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Some preserver problems on algebras of smooth functions

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Abstract. We study bijections between algebras of smooth functions preserving certain parts of its structure. In particular, we show that multiplicative bijections are implemented by diffeomorphisms and they are automatically algebra isomorphisms. This confirms a conjecture by Mrčun and Šemrl.

1. Introduction

This paper deals with bijections of algebras of (real) smooth functions that preserve certain parts of its structure.

By a manifold we understand a Hausdorff topological space X locally homeomorphic to a fixed Banach space E of dimension at least 1. We are not assuming X second-countable, connected or paracompact. A C^k manifold, $1\leq k\leq\infty$, is a manifold with an atlas whose transition maps are all k-times continuously (Fréchet) differentiable, with the usual convention when $k=\infty$. We require moreover that E admits a C^k bump function (one with nonempty bounded support). This condition is automatically satisfied when E is finite-dimensional. For the general case, please consult [3] or [8, Chapter III]. A less intimidating reference is [4].

It is worth noticing that in this setting the form of the isomorphisms between algebras of smooth functions has been elucidated only very recently, even in finite dimensions. Indeed, as observed by Weinstein (2003), the commonly known proofs that algebra isomorphisms between algebras of smooth functions are all induced by composition with a C^k diffeomorphism between the underlying manifolds strongly depend either on a second countability assumption or on paracompactness. (See [5]

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for a ‘typical’ proof in infinite dimensions.) Soon afterwards, (2005) Grabowski and Mrčun filled the gap with two essentially different proofs appearing in [6] and [10].

Here, we generalise this result in two ways. We will show that every additive bijection $T\colon C^k(Y)\to C^k(X)$ preserving the order in both directions has the form

$$Tf(x)=a(x)f(\tau(x)),\quad f\in C^k(Y),\ x\in X,$$

where $a=T1$ is strictly positive and $\tau\colon X\to Y$ is a C^k diffeomorphism.

Also, we prove that every multiplicative isomorphism $T\colon C^k(Y)\to C^k(X)$ is automatically linear (actually what we shall see is that T is just composition with a C^k diffeomorphism). This settles a conjecture by Mrčun and Šemrl who discovered the result and proved it for finite k in [11] (see [2] for related material).

Both results depend on a not very satisfying and rather incomplete, yet useful, description of the bijections $T\colon C^k(Y)\to C^k(X)$ preserving the order in both directions (nothing more is assumed) that could be considered as the main result of this note (Theorem 1).

The paper though self-contained is based on the ideas of Shirota’s seminal paper [12]. However the functional representation for T in Theorem 1 originates with Kaplansky’s classical [7]. See [1] for further references.

2. Order preserving bijections

Let us present our main result. The order we consider on $C^k(X)$ is that inherited from $\mathbb{R}$, so $f\le g$ means $f(x)\le g(x)$ for every $x\in X$.

Theorem 1. *Let $T\colon C^k(Y)\to C^k(X)$ be a bijective mapping preserving the order in both directions. Then there is a homeomorphism $\tau\colon X\to Y$ such that*

$$Tf(x)=t(x,f(\tau(x))),\quad f\in C^k(Y),\ x\in X,$$

where $t\colon X\times\mathbb{R}\to\mathbb{R}$ is given by $t(x,c)=Tc(x)$.

The proof consists of several steps. First of all we need to introduce some relations in the space of smooth functions. This will be done in a somewhat artificial way because $C^k(X)$ is not a lattice.

Given $f\in C^k(X)$, the *support* of f, denoted $\operatorname{supp} f$, is the closure of the set $\{x\in X: f(x)\neq 0\}$. We write U_f for the interior of $\operatorname{supp} f$. This is clearly a regular open set (one which equals the interior of its closure). Whether every regular open set can be obtained in this way is open to reflection.

Set $C^k_+(X)=\{f\in C^k(X): f\geq 0\}$. Our immediate aim is to show that, for $f,g\in C^k_+(X)$, the relations $U_f\subset U_g$ and $\overline{U}_f\subset U_g$ can be expressed within the order structure of $C^k_+(X)$. To this end, given $f_1,f_2\in C^k_+(X)$, put

$$f_1\cap f_2=\{f\in C^k_+(X): f\leq f_i \text{ for } i=1,2\}.$$

Also, let us declare $f\subset g$ when for every $h\in C^k_+(X)$ one has $f\cap h=\{0\}$ whenever $g\cap h=\{0\}$. Finally we write $f\Subset g$ if, whenever the family $\{h_\alpha\}_\alpha$ has an upper bound in $C^k_+(X)$ and $h_\alpha\subset f$ for all α, there is an upper bound $h\in C^k_+(X)$ such that $h\subset g$.

Step 1.1. *Let $f,g\in C^k_+(X)$. Then*
(a) *$f\cap g=\{0\}$ if and only if $U_f\cap U_g=\varnothing$.*
(b) *$f\subset g$ if and only if $U_f\subset U_g$ which holds if and only if* $\operatorname{supp} f\subset \operatorname{supp} g$.
(c) *If $f\Subset g$, then* $\operatorname{supp} f\subset U_g$ *and $\overline{U}_f\subset U_g$.*
(d) *If there is $u\in C^k_+(X)$ such that $u=0$ off U_g and $u=1$ on U_f, then $f\Subset g$.*

Proof. (a) The 'if' part is clear, so assume $U_f\cap U_g\neq\varnothing$. After a moment's reflection we see there is some point where both f and g are strictly positive. It follows that $f\cap g$ contains some nonzero function.

(b) By the very definition, we have $f\subset g$ if and only if $g\cap h=\{0\}$ implies $f\cap h=\{0\}$. By part (a), this is equivalent to '$U_g\cap U_h=\varnothing$ implies $U_f\cap U_h=\varnothing$', which is clearly equivalent to $U_f\subset U_g$. The last equivalence is obvious.

(c) Assume $f\Subset g$. For each $x\in U_f$ pick $h_x\in C^k_+(X)$ such that $h_x\leq 1$ and $h_x(x)=1$, with $h_x\subset f$. Now, the hypothesis gives an upper bound h for the family $\{h_x\}_x$ such that $h\subset g$. Clearly $h\geq 1$ on U_f and thus $\overline{U}_f\subset U_h\subset U_g$.

(d) Assume $u=0$ off U_g and $u=1$ on U_f. If $h_\alpha\subset f$ and h is an upper bound for $\{h_\alpha\}_\alpha$, then uh is also an upper bound and quite clearly $uh\subset f$. $\square$

Throughout this section T will be as in Theorem 1. Besides, we assume $T0=0$. This causes no loss of generality since $f\mapsto Tf-T0$ preserves the order in both directions and sends 0 to 0.

Step 1.2. *Given $f,g,h\in C^k_+(X)$ one has $f\leq g$ on U_h if and only if $g\cap u$ contains $f\cap u$ for every $u\subset h$.*

Therefore, given $f,g,h\in C^k_+(Y)$, one has $f\leq g$ on U_h if and only if $Tf\leq Tg$ on U_{Th}. The same is true if we replace $\leq$ by $\geq$ or $=$.

Proof. If $f\leq g$ on U_h and $u\subset h$, then it is straightforward that every function lower than f and u is lower than g, so $f\cap u\subset g\cap u$.

As for the converse, assume $f(x)>g(x)$ for some $x\in U_h$. Take some $v\in C_+^k(X)$ such that $v\leq 1$ and $v(x)=1$ and having support in the set $U_h\cap\{z: f(z)>g(z)\}$. Then $u=vf$ belongs to $u\cap f$ but not to $u\cap g$. The 'therefore' part is now obvious. □

Let $R^k(X)$ denote the class of those regular open sets of X arising as U_h for $h\in C_+^k(X)$, and similarly for Y. We consider in $R^k(X)$ the (partial) order given by inclusion. It is not hard to see that $R^k(X)$ then becomes a lattice, but we will not use this fact.

Notice that our assumptions on the model Banach space already imply that $R^k(X)$ is a base for the topology of X. We shall use this fact without further mention in the sequel. We consider the mapping $\mathfrak{T}\colon R^k(X)\to R^k(Y)$ sending U_{Th} to U_h. In view of Step 1.1(b), $\mathfrak{T}$ is a well-defined bijection preserving (the order given by) inclusion in both directions.

Step 1.3. *Given $f,g\in C^k(X)$ and $U\in R^k(X)$ one has $f\leq g$ on $\mathfrak{T}(U)$ if and only if $Tf\leq Tg$ on U. The same is true replacing $\leq$ by $\geq$ or $=$.*

Proof. The case $f,g\geq 0$ is contained in Step 1.2. Thus we have the following: for each $U\in R^k(X)$ there is $V=\mathfrak{T}(U)$ in $R^k(Y)$ such that whenever $f,g\in C_+^k(Y)$ one has $f\leq g$ on V if and only if $Tf\leq Tg$ on U. Moreover this property characterises V in $R^k(Y)$.

But 0 plays no special rôle here (that is, in the ordered set of smooth functions) and we have an analogous statement for each fixed $u\in C^k(Y)$: there is a bijection $\mathfrak{T}_u\colon R^k(X)\to R^k(Y)$ preserving inclusions in both directions such that whenever $f,g\in C^k(Y)$ satisfy $f,g\geq u$ and $U\in R^k(X)$ one has $f\leq g$ in $V=\mathfrak{T}_u(U)$ if and only if $Tf\leq Tg$ on U. And, moreover, this property characterises V amongst the regular open sets of $R^k(Y)$.

We want so see that $\mathfrak{T}_u$ does not depend on u. First, is easily checked that $\mathfrak{T}_u=\mathfrak{T}_v$ if $v\leq u$. Now, for arbitrary $u,v\in C^k(Y)$, the function $w=\sqrt{1+u^2+v^2}$ is in $C^k(Y)$ and dominates both u and v, so

$$\mathfrak{T}_u=\mathfrak{T}_w=\mathfrak{T}_v=\mathfrak{T}_0=\mathfrak{T}$$

and $\mathfrak{T}_u=\mathfrak{T}$ is independent of u.

To complete the proof, take $f,g\in C^k(Y)$ and $U\in R^k(X)$. Set $V=\mathfrak{T}(U)$ and $u=-\sqrt{1+f^2+g^2}$, so that $u\leq f,g$. As $V=\mathfrak{T}_u(U)$ we have $f\leq g$ on V if and only if $Tf\leq Tg$ on U and we are done. □

Next, consider the set-valued map $\tau\colon X\to 2^Y$ given by

$$\tau(x)=\bigcap\mathfrak{T}(U),$$

where the intersection is taken over those $U\in R^k(X)$ containing the point x.

Step 1.4. *The set $\tau(x)$ is a singleton for every $x\in X$. The map sending x to the only point in $\tau(x)$ is a homeomorphism (still denoted τ).*

Proof. Let $\{U_n\}_{n=1}^\infty$ be a neighbourhood base at x taken from $R^k(X)$ satisfying $\overline{U}_{n+1}\subset U_n$. We require moreover that for each n there is $h\in C_+^k(X)$, depending on n, such that $U_n=U_h$ and $h=1$ on U_{n+1}. This guarantees that if $V_n=\mathfrak{T}(U_n)$, then $\overline{V}_{n+1}\subset V_n$, by parts (c) and (d) of Step 1.1. Therefore,

$$\tau(x)=\bigcap_{n=1}^\infty \mathfrak{T}(U_n)=\bigcap_{n=1}^\infty V_n=\bigcap_{n=1}^\infty \overline{V}_n.$$

Let us see that $\tau(x)$ cannot be empty. Assume on the contrary that $\tau(x)=\varnothing$. For each n, let $f_n\in C_+^k(X)$ be such that $f_n(x)=n$ and $\operatorname{supp} f_n\subset U_n$. Take g_n such that $Tg_n=f_n$. Then $U_{g_n}\subset V_n$. We claim that $\{g_n\}_{n=1}^\infty$ is locally finite. Indeed, pick $y\in Y$. As $y\notin\bigcap_n \overline{V}_n$ there is $m\in\mathbb{N}$ such that $y\notin\overline{V}_m$ and we can choose a neighbourhood of y, say W, that does not meet $\overline{V}_m$. As $\operatorname{supp} g_n\subset\overline{V}_n$ we see that g_n vanishes on W as long as $n\geq m$.

Therefore the sequence $\{g_n\}_{n=1}^\infty$ has an upper bound in $C^k(Y)$, namely $\sum_{n=1}^\infty g_n$, while $\{f_n\}_{n=1}^\infty$ lacks it. A contradiction.

We see that $\tau(x)$ has exactly one point. If $y\in\tau(x)$, then by the very definition of τ, we have $y\in\mathfrak{T}(U)$ as long as $U\in R^k(X)$ contains x. Let $S\colon C^k(X)\to C^k(Y)$ be the inverse of T, $\mathfrak{S}\colon R^k(Y)\to R^k(X)$ the order isomorphism associated to S and $\sigma\colon Y\to 2^X$ the set-valued function associated to $\mathfrak{S}$. Taking into account the trivial fact that $\mathfrak{S}$ is nothing but the inverse of $\mathfrak{T}$, we get

$$\sigma(y)=\bigcap_{y\in V}\mathfrak{S}(V)\subset\bigcap_{x\in U}\mathfrak{S}(\mathfrak{T}(U))=\bigcap_{x\in U}U=\{x\}.$$

And since $\sigma(y)\neq\varnothing$ we see that $x=\sigma(y)$. Hence, for $U\in R^k(X)$, we have $x\in U$ if and only if $y\in\mathfrak{T}(U)$. Equivalently, given $V\in R^k(Y)$, one has $y\in V$ if and only if $x\in\mathfrak{S}(V)$. But $\mathfrak{S}$ is an order isomorphism onto $R^k(Y)$ (which separates points of Y) and so there is at most one y satisfying that condition.

This shows that, for every $x\in X$, $\tau(x)$ is a singleton. That the map $X\to Y$, sending x into the only element of $\tau(x)$, is continuous is trivial. That this map is a homeomorphism follows by symmetry: the map sending each y into the only element of $\sigma(y)$ is the inverse of $\tau\colon X\to Y$. $\square$

Step 1.5. *Let $f,g\in C^k(Y)$ and $x\in X$. If $f(\tau(x))=g(\tau(x))$, then $Tf(x)=Tg(x)$.*

Proof. Write $y=\tau(x)$. It clearly suffices to see that if $f(y)\le g(y)$, then $Tf(x)\le Tg(x)$. Or else, that if $Tf(x)>Tg(x)$, then $f(y)>g(y)$. So, assume on the contrary that $Tf(x)>Tg(x)$, but $f(y)\le g(y)$. As $Tf\ge Tg$ on some neighbourhood of x we know from Steps 1.2 and 1.3 that $f\ge g$ on some neighbourhood of y, whence $f(y)=g(y)$ and $Df=Dg$ at y—where $f-g$ attains a local minimum.

Let h be any function in $C^k(Y)$ such that $h(y)=f(y)$ and $Dh(y)\ne Df(y)$. Then every neighbourhood of y contains points (and so an open set) where $h>f$ as well as points where $h<g$. It follows that $Tf(x)\le Th(x)\le Tg(x)$, a contradiction. □

This completes the proof of Theorem 1.

Applications. Our first application, in the spirit of [9], shows that the behaviour of a bijection that preserves order depends largely on the action on constant functions.

Corollary 1. *Let $T\colon C^k(Y)\to C^k(X)$ be a bijection preserving the order in both directions.*

(a) *T is linear (or additive) if and only if it is linear (additive) on the set of constant functions on Y. In this case $Tf(x)=a(x)f(\tau(x))$, where $a=T1$ is strictly positive and τ is a C^k diffeomorphism.*

(b) *T is an algebra (or ring) isomorphism if and only if it sends each constant on Y into the same constant on X. If so, $Tf(x)=f(\tau(x))$ and τ is a C^k diffeomorphism.*

It is worth noticing that a ring isomorphism between algebras of smooth functions must preserve order, so Corollary 1 implies the following result.

Corollary 2. (Grabowski [6], Mrčun [10]) *Every ring isomorphism between $C^k(Y)$ and $C^k(X)$ is given by composition with a C^k diffeomorphism between the underlying manifolds.*

3. Multiplicative bijections

In this section we move to the multiplicative structure of smooth functions. The following result settles a conjecture in [11], where the case $k<\infty$ is proved.

Theorem 2. *Every multiplicative bijection $C^k(Y)\to C^k(X)$ is induced by composition with a C^k diffeomorphism. Therefore they are all linear.*

As before, we break the proof into a number of steps. From now on we assume that T is a bijection satisfying $T(fg)=Tf\cdot Tg$ for all $f,g\in C^k(Y)$. We use the same notation as in the proof of Theorem 1, with the only exception that we will use sets U_f for arbitrary f. Note, however, that U_f is the same as U_{f^2}, so this leads to the same class of regular open sets.

First of all, we remark that $T0=0$ and $T1=1$. Also, T preserves the set of strictly positive functions (they are just the invertible squares). Moreover, T acts in a local way, meaning that, given f, g and h in $C^k(Y)$, one has $f=g$ on U_h if and only if $Tf=Tg$ on U_{Th}. And this is so because the former condition is equivalent to $fh=gh$. It follows that $T(-1)=-1$ since -1 is the only idempotent which does not agree with 1 on some nonempty open set. Next notice that $U_f\subset U_g$ if and only if $U_{Tf}\subset U_{Tg}$ since the former just means that $gh=0$ implies $fh=0$ for every $h\in C^k(Y)$.

So, as we did in Section 1, we can define an order isomorphism $\mathfrak{T}\colon R^k(X)\to R^k(Y)$ taking $\mathfrak{T}(U_{Th})=U_h$ for $h\in C^k(Y)$. Now, consider the set-valued map $\tau\colon X\to 2^Y$ given by

$$\tau(x)=\bigcap \mathfrak{T}(U),$$

where the intersection is taken over those $U\in R^k(X)$ containing the point x.

Step 2.1. *The set $\tau(x)$ is a singleton for every $x\in X$. The map sending x to the only point in $\tau(x)$ is a homeomorphism between X and Y.*

Proof. Let $\{U_n\}_{n=1}^\infty$ be a neighbourhood base at x consisting of regular open sets of $R^k(X)$, with $\overline{U}_{n+1}\subset U_n$. We require moreover that for each n there is $h\in C^k(X)$, depending on n, such that $h=1$ on U_{n+1} and $h=0$ outside U_n. This guarantees that if $V_n=\mathfrak{T}(U_n)$, then $\overline{V}_{n+1}\subset V_n$, and so

$$\tau(x)=\bigcap_{n=1}^{\infty}\mathfrak{T}(U_n)=\bigcap_{n=1}^{\infty}V_n=\bigcap_{n=1}^{\infty}\overline{V}_n.$$

Let us see that $\tau(x)$ is not empty. If we assume the contrary, passing to a subsequence if necessary we find a sequence $\{y_n\}_{n=1}^\infty$ with $y_n\in V_n\setminus\overline{V}_{n+1}$ and a sequence of regular open sets $\{W_n\}_{n=1}^\infty$ having disjoint closures, with $y_n\in W_n$.

Now, for even n, let $h_n\in C^k(Y)$ be a hat function around y_n (that is, one which agrees with 1 on a neighbourhood of the point) with support in W_n and set

$$f=\sum_{n\text{ even}}h_n.$$

As the sum is locally finite, f is in $C^k(Y)$ and has the following property: every V_n contains an open set where $f=1$ and also an open set where f vanishes. Therefore Tf is discontinuous at y and we have reached a contradiction.

That $\tau(x)$ has exactly one point and that sending x to that point defines a homeomorphism is shown as in Step 1.4. □

We pause for the construction of certain real-valued functions of a single real variable which we will use later.

Step 2.2. *Let* $0<r<1$ *be fixed. There exists* C^∞ *smooth functions* $\varphi, \phi\colon \mathbb{R}\to[0,1)$ *having the following properties*:

(a) *For every* $s\neq r$ *and* N *there is a nonempty open interval* I *lying between* r *and* s *and an integer* $p\geq N$ *such that* $\varphi(t)=t^p$ *for all* $t\in I$.

(b) *Every neighbourhood of* 0 *contains nonempty open sets where* $\phi(t)=r^p$ *for arbitrarily large* p.

Proof. (a) We start with a C^∞ smooth hat function h around the origin having support in $(-1,1)$. Set $r_n=r+(-\frac{1}{2})^n$ and let V_n be the open interval of radius $1/2^{n+1}$ centred at r_n. Notice that $V_n\cap V_m$ is empty unless $n=m$. We define h_n by $h_n(t)=h(2^{n+1}(t-r_n))$. This is a hat function around r_n with support in V_n. Finally, for $p\colon \mathbb{N}\to\mathbb{N}$ we define φ by

$$\varphi=\sum_{n=n_0}^{\infty}\imath^{p(n)}h_n,$$

where $\imath$ denotes the identity on $\mathbb{R}$ and n_0 is so that $V_n\subset(0,1)$ for every $n\geq n_0$.

The proof will be complete if we show that a good choice of the sequence $p(n)\to\infty$ makes φ smooth.

After all, let us remark that $\varphi(r)=0$ and that φ is C^∞ smooth at all points with the only possible exception of r. So it suffices to see that

$$\frac{D^m\varphi(t)}{r-t}\to 0,\quad \text{as } t\to r, \tag{1}$$

for each $m\in\mathbb{N}$. Indeed this implies that $D^m\varphi(0)=0$ for all m.

We use the following notation in the remainder of the proof. If u is a function whose domain contains A, we put

$$\|u\|_A=\sup_{t\in A}|u(t)|.$$

As φ vanishes outside the V_n's and taking into account that $\varphi=\imath^{p(n)}h_n$ on V_n and that $|r-t|\geq 2^{-(n+1)}$ for $t\in V_n$ we see that (1) is implied by the condition

$$\text{(2)}\qquad \frac{\|D^m(\imath^{p(n)}h_n)\|_{V_n}}{2^{-(n+1)}}\to 0, \quad \text{as } n\to\infty.$$

But $D^m(\imath^{p(n)}h_n)=\sum_{l=0}^m\binom{m}{l}D^l\imath^{p(n)}D^{m-l}h_n$, and it suffices to see that, whenever $0\leq l\leq m$,

$$\text{(3)}\qquad 2^{n+1}\|D^l\imath^{p(n)}D^{m-l}h_n\|_{V_n}$$

converges to zero as $n\to\infty$. On the other hand

$$D^{m-l}h_n(z)=2^{n+1}...2^{n+1}D^{m-l}h(2^{n+1}(z-r_n))$$

(the factor 2^{n+1} appearing $m-l$ times), so

$$\|D^{m-l}h_n\|_{V_n}=2^{(n+1)(m-l)}\|D^{m-l}h\|_{[-1,1]},$$

while for $p(n)>l$ one has

$$D^l\imath^{p(n)}=p(n)(p(n)-1)...(p(n)-l+1)\imath^{p(n)-l}.$$

Thus, if we take $r<\rho<1$, the sequence in (3) is dominated by

$$\|D^{m-l}h\|_{[-1,1]}2^{n+1}2^{(n+1)(m-l)}p(n)^l\rho^{p(n)-l},$$

which goes to zero as long as $p(n)/n\to\infty$. So, taking $p(n)=n^2$ suffices.

Part (b) is simpler. Let $h_n(t)=h\big(2^{n+1}\big(t-\big(-\frac{1}{2}\big)^n\big)\big)$. Given $p\colon\mathbb{N}\to\mathbb{N}$, define $\phi(t)=\sum_{n=1}^\infty r^{p(n)}h_n$. As before we want so see that $D^m\phi(t)/t\to 0$, as $t\to 0$, if $p(n)\to\infty$ fast enough. Let V_n be the open interval of radius $\big(\frac{1}{2}\big)^{n+1}$ centred at $\big(-\frac{1}{2}\big)^n$. It suffices to see that

$$2^{n+1}\|D^m r^{p(n)}h_n\|_{V_n}=2^{n+1}r^{p(n)}2^{m(n+1)}\|D^m h\|_{[-1,1]}$$

goes to zero as n increases. Again, taking $p(n)=n^2$ suffices. □

Step 2.3. *If* $0<f\leq 1$, *then* $0<Tf\leq 1$.

Proof. Assume on the contrary that $0<f\leq 1$, but $Tf(z)>1$ for some $z\in X$. Take $U\in R^k(X)$ such that $Tf>R$ on U, with $R>1$. Choose $y\in\mathfrak{T}(U)$ such that $f(y)<1$. We will construct a function $g\in C^k(Y)$ having the following property:

For every neighbourhood W of y and every $N>0$ there is an integer $p\geq N$ and an open subset of W where $g=f^p$.

Then, if $\tau(x)=y$, the function Tg is unbounded near x since each neighbourhood of x contains points where $Tg>R^N$ for all N.

We construct g using the functions of Step 2.2 as follows. Suppose f is non-constant on every neighbourhood of y. Then $g=\varphi\circ f$, where φ is as part (a), with $r=f(y)$.

If f is constant on a neighbourhood of y, let $u\in C^k(Y)$ be such that $u(y)=0$ and $Du(y)\neq 0$. Then take $g=\phi\circ u$, with ϕ as in part (b) and $r=f(y)$. □

An immediate consequence is that if $0<f\leq g$, with $f,g\in C^k(Y)$, then $0<Tf\leq Tg$. Indeed the hypothesis means that $0<f/g\leq 1$, so $0<T(f/g)=Tf/Tg\leq 1$. Hence T defines a bijection between $C^k(Y,(0,\infty))$ and $C^k(X,(0,\infty))$ preserving the order in both directions. As $\mathbb{R}$ is order-diffeomorphic with $(0,\infty)$ we can apply Theorem 1 to get the following consequence.

Step 2.4. *There is a strictly positive function* $p\colon X\to\mathbb{R}$ *such that*

$$Tf(x)=|f(\tau(x))|^{p(x)}\operatorname{sign}(f(\tau(x))) \tag{4}$$

for every $f\in C^k(Y)$.

Proof. First suppose $f>0$. We know from Theorem 1 that

$$Tf(x)=t(x,f(\tau(x))),$$

where $t(x,c)=Tc(x)$, with $c>0$. Actually only Step 1.5 is needed here. Next notice that for fixed x the map $c\mapsto Tc(x)$ is multiplicative and increasing on $(0,\infty)$, so $Tc(x)=c^p$, with $p>0$ depending on x. This proves (4) when $f>0$. As T acts locally, the formula is true whenever $f(\tau(x))>0$, even if f assumes negative values. On the other hand, $T(-f)=-Tf$, so (4) holds true for all $f\in C^k(Y)$ and every $x\in X$ provided $f(\tau(x))\neq 0$. In particular, we see that $f(\tau(x))\neq 0$ implies $Tf(x)\neq 0$, that is, $Tf(x)=0$ implies $f(\tau(x))=0$. Applying the same reasoning to the inverse of T we see that, in fact, $Tf(x)=0$ if and only if $f(\tau(x))=0$. Hence (4) is true for all $f\in C^k(Y)$ and all $x\in X$. □

Step 2.5. *One has* $p(x)=1$ *for all* x, *and* τ *is a* C^k *diffeomorphism.*

Proof. First, p is in $C^k(X)$, since $2^p=T2$ is. It follows that $f\circ\tau$ belongs to $C^k(X)$ for $f\in C^k(Y)$ and $f>0$, hence τ is C^k smooth and, by symmetry, it is a C^k diffeomorphism.

Finally, let us check $p(x)=1$ for all $x\in X$. By symmetry one only has to see that $p(x)\geq 1$ for all x in X. As τ is a C^k diffeomorphism we infer from (4) that

$$Lf(x) = |f(x)|^{p(x)}\,\mathrm{sign}(f(x))$$

defines a multiplicative automorphism of $C^k(X)$. Suppose $p(x)<1$. Pick $f\in C^k(X)$ with $f(x)=0$ and $Df(x)\neq 0$. We claim that $Lf=\mathrm{sign}(f)|f|^p$ is not differentiable at x. Indeed, let $u\colon\mathbb{R}\to X$ be a smooth curve passing through x at $t=0$, with $Df(x)u'(0)>0$. Then $Lf\circ u$ has no derivative at 0 since, taking $0<c<Df(x)(u'(0))$ and $p(x)<p_0<1$, one has

$$\lim_{t\to 0^+}\frac{Lf(u(t))-Lf(u(0))}{t}=\lim_{t\to 0^+}\frac{(f(u(t)))^{p(u(t))}}{t}\geq\lim_{t\to 0^+}\frac{(ct)^{p_0}}{t}=\infty.$$

This completes the proof. □

Concluding remarks

We close with a couple of questions arising from the content of this note.

(a) In Theorem 1, what can be said about τ? We do not know if τ must be differentiable even in the case $Y=X=\mathbb{R}$. It is apparent that the main obstruction is 'decoupling' the actions of $t(\,\cdot\,,\cdot\,)$ and τ.

(b) Does Corollary 2 remain true if the 'model' Banach space E fails to have a bump? We do not know the answer even if Y and X are in fact Banach spaces. See [5] for some affirmative results.

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