} r^{-k}\, d\sigma_{k,l}. \tag{26}$$

(3) First consider the case when all measures $\sigma_{k,l}$ have supports in the interval $[\rho, R]$ with $R < \infty$ (cf. Remark 2.6). Then (26) and the Riesz representation theorem for compact spaces yield a representing measure μ with support in the annulus $\{x \in \mathbb{R}^d : \rho \le |x| \le R\}$. Clearly μ is a moment measure. The pseudo-positivity of μ will be proved in (5) below.

(4) In the case that $\sigma_{k,l}$ have supports in $[0, \infty)$, we apply the Riesz representation theorem given in [6, p. 41, Theorem 2.5]: there exists a unique signed measure σ such that $T_0(g) = \int_{\mathbb{R}^d} g\, d\sigma$ for all $g \in C_c(\mathbb{R}^d)$. Next we will show that the polynomials are integrable with respect to the variation of the representation measure σ. Let $\sigma = \sigma_+ - \sigma_-$ be the Jordan decomposition of σ. Following the techniques of Theorems 2.4 and 2.5 in [6, p. 42], we have the equality

$$\int_{\mathbb{R}^d} g(x)\, d\sigma_+ = \sup\{T_0(h) : h \in C_c(\mathbb{R}^d) \text{ with } 0 \le h \le g\} \tag{27}$$

which holds for any non-negative function $g \in C_c(\mathbb{R}^d)$. Let u_R be the cut-off function defined in (19). We want to estimate $\int_{\mathbb{R}^d} g(x)\, d\sigma_+$ for the function $g := |x|^N u_R(|x|^2)$. In view of (27), let $h \in C_c(\mathbb{R}^d)$ with $0 \le h(x) \le |x|^N u_R(|x|^2)$ for all $x \in \mathbb{R}^d$. Then for the Laplace–Fourier coefficient $h_{k,l}$ of h we have the estimate

$$|h_{k,l}(r)| \le \sqrt{\int_{\mathbb{S}^{d-1}} |h(r\theta)|^2\, d\theta} \sqrt{\int_{\mathbb{S}^{d-1}} |Y_{k,l}(\theta)|^2\, d\theta} \le r^N u_R(r^2) \sqrt{\omega_{d-1}}.$$

According to (24),

$$T_0(h) \le |T_0(h)| \le \sqrt{\omega_{d-1}} \sum_{k=0}^{\infty} \sum_{l=1}^{a_k} \int_0^{\infty} r^N r^{-k}\, d\sigma_{k,l} =: D_N.$$

From (27) it follows that $\int_{\mathbb{R}^d} |x|^N u_R(|x|^2)\, d\sigma_+ \le D_N$ for all $R > 0$ (note that D_N does not depend on R). By the monotone convergence theorem (note that $u_R(x) \le u_{R+1}(x)$ for all $x \in \mathbb{R}^d$) we obtain

$$\int_{\mathbb{R}^d} |x|^N\, d\sigma_+ = \lim_{R \to \infty} \int_{\mathbb{R}^d} |x|^N u_R(|x|^2)\, d\sigma_+ \le D_N.$$

Similarly one shows that $\int_{\mathbb{R}^d} |x|^N d\sigma_- < \infty$ by considering the functional $S = -T_0$. It follows that all polynomials are integrable with respect to σ_+ and σ_-. Using similar arguments it is not difficult to see that for all $g \in C^\times(\mathbb{R}^d) \cap C_{\mathrm{pol}}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} g(x)\, d\sigma = \widetilde{T}(g). \tag{28}$$

(5) It remains to prove that σ is pseudo-positive. Let $h \in C_c([0,\infty))$ be a non-negative function. The Laplace–Fourier coefficients $f_{k',l'}$ of $f(x) := h(|x|) Y_{k,l}(x)$ are given by $f_{k',l'}(r) = \delta_{k,k'} \delta_{l,l'} h(r) r^k$ and by (28) it follows that

$$\int_{\mathbb{R}^d} h(|x|) Y_{k,l}(x)\, d\sigma = \widetilde{T}(f) = \int_0^\infty f_{k,l}(r) r^{-k}\, d\sigma_{k,l} = \int_0^\infty h(r)\, d\sigma_{k,l}.$$

Since $\sigma_{k,l}$ are non-negative measures, the last term is non-negative, thus σ is pseudo-positive. The proof is complete. □

The following theorem is the main result of the present section and is an immediate consequence of Theorem 2.5. It provides a simple sufficient condition for the pseudo-positive definite functional on $\mathbb{C}[x_1, ..., x_d]$ defined in (16) to possess a pseudo-positive representing measure. Let us note that not every pseudo-positive definite functional has a pseudo-positive representing measure, see Theorem 5.7.

Theorem 2.7. *Let $T : \mathbb{C}[x_1, ..., x_d] \to \mathbb{C}$ be a pseudo-positive definite functional. Let $\sigma_{k,l}$, $k \in \mathbb{N}_0$, $l = 1, ..., a_k$, be non-negative measures with supports in $[0,\infty)$ representing the functional T as obtained in Proposition 2.2. If for any $N \in \mathbb{N}_0$,*

$$\sum_{k=0}^{\infty} \sum_{l=1}^{a_k} \int_0^\infty r^N r^{-k}\, d\sigma_{k,l} < \infty, \tag{29}$$

then there exists a pseudo-positive, signed moment measure σ such that

$$T(f) = \int_{\mathbb{R}^d} f\, d\sigma \quad \text{for all } f \in \mathbb{C}[x_1, ..., x_d].$$

It would be interesting to see whether the summability condition (29) may be weakened, cf. also the discussion at the end of Section 5.

By the uniqueness of the representing measure in the Riesz representation theorem for compact spaces we conclude from Theorem 2.5:

Corollary 2.8. *Let μ be a signed measure with compact support. Then μ is pseudo-positive if and only if μ is pseudo-positive definite as a functional on $\mathbb{C}[x_1, ..., x_d]$.*

Let us remark that Corollary 2.8 does not hold without the compactness assumption which follows from well-known arguments in the univariate case: Indeed, let ν_1 be a non-negative moment measure on $[0,\infty)$ which is not determined in the sense of Stieltjes; hence there exists a non-negative moment measure ν_2 on $[0,\infty)$ such that $\nu_1(p)=\nu_2(p)$ for all univariate polynomials. Since $\nu_1\neq\nu_2$ there exists a continuous function $h\colon[0,\infty)\to[0,\infty)$ with compact support such that $\nu_1(h)\neq\nu_2(h)$. Without loss of generality assume that

$$\int_0^\infty h(r)\,d\nu_1-\int_0^\infty h(r)\,d\nu_2<0. \tag{30}$$

For $i=1,2$ define $d\mu_i=d\theta\,d\nu_i$, so for any $f\in C(\mathbb{R}^d)$ of polynomial growth

$$\int_{\mathbb{R}^d} f\,d\mu_i=\int_0^\infty\int_{\mathbb{S}^{d-1}} f(r\theta)\,d\theta\,d\nu_i.$$

For a polynomial f let f_0 be the first Laplace–Fourier coefficient. Then $\int_{\mathbb{R}^d} f\,d\mu_i=\int_0^\infty f_0(r)\,d\nu_i$ for $i=1,2$. Since $\nu_1(p)=\nu_2(p)$ for all univariate polynomials it follows that $\int_{\mathbb{R}^d} f\,d\mu_1=\int_{\mathbb{R}^d} f\,d\mu_2$ for all polynomials. Then $\mu:=\mu_1-\mu_2$ is a signed measure which is pseudo-positive definite since $\mu(P)=0$ for all polynomials P. It is not pseudo-positive since $\mu_0(h)=\int_{\mathbb{R}^d} h(|x|)\,d\mu<0$ by (30).

3. The truncated moment problem for pseudo-positive definite functionals

The classical *truncated moment problem* of order $2n-1$ for a sequence of real numbers $s_0,s_1,s_2,\dots$ asks for conditions providing the existence of a non-negative measure σ_n on the real line such that

$$s_k=\int_{-\infty}^\infty t^k\,d\sigma_n(t)\quad\text{for } k=0,\dots,2n-1, \tag{31}$$

cf. [1, p. 30]. Let $\mathcal{P}_{\le m}$ denote the space of all univariate polynomials of degree $\le m$, and let us associate to the numbers $s_0,\dots,s_{2n}$ the linear functional $T_n\colon\mathcal{P}_{\le 2n}\to\mathbb{R}$ defined by

$$T_n(t^k):=s_k\quad\text{for } k=0,\dots,2n.$$

A necessary and sufficient condition for the existence of a non-negative measure σ_n on the real line satisfying (31) is that T_n is *positive definite on* $\mathcal{P}_{\le 2n}$ which means that

$$T_n(p^*(t)p(t))\ge 0\quad\text{for all } p\in\mathcal{P}_{\le n},$$

see [1, p. 30]. Moreover, if T_n is strictly positive definite on $\mathcal{P}_{\leq 2n}$ (i.e. it is true that $T_n(p^*(t)p(t))>0$ for all $p\in\mathcal{P}_{\leq n}$, $p\neq 0$) then one can find a whole continuum of solutions to the truncated problem of order $2n-1$.

A classical argument based on the Helly theorem shows that the solutions σ_n of the truncated moment problem of order $2n-1$ for $n\in\mathbb{N}_0$ converge to a solution σ of the moment problem. For discussions of truncated multivariate moment problems we refer to [12] and [29].

We now formulate a truncated moment problem in our framework. A basic question is of course which moments are assumed to be known. Our formulation will depend on two parameters, namely $n\in\mathbb{N}_0$ and $k_0\in\mathbb{N}_0\cup\{\infty\}$. We define the space $U_n(k_0)$ as the set of all polynomials $f\in\mathbb{C}[x_1, ..., x_d]$ such that the Laplace–Fourier series (cf. (12))

$$f(x)=\sum_{k=0}^{\deg f}\sum_{l=1}^{a_k} p_{k,l}(|x|^2)Y_{k,l}(x)$$

satisfies the restriction

$$\deg p_{k,l}\leq n \text{ for } k=0, ..., k_0 \quad \text{and} \quad p_{k,l}=0 \text{ for all } k\in\mathbb{N}_0 \text{ with } k>k_0.$$

A functional $T_n\colon U_{2n}(k_0)\to\mathbb{C}$ is called *pseudo-positive definite with respect to the orthonormal basis* $Y_{k,l}$, $l=1, ..., a_k$, $k\in\mathbb{N}_0$, $k\leq k_0$, if the component functionals $T_{n,k,l}\colon \mathcal{P}_{\leq 2n}\to\mathbb{C}$ defined by

$$T_{n,k,l}(p):=T_n(p(|x|^2)Y_{k,l}(x)) \quad \text{for } p\in\mathcal{P}_{\leq 2n}$$

satisfy

$$T_{n,k,l}(p^*p)\geq 0 \quad \text{for all } p\in\mathcal{P}_{\leq n}, \tag{32}$$

$$T_{n,k,l}(tp^*(t)p(t))\geq 0 \quad \text{for all } p\in\mathcal{P}_{\leq n-1}. \tag{33}$$

If $k_0<\infty$, the space $U_n(k_0)$ is obviously finite-dimensional and in this case we can solve the truncated moment problem.

Theorem 3.1. *Suppose that n and k_0 are natural numbers. If $T_n\colon U_{2n}(k_0)\to\mathbb{C}$ is pseudo-positive definite with respect to the orthonormal basis $Y_{k,l}$, $l=1, ..., a_k$, $k\in\mathbb{N}_0$, then there exists a pseudo-positive measure σ such that*

$$T_n(P)=\int_{\mathbb{R}^d} P(x)\,d\sigma(x)$$

for all $P\in U_{2n-1}(k_0)$.

Proof. Let $k\in\{0,...,k_0\}$ and let $T_{n,k,l}\colon \mathcal{P}_{\leq 2n}\to\mathbb{C}$ be the component functional. In the first case assume that there exists a polynomial $p_m\in\mathcal{P}_{\leq n}$, $p_m\neq 0$, with $T_{n,k,l}(p_m^* p_m)=0$. We may assume that p_m has minimal degree, say $m\leq n$. Then $T_{n,k,l}(p^*p)>0$ for all $p\in\mathcal{P}_{\leq m-1}$, $p\neq 0$. Using the Gauss–Jacobi quadrature for the functional $T_{n,k,l}$ restricted to $\mathcal{P}_{\leq 2m}$ it follows that there exist points $t_{1,k,l}<...<t_{m,k,l}\in\mathbb{R}$ and weights $\alpha_{1,k,l},...,\alpha_{m,k,l}>0$ such that the measure $\sigma_{k,l}:=\alpha_{1,k,l}\delta_{t_{1,k,l}}+...+\alpha_{m,k,l}\delta_{t_{m,k,l}}$ coincides with $T_{n,k,l}$ on $\mathcal{P}_{\leq 2m-1}$. Moreover, condition (33) implies that $t_{1,k,l}>0$. By the Cauchy–Schwarz inequality we have for all $q\in\mathcal{P}_{\leq 2n-m}$,

$$|T_{n,k,l}(qp_m(t))|^2\leq T_{n,k,l}(q^*q)T_{n,k,l}(p_m^*p_m)=0.$$

It follows that $T_{n,k,l}$ and $\sigma_{k,l}$ coincide on $\mathcal{P}_{\leq 2n-1}$. Hence we have proved that there exists a non-negative moment measure $\sigma_{k,l}$ with support in $[0,\infty)$ such that $T_{n,k,l}(p)=\int_0^\infty p(t)\,d\sigma_{k,l}(t)$ for all $p\in\mathcal{P}_{\leq 2n-1}$, and (since $t_{1,k,l}>0$)

$$\int_0^\infty r^{-k}\,d\sigma_{k,l}<\infty. \tag{34}$$

In the second case, we have $T_{n,k,l}(p^*p)>0$ for all $p\in\mathcal{P}_{\leq n}$, $p\neq 0$. Using the Gauss–Jacobi quadrature again one obtains a non-negative moment measure $\sigma_{k,l}$ with support in $[0,\infty)$, such that $T_{n,k,l}(p)=\int_0^\infty p(t)\,d\sigma_{k,l}(t)$ for all $p\in\mathcal{P}_{\leq 2n-1}$, satisfying (34).

Let $\sigma_{k,l}$ for $k=0,...,k_0$ be as above and define $\sigma_{k,l}=0$ for $k>k_0$. Define a functional $T\colon\mathbb{C}[x_1,...,x_d]\to\mathbb{C}$ by

$$T(f):=\sum_{k=0}^{\deg f}\sum_{l=1}^{a_k}\int_0^\infty f_{k,l}(r)r^{-k}\,d\sigma_{k,l}.$$

By Theorem 2.5 (note that the summability condition is satisfied) there exists a pseudo-positive moment measure σ with the same moments as T. The proof is accomplished by the fact that T and T_n agree on the subspace $U_{2n-1}(k_0)$. □

Now we consider the case $k_0=\infty$, so the space $U_n(k_0)$ is infinite-dimensional. Using the same method of proof one obtains the following result.

Theorem 3.2. *Suppose that n is a natural number and that $T_n\colon U_{2n}(\infty)\to\mathbb{C}$ is pseudo-positive definite with respect to the orthonormal basis $Y_{k,l}$, $l=1,...,a_k$, $k\in\mathbb{N}_0$. Assume that the non-negative measures $\sigma_{k,l}$ constructed in the proof of Theorem 3.1 satisfy the following conditions*

$$C_N:=\sum_{k=0}^{\infty}\sum_{l=1}^{a_k}\int_0^\infty r^N r^{-k}\,d\sigma_{k,l}<\infty$$

for any $N \in \mathbb{N}_0$. *Then there exists a pseudo-positive, signed moment measure* σ *such that*

$$T(f) = \int_{\mathbb{R}^n} f \, d\sigma \quad \textit{for all } f \in U_{2n-1}(\infty).$$

Remark 3.3. Let us note that (in the case $k_0 = \infty$) the space $U_{2n}(\infty)$ coincides with the set of all polynomials h which are polyharmonic of order $n+1$, i.e. satisfy $\Delta^{n+1} h = 0$, where Δ is the Laplace operator and Δ^j is its jth iterate. Apparently for the first time such representing measures have been considered more systematically in [26]. In the case $n=0$ the problem we consider is equivalent to the inverse magnetic problem, cf. [31].

4. Determinacy for pseudo-positive definite functionals

Let $M^*(\mathbb{R}^d)$ be the set of all *signed moment measures*, and $M_+^*(\mathbb{R}^d)$ be the set of *non-negative moment measures* on $\mathbb{R}^d$. On $M^*(\mathbb{R}^d)$ we define an equivalence relation: we say that $\sigma \sim \mu$ for two elements $\sigma, \mu \in M^*(\mathbb{R}^d)$ if and only if $\int_{\mathbb{R}^d} f \, d\sigma = \int_{\mathbb{R}^d} f \, d\mu$ for all $f \in \mathbb{C}[x_1, ..., x_d]$.

Definition 4.1. Let $\mu \in M^*(\mathbb{R}^d)$ be a pseudo-positive measure. We define

$$V_\mu = \{\sigma \in M^*(\mathbb{R}^d) : \sigma \text{ is pseudo-positive and } \sigma \sim \mu\}.$$

We say that the measure $\mu \in M^*(\mathbb{R}^d)$ is *determined in the class of pseudo-positive measures* if V_μ has only one element, i.e. is equal to $\{\mu\}$.

Recall that a positive definite functional $\phi\colon \mathcal{P}_1 \to \mathbb{R}$ is *determined in the sense of Stieltjes* if the set

$$W_\phi^{\mathrm{Sti}} := \left\{ \tau \in M_+^*([0,\infty)) : \int_0^\infty r^m \, d\tau = \phi(r^m) \text{ for all } m \in \mathbb{N}_0 \right\} \tag{35}$$

has exactly one element, cf. [7, p. 210].

According to Proposition 2.4, we can associate to a pseudo-positive measure μ the sequence of non-negative measures $\mu_{k,l}$, $k \in \mathbb{N}_0$, $l = 1, ..., a_k$, with support in $[0, \infty)$. The measures $\mu_{k,l}$ contain all information about μ. Indeed, we prove the following result.

Proposition 4.2. *Let* μ *and* σ *be pseudo-positive measures and let* $\mu_{k,l}$ *and* $\sigma_{k,l}$ *be as in Proposition* 2.4. *If* $\mu_{k,l} = \sigma_{k,l}$ *for all* $k \in \mathbb{N}_0$, $l = 1, ..., a_k$, *then* $\mu = \sigma$.

Proof. Let $h\in C_c[0,\infty)$. Then, using the assumption $\mu_{k,l}=\sigma_{k,l}$, we obtain that

$$\int_{\mathbb{R}^d} h(|x|)Y_{k,l}(x)\,d\mu=\int_0^\infty h(t)\,d\mu_{k,l}=\int_{\mathbb{R}^d} h(|x|)Y_{k,l}(x)\,d\sigma.$$

Since each $f\in C^\times(\mathbb{R}^d)\cap C_c(\mathbb{R}^d)$ is a finite linear combination of functions of the type $h(|x|)Y_{k,l}(x)$ with $h\in C_c[0,\infty)$, we obtain that $\int_{\mathbb{R}^d} f\,d\mu=\int_{\mathbb{R}^d} f\,d\sigma$ for all $f\in C^\times(\mathbb{R}^d)\cap C_c(\mathbb{R}^d)$. We apply Proposition 4.3 to see that μ is equal to σ. $\square$

The following result is proved in [7, Proposition 3.1].

Proposition 4.3. *Let μ and σ be signed measures on $\mathbb{R}^d$. If $\int_{\mathbb{R}^d} f\,d\mu=\int_{\mathbb{R}^d} f\,d\sigma$ for all $f\in C^\times(\mathbb{R}^d)\cap C_c(\mathbb{R}^d)$, then μ is equal to σ.*

We can characterize V_μ in the case that only finitely many $\mu_{k,l}$ are non-zero.

Theorem 4.4. *Let μ be a pseudo-positive measure on $\mathbb{R}^n$ such that $\mu_{k,l}=0$ for all $k>k_0$, $l=1,\dots,a_k$. Then V_μ is affinely isomorphic to the set*

$$\bigoplus_{k=0}^{k_0}\bigoplus_{l=1}^{a_k}\left\{\rho_{k,l}\in W^{\mathrm{Sti}}_{\mu^{\psi}_{k,l}}:\int_0^\infty t^{-k/2}\,d\rho_{k,l}<\infty\right\}, \tag{36}$$

where the isomorphism is given by $\sigma\mapsto(\sigma^{\psi}_{k,l})_{k=1,..,k_0,l=1,\dots,a_k}$ and the map $\psi\colon[0,\infty)\to[0,\infty)$ is defined by $\psi(t)=t^2$, cf. (9).

Proof. Let σ be in V_μ. Let $\sigma_{k,l}$ and $\mu_{k,l}$ be the unique moment measures obtained in Proposition 2.4. Then

$$\int_0^\infty h(t)\,d\sigma^{\psi}_{k,l}=\int_0^\infty h(t^2)\,d\sigma_{k,l}=\int_{\mathbb{R}^n} h(|x|^2)Y_{k,l}(x)\,d\sigma(x)$$

for all $h\in C_{\mathrm{pol}}[0,\infty)$, and an analogous equation is valid for $\mu_{k,l}$ and μ. Taking polynomials $h(t)$ we see that $\sigma_{k,l}\in W^{\mathrm{Sti}}_{\mu^{\psi}_{k,l}}$ using the assumption that $\mu\sim\sigma$. Using a simple approximation argument it is easy to see from (18) that

$$\int_0^\infty t^{-k/2}\,d\sigma^{\psi}_{k,l}=\int_{\mathbb{R}^n} Y_{k,l}\left(\frac{x}{|x|}\right)d\sigma(x).$$

Since $x\mapsto Y_{k,l}(x/|x|)$ is bounded on $\mathbb{R}^n$, say by M, we obtain the estimate

$$\left|\int_0^\infty t^{-k/2}\,d\sigma^{\psi}_{k,l}\right|\le M\int_{\mathbb{R}^n} 1\,d|\sigma|<\infty.$$

It follows that $(\sigma_{k,l}^{\psi})_{k=1,..,k_0,l=1,...,a_k}$ is contained in the set defined by (36).

Let now $\rho_{k,l}\in W^{\mathrm{Sti}}_{\mu^{\psi}_{k,l}}$ be given such that $\int_0^\infty t^{-k/2}\,d\rho_{k,l}<\infty$ for $k=1,...,k_0$, $l=1,...,a_k$. Define $\sigma_{k,l}=\rho_{k,l}^{\psi^{-1}}$ and $\sigma_{k,l}=0$ for $k>k_0$. Then by Theorem 2.5 there exists a measure $\tau\in V_\mu$ such that $\tau_{k,l}=\sigma_{k,l}$. This shows the surjectivity of the map. Let now σ and τ belong to V_μ with $\sigma_{k,l}^{\psi}=\tau_{k,l}^{\psi}$ for $k=1,...,k_0$, $l=1,...,a_k$. The property $\sigma\in V_\mu$ implies that $\sigma_{k,l}^{\psi}\in W^{\mathrm{Sti}}_{\mu^{\psi}_{k,l}}$ for all $k\in\mathbb{N}_0$, $l=1,...,a_k$, hence $\sigma_{k,l}^{\psi}=0$ for $k>k_0$, and similarly $\tau_{k,l}^{\psi}=0$. Thus $\sigma_{k,l}=\tau_{k,l}$ for all $k\in\mathbb{N}_0$, $l=1,...,a_k$, and this implies that $\sigma=\tau$ by Proposition 4.2. □

The following is a sufficient condition for a functional T to be determined in the class of pseudo-positive measures.

Theorem 4.5. *Let $T:\mathbb{C}[x_1,...,x_d]\to\mathbb{R}$ be a pseudo-positive definite functional. If the functionals $T_{k,l}:\mathbb{C}[x_1]\to\mathbb{C}$ are determined in the sense of Stieltjes then there exists at most one pseudo-positive, signed moment measure μ on $\mathbb{R}^d$ with*

$$T(f)=\int_{\mathbb{R}^d} f\,d\mu \quad \textit{for all } f\in\mathbb{C}[x_1,...,x_d]. \tag{37}$$

Proof. Let us suppose that μ and σ are pseudo-positive, signed moment measures on $\mathbb{R}^d$ representing T. Taking $f=|x|^{2N}Y_{k,l}(x)$ we obtain from (37) that

$$\int_{\mathbb{R}^d}|x|^{2N}Y_{k,l}(x)\,d\mu=T_{k,l}(t^N)=\int_{\mathbb{R}^d}|x|^{2N}Y_{k,l}(x)\,d\sigma$$

for all $N\in\mathbb{N}_0$. Let $\mu_{k,l}$ and $\sigma_{k,l}$ be as in Proposition 2.4, and consider the function $\psi:[0,\infty)\to[0,\infty)$ defined by $\psi(t)=t^2$. Then the image measures $\mu_{k,l}^{\psi}$ and $\sigma_{k,l}^{\psi}$ are non-negative measures with supports on $[0,\infty)$ such that $\int_0^\infty t^N\,d\mu_{k,l}^{\psi}=T_{k,l}(t^N)=\int_0^\infty t^N\,d\sigma_{k,l}^{\psi}$. Our assumption implies that $\mu_{k,l}^{\psi}=\sigma_{k,l}^{\psi}$, so $\mu_{k,l}=\sigma_{k,l}$. Proposition 4.2 implies that μ is equal to σ. □

In the following we want to prove the converse of the last theorem, which is more subtle. We now need some special results about *Nevanlinna extremal measures*. Let us introduce the following notation: for a non-negative measure $\phi\in M_+^*(\mathbb{R})$ we put([1])

$$[\phi]:=\{\sigma\in M_+^*(\mathbb{R}):\sigma\sim\phi\}.$$

([1]) Here in order to avoid mixing of the notations, we retain the notation $[\phi]$ from the one-dimensional case in [7].

Proposition 4.6. *Let ν be a non-negative moment measure on $\mathbb{R}$ with support in $[0,\infty)$ which is not determined in the sense of Stieltjes, or applying the notation* (35) $W_\nu^{\mathrm{Sti}} \neq \{\nu\}$. *Then there exist uncountably many $\sigma \in W_\nu^{\mathrm{Sti}}$ such that $\int_0^\infty u^{-k}\,d\sigma < \infty$ for all $k \in \mathbb{N}_0$.*

Proof. In the proof we will borrow some arguments about the Stieltjes problem as given in [10] or [23]. As in the proof of Proposition 4.1 in [23] let $\varphi\colon(-\infty,\infty)\to[0,\infty)$ be defined by $\varphi(x)=x^2$. If λ is a measure on $\mathbb{R}$ define a measure λ^- by $\lambda^-(A):=\lambda(-A)$ for each Borel set A where $-A:=\{-x:x\in A\}$. The measure is *symmetric* if $\lambda^-=\lambda$. For each $\tau\in W_\nu^{\mathrm{Sti}}$ define a measure $\tilde\tau:=\frac{1}{2}(\tau^\varphi+(\tau^\varphi)^-)$ which is clearly symmetric, in particular $\tilde\nu$ is symmetric. As pointed out in [23], the map $\tilde{\cdot}\colon W_\nu^{\mathrm{Sti}}\to[\tilde\nu]$ is injective and the image is exactly the set of all symmetric measures in the set $[\tilde\nu]$. The inverse map of $\tilde{\cdot}$ defined on the image space is just the map $\sigma\mapsto\sigma^\varphi$.

It follows that $\tilde v$ is not determined, so we can make use of the Nevanlinna theory for the indeterminate measure $\tilde\nu$, see p. 54 in [1]. We know by formulas II.4.2 (9) and II.4.2 (10) in [1] that for every $t\in\mathbb{R}$ there exists a unique Nevanlinna-extremal measure σ_t such that

$$\int_{-\infty}^{\infty}\frac{d\sigma_t(u)}{u-z}=-\frac{A(z)t-C(z)}{B(z)t-D(z)},$$

where $A(z)$, $B(z)$, $C(z)$ and $D(z)$ are entire functions. Since the support of σ_t is the zero-set of the entire function $B(z)t-D(z)$ it follows that the measure σ_t has no mass at 0 for $t\neq0$, and now it is clear that $\sigma_t([-\delta,\delta])=0$ for $t\neq0$ and suitable $\delta>0$ (this fact is pointed out at least in the reference [7, p. 210]). It follows that

$$\int_{-\infty}^{\infty}|u|^{-k}\,d\sigma_t<\infty \tag{38}$$

since the function $u\mapsto|u|^{-k}$ is bounded on $\mathbb{R}\setminus[-\delta,\delta]$ for each $\delta>0$. Using the fact that the functions $A(z)$ and $B(z)$ of the Nevanlinna matrix are odd, while the functions $B(z)$ and $C(z)$ are even, one derives that the measure $\rho_t:=\frac{1}{2}\sigma_t+\frac{1}{2}\sigma_{-t}$ is symmetric. Further from the equation $A(z)D(z)-B(z)C(z)=1$ it follows that $\rho_t\neq\rho_s$ for positive numbers $t\neq s$. By the above we know that $\rho_t^\varphi\neq\rho_s^\varphi$. This finishes the proof. □

Theorem 4.7. *Let μ be a pseudo-positive signed measure on $\mathbb{R}^d$ such that the summability assumption* (8) *holds. Then V_μ contains exactly one element if and only if each $\mu_{k,l}^\psi$ is determined in the sense of Stieltjes.*

Proof. Let $\mu_{k,l}$ be the component measures as defined in Proposition 2.4. Assume that $V_\mu=\{\mu\}$ but that some $\tau:=\mu_{k_0,l_0}^{\psi}$ is not determined in the sense of Stieltjes, where $\psi(t)=t^2$ for $t\in[0,\infty)$. By Proposition 4.6 there exists a measure $\sigma\in W_\tau^{\mathrm{Sti}}$ such $\sigma\neq\tau$ and $\int_0^\infty r^{-k}\,d\sigma<\infty$. By Theorem 2.5 there exists a pseudo-positive moment measure $\tilde{\mu}$ representing the functional

$$\widetilde{T}(f):=\sum_{\substack{k=0\\ k\neq k_0}}^{\infty}\sum_{\substack{l=1\\ l\neq l_0}}^{a_k}\left(\int_0^\infty f_{k,l}(r)r^{-k}\,d\mu_{k,l}+\int_0^\infty f_{k_0,l_0}(r)r^{-k}\,d\sigma^{\psi^{-1}}\right).$$

Then $\tilde{\mu}$ is different from μ since $\sigma^{\psi^{-1}}\neq\mu_{k_0,l_0}$ and $\tilde{\mu}\in V_\mu$ since $\sigma\in W_\tau^{\mathrm{Sti}}$. This contradiction shows that μ_{k_0,l_0}^{ψ} is determined in the sense of Stieltjes. The sufficiency follows from Theorem 4.5. The proof is complete. □

5. Miscellaneous results

In this section we provide some examples and results on pseudo-positive measures which throw more light on these new notions.

5.1. The univariate case

As we mentioned in the introduction the non-negative spherically symmetric measures are pseudo-positive and, as it is easy to see from (3), our theory reduces to the classical Stieltjes moment problem. Other pseudo-positive measures μ for which our theory reduces essentially to the Stieltjes one-dimensional moment problem are those having only one non-zero component measure $\mu_{k,l}$; this is the problem $\int_0^\infty r^{k+2j}\,d\mu_{k,l}(r)=c_{j,k,l}$ for $j=0,1,2,\ldots$, (cf. (18) and (3)).

On the other hand it is instructive to consider the univariate case of our theory: then $d=1$, $\mathbb{S}^0=\{-1,1\}$, and the normalized measure is $\omega_0(\theta)=\frac{1}{2}$ for all $\theta\in\mathbb{S}^0$. The harmonic polynomials are the linear functions, their basis are the two functions defined by $Y_0(x)=1$ and $Y_1(x)=x$ for all $x\in\mathbb{R}$. The following is now immediate from the definitions.

Proposition 5.1. *Let $d=1$. A functional $T\colon\mathbb{C}[x]\to\mathbb{C}$ is pseudo-positive definite if and only if $T(p^*(x^2)p(x^2))\geq 0$ and $T(xp^*(x^2)p(x^2))\geq 0$ for all $p\in\mathbb{C}[x]$.*

Recall that a functional $T\colon\mathbb{C}[x]\to\mathbb{C}$ defines a *Stieltjes moment sequence* if $T(q^*(x)q(x))\geq 0$ and $T(xq^*(x)q(x))\geq 0$ for all $q\in\mathbb{C}[x]$, hence this property implies pseudo-positive definiteness; the next example shows that the converse is not true.

Example 5.2. Let σ be a non-negative finite measure on the interval $[a,b]$ with $a>0$. Then the functional $T\colon \mathbb{C}[x]\to\mathbb{C}$ defined by

$$T(f)=\int_a^b f(x)\,d\sigma-\int_a^b f(-x)\,d\sigma$$

is pseudo-positive definite but not positive definite.

Proof. As pointed out in [28, Chapter 4.1], the Laplace–Fourier expansion of f is given by $f(r\theta)=f_0(r)Y_0(\theta)+f_1(r)Y_1(\theta)$ for $x=r\theta$ with $r=|x|$ and $\theta\in\mathbb{S}^0$, where

$$f_0(r)=\int_{\mathbb{S}^0} f(r\theta)Y_0(\theta)\,d\omega_0(\theta)=\frac{f(r)+f(-r)}{2},$$

$$f_1(r)=\int_{\mathbb{S}^0} f(r\theta)Y_1(\theta)\,d\omega_0(\theta)=\frac{f(r)-f(-r)}{2}.$$

Since f_0 is even, f_1 is odd and $f=f_0+f_1$ we infer that $T(f)=2\int_a^b f_1(r)\,d\sigma$. By Proposition 2.3, T is pseudo-positive definite. As $T(1)=0$ and $T\neq 0$ it is clear that T is not positive definite. □

5.2. A criterion for pseudo-positivity

The following is a simple criterion for pseudo-positivity.

Proposition 5.3. *Let μ be a signed moment measure on $\mathbb{R}^d$. Assume that μ has a density $w(x)$ with respect to the Lebesgue measure dx such that $\theta\mapsto w(r\theta)$ is in $L^2(\mathbb{S}^{d-1})$ for each $r>0$. If the Laplace–Fourier coefficients of w,*

$$w_{k,l}(r):=\int_{\mathbb{S}^{d-1}} w(r\theta)Y_{k,l}(\theta)\,d\theta,$$

are non-negative then μ is pseudo-positive and

$$d\mu_{k,l}(r)=r^{k+d-1}w_{k,l}(r), \tag{39}$$

$$\int_0^\infty r^{-k}\,d\mu_{k,l}(r)=\int_0^\infty w_{k,l}(r)r^{d-1}\,dr \tag{40}$$

if the last integral exists. The measures $\mu_{k,l}$ are defined by means of equality (18).

Proof. Since μ has a density $w(x)$ we can use polar coordinates to obtain, for $f\in C_{\text{pol}}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} f\,d\mu=\int_{\mathbb{R}^d} f(x)w(x)\,dx=\int_0^\infty\int_{\mathbb{S}^{d-1}} f(r\theta)w(r\theta)r^{d-1}\,d\theta\,dr. \tag{41}$$

For any $h\in C_{\text{pol}}[0,\infty)$ we put $f(x)=h(|x|)Y_{k,l}(x)$. Then we obtain

$$\int_{\mathbb{R}^d} h(|x|)Y_{k,l}(x)\,d\mu=\int_0^\infty\int_{\mathbb{S}^{d-1}} h(r)r^{k+d-1}Y_{k,l}(\theta)w(r\theta)\,d\theta\,dr. \tag{42}$$

Since $\theta\mapsto w(r\theta)$ is in $L^2(\mathbb{S}^{d-1})$, we know that $w_{k,l}(r)=\int_{\mathbb{S}^{d-1}} w(r\theta)Y_{k,l}(\theta)\,d\theta$. Hence, by the definition of $\mu_{k,l}$, we obtain

$$\int_0^\infty h(r)\,d\mu_{k,l}:=\int_{\mathbb{R}^d} h(|x|)Y_{k,l}(x)\,d\mu=\int_0^\infty h(r)w_{k,l}(r)r^{k+d-1}\,dr. \tag{43}$$

Thus the measure μ is pseudo-positive, and (39) follows. Let us prove (40): we define the cut-off functions $h_m\in C_{\text{pol}}[0,\infty)$ such that $h_m(t)=t^{-k}$ for $t\geq 1/m$ and such that $h_m\leq h_{m+1}$. Now use (43) and the monotone convergence theorem to obtain (40). □

5.3. Examples in the two-dimensional case

Let us consider the case $d=2$, and take the usual orthonormal basis of solid harmonics, defined by $Y_0(e^{it})=1/2\pi$ and

$$Y_{k,1}(re^{it})=\frac{1}{\sqrt{\pi}}r^k\cos kt \text{ and } Y_{k,2}(re^{it})=\frac{1}{\sqrt{\pi}}r^k\sin kt \quad \text{for } k\in\mathbb{N}. \tag{44}$$

We define a density $w^{(\alpha)}\colon\mathbb{R}^n\to[0,\infty)$, depending on the parameter $\alpha>0$, by

$$\begin{aligned} w^{(\alpha)}(re^{it})&:=(1-r^\alpha)P(re^{it}) && \text{for } 0\leq r<1,\\ w^{(\alpha)}(re^{it})&=0 && \text{for } r\geq 1; \end{aligned}$$

here the function $P(re^{it})$ is the Poisson kernel for $0\leq r<1$ given by (see e.g. 5.1.16 in [2, p. 243])

$$P(re^{it}):=\frac{1-r^2}{1-2r\cos t+r^2}=1+\sum_{k=1}^\infty 2r^k\cos kt. \tag{45}$$

By Proposition 5.3, the measure $d\mu^\alpha:=w^{(\alpha)}(x)\,dx$ is pseudo-positive. For $k>0$, by (40) and (44) we obtain

$$\int_{\mathbb{R}^d} r^{-k}\,d\mu^\alpha_{k,1}=2\sqrt{\pi}\int_0^1 r^{k+1}(1-r^\alpha)\,dr=\frac{2\sqrt{\pi}\alpha}{(k+2)(\alpha+k+2)}.$$

It follows that $w^{(\alpha)}(x)\,dx$ satisfies the summability condition (8).

On the other hand, there exist pseudo-positive measures which do not satisfy the summability condition (8).

Proposition 5.4. *Let $w(re^{it}) := P(re^{it})$ for $0 \le r < 1$ and $w(re^{it}) := 0$ for $r \ge 1$, where $P(x)$ is given by (45). Then $d\mu := w(x)\,dx$ is a pseudo-positive, non-negative moment measure which does not satisfy the summability condition (8).*

Proof. It follows from (40) for $k \ge 1$

$$\int r^{-k}\,d\mu_{k,1} = \int_0^\infty w_{k,1}(r) r^{d-1}\,dr = 2\sqrt{\pi}\int_0^1 r^{k+1}\,dr = \frac{2\sqrt{\pi}}{k+2},$$

so we see that the summability condition (8) is not fulfilled. □

5.4. The summability condition

The next result shows that the spectrum of the measures $\sigma_{k,l}$ is contained in the spectrum of the representation measure μ.

Theorem 5.5. *Let $\sigma_{k,l}$ be non-negative measures on $[0,\infty)$. If the functional $T: \mathbb{C}[x_1, ..., x_d] \to \mathbb{C}$ defined by (16) possesses a representing moment measure μ with compact support then*

$$\sigma_{k,l}(\{|x|^2\}) \le \max_{\theta\in\mathbb{S}^{d-1}} |Y_{k,l}(\theta)|\,|x|^k |\mu|(|x|^2\mathbb{S}^{d-1})$$

for any $x \in \mathbb{R}^d$, where $|\mu|$ is the total variation and $|x|^2\mathbb{S}^{d-1} = \{|x|^2\theta : \theta \in \mathbb{S}^{d-1}\}$.

Proof. Let the support of μ be contained in B_R. Let $x_0 \in \mathbb{R}^d$ be given. For every univariate polynomial $p(t)$ with $p(|x_0|^2) = 1$ we have

$$\sigma_{k,l}(\{|x_0|^2\}) \le \int_0^\infty p(r^2)\,d\sigma_{k,l} \le \int_{\mathbb{R}^d} \left|p(|x|^2) Y_{k,l}(x)\right| d|\mu|$$
$$\le \max_{\theta\in\mathbb{S}^{d-1}} |Y_{k,l}(\theta)| \int_{\mathbb{R}^d} \left|p(|x|^2)\right| |x|^k\,d|\mu|.$$

Now choose a sequence of polynomials p_m with $p_m(|x_0|^2) = 1$ which converges on $[0,R]$ to the function f defined by $f(|x_0|^2) = 1$ and $f(t) = 0$ for $t \ne |x_0|^2$. Since $|\mu|$ has support in B_R Lebesgue's convergence theorem shows that

$$\sigma_{k,l}(\{|x_0|^2\}) \le \max_{\theta\in\mathbb{S}^{d-1}} |Y_{k,l}(\theta)| \int_{\mathbb{R}^d} |f(x)|\,|x|^k\,d|\mu|.$$

This implies our statement. □

The following result shows that the summability condition is sometimes equivalent to the existence of a pseudo-positive representing measure.

Corollary 5.6. *Let $d = 2$. Let $\sigma_{k,l}$ be non-negative measures on $[0,\infty)$ and assume that they have disjoint and at most countable supports. If the functional*

$T\colon \mathbb{C}[x_1,x_2]\to\mathbb{C}$ defined by (16) possesses a representing moment measure with compact support then

$$\sum_{k=0}^{\infty}\sum_{l=1}^{a_k}\int_0^{\infty} r^{-k}\,d\sigma_{k,l}(r)<\infty.$$

Proof. Let $\Sigma_{k,l}$ be the support set of $\sigma_{k,l}$. The last theorem shows that $\sigma_{k,l}(\{0\})=0$, hence $0\notin\Sigma_{k,l}$. Moreover it tells us that

$$\int_0^{\infty} r^{-k}\,d\sigma_{k,l}(r)\le \max_{\theta\in\mathbb{S}^{d-1}}|Y_{k,l}(\theta)|\cdot\sum_{r\in\Sigma_{k,l}}|\mu|(r\mathbb{S}^{d-1}).$$

Since $d=2$ we know that $\max_{\theta\in\mathbb{S}^{d-1}}|Y_{k,l}(\theta)|\le 1$. Hence

$$\sum_{k=0}^{\infty}\sum_{l=1}^{a_k}\int_0^{\infty} r^{-k}\,d\sigma_{k,l}(r)\le\sum_{k=0}^{\infty}\sum_{l=1}^{a_k}\sum_{r\in\Sigma_{k,l}}|\mu|(r\mathbb{S}^{d-1})\le|\mu|(\mathbb{R}^d),$$

where the last inequality follows from the fact that $\Sigma_{k,l}$ are pairwise disjoint. □

Recall that the converse of the last theorem holds under the additional assumption that the supports of all $\sigma_{k,l}$ are contained in some interval $[0,R]$.

Theorem 5.7. *There exists a functional $T\colon\mathbb{C}[x_1,...,x_d]\to\mathbb{C}$ which is pseudo-positive definite but does not possess a pseudo-positive representing measure.*

Proof. Let σ be a non-negative measure over $[0,R]$. Let $f\in\mathbb{C}[x_1,..,x_d]$ and let $f_{k,l}$ be the Laplace–Fourier coefficient of f. By Proposition 2.3 it is clear that

$$T(f):=\int_0^R f_{1,1}(r)r^{-1}\,d\sigma(r)$$

is pseudo-positive definite. We take now for σ the Dirac functional at $r=0$. Suppose that T has a signed representing measure μ which is pseudo-positive. Then the measure μ_{11} is non-negative, and it is defined by the equation $\int_0^\infty h(r)\,d\mu_{11}(r):=\int_{\mathbb{R}^n} h(|x|)Y_{11}(x)\,d\mu$ for any continuous function $h\colon[0,\infty)\to\mathbb{C}$ with compact support. Take now $h(r)=r^2$. Then by Proposition 2.4,

$$\int_0^{\infty} r^2\,d\mu_{11}(r)=\int_{\mathbb{R}^n}|x|^2Y_{11}(x)\,d\mu=T(|x|^2Y_{11}(x))=0.$$

It follows that μ_{11} has support $\{0\}$. On the other hand, if we take a sequence of functions $h_m\in C_c([0,\infty))$ such that $h_m\to 1_{\{0\}}$, then we obtain

$$\mu_{11}(\{0\})=\lim_{m\to\infty}\int_{\mathbb{R}^n}h_m(|x|)Y_{11}(x)\,d\mu.$$

But $h_m(|x|)Y_{11}(x)$ converges to the zero-function, and Lebesgue's theorem shows that $\mu_{11}(\{0\})=0$, so $\mu_{11}=0$. This is a contradiction since

$$\int_0^\infty 1\, d\mu_{11}(r) = \int_{\mathbb{R}^n} Y_{11}(x)\, d\mu = T(Y_{11}) = \int_0^R 1\, d\sigma(r) = 1.$$

The proof is complete. □

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