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A normality criterion involving rotations and dilations in the argument

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Abstract. We show that a family $\mathcal{F}$ of analytic functions in the unit disk $\mathbb{D}$ all of whose zeros have multiplicity at least k and which satisfy a condition of the form

$$f^n(z)f^{(k)}(xz)\neq 1$$

for all $z\in\mathbb{D}$ and $f\in\mathcal{F}$ (where $n\geq 3$, $k\geq 1$ and $0<|x|\leq 1$) is normal at the origin. The proof relies on a modification of Nevanlinna theory in combination with the Zalcman–Pang rescaling method. Furthermore we prove the corresponding Picard-type theorem for entire functions and some generalizations.

1. Introduction and statement of results

In 1959 W.K. Hayman [10, Corollary on p. 36 and Theorem 10] proved that if f is a transcendental meromorphic function in $\mathbb{C}$ and $n\geq 3$ is an integer, then $f^n f'$ assumes all values in $\mathbb{C}\setminus\{0\}$ infinitely often; if f is entire, this holds also for $n=2$.

In 1979, E. Mues [13] extended this result (for meromorphic f) to the case $n=2$; the case $n=1$ was settled by W. Bergweiler and A. Eremenko [1] and independently by H. Chen and M.-L. Fang [3] in 1995. For entire functions the case $n=1$ goes back to J. Clunie [4].

X.-C. Pang and L. Zalcman [15] showed that in the entire case an analogous result also holds for the differential polynomial $f^n f^{(k)}$ provided that all zeros of f have multiplicity at least k; their key idea was to use the well-known Zalcman–Pang rescaling lemma (an extension of Lemma 2.10) to reduce considerations to functions of exponential type. A similar result for meromorphic functions (and involving "small" exceptional functions instead of exceptional values) was proved by

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J.-P. Wang [17, Theorem 2] in 2003. A further extension to more general differential polynomials is due to W. Döringer [5, Corollary 1]; he has shown that if f is a transcendental meromorphic function in $\mathbb{C}$ and M is an arbitrary normalized differential monomial, then $f^n M[f]$ assumes all values in $\mathbb{C}\setminus\{0\}$ infinitely often provided that $n\geq 3$; if f is entire, this holds also for $n=2$.

According to Bloch's principle, to every "Picard-type" theorem there should belong a corresponding normality criterion. The normality result corresponding to the aforementioned Picard-type theorems was proved by L. Yang and K.-H. Chang [18] in 1965, I. B. Oshkin [14] in 1982 and Pang and Zalcman [15] in 1999.

Theorem A. ([14], [15] and [18]) *Let n and k be natural numbers and $\mathcal{F}$ be a family of analytic functions in a domain D all of whose zeros have multiplicity at least k. Assume that $f^n f^{(k)}-1$ is non-vanishing for each $f\in\mathcal{F}$. Then $\mathcal{F}$ is normal in D.*

In view of the various Picard-type theorems and normality results for differential polynomials known so far one might ask whether these results remain valid for "generalized" differential polynomials admitting rotations and dilations in the argument of some terms, i.e. whether conditions like $P_1[f](z)P_2[f](xz)+P_3[f](yz)\neq 1$ (where P_1, P_2 and P_3 are appropriate differential polynomials and $0<|x|, |y|\leq 1$) constitute normality or force entire (meromorphic) functions to be constant.([1]) This would be an indication that Bloch's principle is a much more far-reaching and versatile phenomenon than known so far.([2]) The first positive results in this direction, concerning the condition $f^n(z)+af^{(k)}(xz)\neq b$, were obtained in [9].

In the present paper we show that at least for analytic functions and n large enough (i.e. $n\geq 3$) normality results and Picard-type theorems of the type discussed above also hold for the "generalized" differential polynomial $f^n(z)f^{(k)}(xz)$ (with $|x|\leq 1$) instead of $f^n f^{(k)}$. In fact, our Picard-type result admits some extensions in the style of Döringer's result mentioned above.

We use the standard notation of Nevanlinna theory [11]. Furthermore, we denote the open resp. closed disk with center c and radius r by $U_r(c)$ resp. $B_r(c)$ and set $\mathbb{D}:=U_1(0)$ for the open unit disk and more generally $\mathbb{D}_r:=U_r(0)$.

Theorem 1.1. *Let $\mathcal{F}$ be a family of analytic functions in $\mathbb{D}$, $n\geq 3$, $k\geq 1$ and $0<|x|\leq 1$. Assume that for each $f\in\mathcal{F}$ the zeros of f are of multiplicity at least k*

([1]) Originally, this question was inspired by the study of the semiduality of certain small sets of analytic functions in the unit disk [7], see also footnote 1 in [9].

([2]) Furthermore, in view of the introduced complex quantities x and y, it would suggest that there might be some connections to the theory of functions of several complex variables.

and that

$$f^n(z)f^{(k)}(xz)\neq 1 \tag{1}$$

for all $z\in\mathbb{D}$. *Then* $\mathcal{F}$ *is normal at* $z=0$.

The question whether $\mathcal{F}$ is normal in the whole of $\mathbb{D}$ remains open. Furthermore, we do not know whether Theorem 1.1 still holds for families of meromorphic functions (and sufficiently large n).

The assumption on the multiplicities is inevitable, even in the "classical" case $x=1$, as the functions $f_j(z):=jz$ show: They satisfy

$$f_j^n(z)f_j^{(k)}(xz)\equiv 0\neq 1$$

for all $j\in\mathbb{N}$, $z\in\mathbb{D}$, $x\in\overline{\mathbb{D}}\setminus\{0\}$, $n\geq 1$ and $k\geq 2$, but the family $\{f_j\}_{j=1}^{\infty}$ is non-normal at $z=0$.

The family of the functions $f_j(z)=e^{jz}$ demonstrates that Theorem 1.1 is no longer valid for $n=1$: For $x=-1$ we have $f_j(z)f_j'(xz)=j\neq 1$ for all $z\in\mathbb{D}$ and all $j\geq 2$, but $\{f_j\}_{j=1}^{\infty}$ is not normal at $z=0$. The case $n=2$ remains unclear.

Finally, Theorem 1.1 does not hold for $x=-n$. (Here, of course one has to assume (1) only for $|z|<1/n$.) Again, this is shown by the functions $f_j(z):=e^{jz}$ which satisfy $f_j^n(z)f_j^{(k)}(-nz)=j^k\neq 1$ for all $z\in\mathbb{C}$ and all $j\geq 2$. However, we do not know whether the condition $|x|\leq 1$ can be weakened; we conjecture that this is not possible.

Theorem 1.2. *Let* f *be a transcendental entire function,* $k_1,\dots,k_s$ *and* n *be natural numbers with* $n\geq s+2$, $x_1,\dots,x_s\in\mathbb{C}$ *with* $0<|x_j|\leq 1$ *for all* $j=1,\dots,s$, *and* $c\not\equiv 0$ *be a meromorphic function satisfying* $T(r,c)=S(r,f)$. *Then the function*

$$\psi(z):=c(z)f^n(z)f^{(k_1)}(x_1z)\dots f^{(k_s)}(x_sz)-1$$

has infinitely many zeros in $\mathbb{C}$.

This result no longer holds for $n\leq s$ since for $f:=\exp$ and $x:=-n/s\in\overline{\mathbb{D}}$ we have $2f^n(z)(f')^s(xz)-1=1\neq 0$ for all $z\in\mathbb{C}$. The case $n=s+1$ remains open.

While the Picard-type result in Theorem 1.2 can be proved using "classical" Nevanlinna theory (in a similar way as in [5] and [17]), the proof of Theorem 1.1 is more complicated. The proof of Theorem A in [15] was based on the Zalcman–Pang rescaling method which has proved to be a very helpful tool in normality theory for many years since it reduces normality results to the corresponding Picard-type theorems which in most cases are easier to prove. Unfortunately, this elegant method seems to fail in our context since the newly introduced rotations and dilations in $f^{(k)}(xz)$ destroy the translation invariance of the assumption (1). For the same reason, the methods from Nevanlinna theory established in Drasin's seminal paper [6] from 1969 cannot be applied immediately.

Therefore, in the proof of Theorem 1.1 (as in [9]) we resort to a modified version of Nevanlinna theory which was developed in [8] and which refines previous ideas of H. Cartan [2] and D. Drasin [6]. Since it makes use of the full generality of Poisson–Jensen–Nevanlinna's formula, it gives us more flexibility in dealing with the so-called initial value terms appearing in applications of Nevanlinna theory to normality problems. For the convenience of the reader, the required tools are summarized in Section 2.

But then again, Zalcman's rescaling lemma is also useful in our proof to simplify the discussion of one special case. And of course, we hope that our method (which surely is quite complicated) gives some kind of inspiration to adjust the Zalcman–Pang rescaling method to problems of the described kind, and therefore leading to a deeper and broader understanding of Bloch's principle.

2. A modification of Nevanlinna theory and some other lemmas

In this section we tacitly assume (unless otherwise stated) that f is a non-constant meromorphic function in the disk $\mathbb{D}_{R_0}$, where $0<R_0\leq\infty$.

Let $\log^+ x:=\max\{\log x, 0\}$ if $x>0$ and $\log^+ 0:=0$.

Definition 2.1. Let $\alpha\in\mathbb{D}_{R_0}$ be such that α is not a pole of f. Let the b_k be the poles of f, each of them taken into account according to its multiplicity. Then for $|\alpha|<r<R_0$ we define

$$m_\alpha(r,f) := \frac{1}{2\pi}\int_0^{2\pi} \log^+|f(re^{it})|\,\mathrm{Re}\,\frac{re^{it}+\alpha}{re^{it}-\alpha}\,dt,$$

$$N_\alpha(r,f) := \sum_{|b_k|<r} \log\left|\frac{r^2-\bar{b}_k\alpha}{r(\alpha-b_k)}\right|,$$

$$T_\alpha(r,f) := m_\alpha(r,f)+N_\alpha(r,f)$$

and call them the *modified proximity function*, *counting function* and *characteristic of* f with respect to α, respectively. In the same way we define $\overline{N}_\alpha(r,f)$; here each pole of f is counted only once.

As we have shown in [8], the results from the "classical" Nevanlinna theory (corresponding to the case $\alpha=0$) remain valid for these modified quantities, the first fundamental theorem being an immediate consequence of the general form of Poisson–Jensen–Nevanlinna's formula.

Remark 2.2. ([8, Remark 1]) For the functions $V_\alpha = m_\alpha$, N_α, $\overline{N}_\alpha$ and T_α the following fundamental estimates hold:

$$\begin{aligned}
&V_\alpha(r,f) \geq 0,\\
&V_\alpha\left(r, \prod_{j=1}^{p} f_j\right) \leq \sum_{j=1}^{p} V_\alpha(r,f_j),\\
&V_\alpha\left(r, \sum_{j=1}^{p} f_j\right) \leq \sum_{j=1}^{p} V_\alpha(r,f_j) + \log p,\\
&N_\alpha\left(r, \sum_{j=1}^{p} f_j\right) \leq \sum_{j=1}^{p} N_\alpha(r,f_j)
\end{aligned}$$

for $f, f_1, \ldots, f_p$ meromorphic in $\mathbb{D}_{R_0}$ and $\alpha \in \mathbb{D}_{R_0}$ with $f(\alpha) \neq \infty$, $f_1(\alpha) \neq \infty$, ..., $f_p(\alpha) \neq \infty$ and $|\alpha| < r < R_0$.

Theorem 2.3. (Generalized first fundamental theorem, [8, Theorem 1]) *Let $\alpha \in \mathbb{D}_{R_0}$ be given such that it is not a zero or pole of f. Then for $|\alpha| < r < R_0$ we have*

$$T_\alpha\left(r, \frac{1}{f}\right) = T_\alpha(r,f) - \log|f(\alpha)|.$$

Theorem 2.4. ([8, Theorem 4]) *Let $\alpha \in \mathbb{D}_{R_0}$ be such that $f(\alpha) \neq \infty$. Then the functions $N_\alpha(r,f)$ and $T_\alpha(r,f)$ are increasing and continuous for $|\alpha| < r < R_0$.*

Theorem 2.5. (Generalized second fundamental theorem, [8, Theorem 5]) *Let f be non-constant and meromorphic in $\mathbb{D}_{R_0}$ and let $\alpha \in \mathbb{D}_{R_0}$ be such that $f(\alpha) \neq \infty$ and $f'(\alpha) \neq 0$. If $c_1, \ldots, c_q \in \mathbb{C}$, $q \geq 2$, are pairwise distinct, then for $|\alpha| < r < R_0$ the estimate*

$$m_\alpha(r,f) + \sum_{k=1}^{q} m_\alpha\left(r, \frac{1}{f - c_k}\right) \leq 2T_\alpha(r,f) - N_{1,\alpha}(r,f) + S_\alpha(r,f)$$

holds, where

$$N_{1,\alpha}(r,f) = N_\alpha\left(r, \frac{1}{f'}\right) + 2N_\alpha(r,f) - N_\alpha(r,f') \geq 0,$$

$$S_\alpha(r,f)=m_\alpha\left(r,\frac{f'}{f}\right)+\sum_{k=1}^{q}m_\alpha\left(r,\frac{f'}{f-c_k}\right)+\log\frac{1}{|f'(\alpha)|}+C_0$$

and C_0 *is an absolute constant independent of* f *and* r.

The estimate in the next lemma is obvious but proves to be crucial in the proof of Theorem 1.1.

Lemma 2.6. *Let* $\alpha\in\mathbb{D}_{R_0}$ *with* $f(\alpha)\neq\infty$ *be given. Then the estimate*

$$\frac{r-|\alpha|}{r+|\alpha|}\,m(r,f)\leq m_\alpha(r,f)\leq\frac{r+|\alpha|}{r-|\alpha|}\,m(r,f)$$

holds for all $r\in]|\alpha|,R_0[$.

There is also a theorem on the logarithmic derivative for the modified proximity function, but since in its general form it is not needed for our purposes we refer to [8, Theorem 7] for it and state only a consequence of this result for non-normal families which is required in the proof of Theorem 1.1.

Lemma 2.7. ([8, Corollary 9]) *Let* $\mathcal{F}$ *be a family of functions analytic in* $\mathbb{D}$ *and assume that* $\mathcal{F}$ *is not normal at* $z_0\in\mathbb{D}$. *Then there exist a sequence* $\{f_j\}_{j=1}^{\infty}\subseteq\mathcal{F}$, *not normal at* z_0, *and constants* $A_k<\infty$ *such that for all* $r_0\in]|z_0|,1[$ *and all* $k\in\mathbb{N}$ *the estimate*

$$m\left(r,\frac{f_j^{(k)}}{f_j}\right)\leq A_k\left(\log\frac{1}{R-r}+\log^+ m(R,f_j)+\log\frac{1}{r_0-|z_0|}+1\right)\tag{2}$$

holds for all but finitely many $j\in\mathbb{N}$ *and* $r_0<r<R<1$.

A slightly weaker form of this result was proved already by Drasin [6, Lemma 8].

To get rid of the terms $\log^+ T_{\alpha_j}(R,f_j)$ and $m(R,f_j)$ in Lemma 2.7 we need the following growth estimate [12, Chapter VIII, Lemma 1.5].

Lemma 2.8. *Let* $-\infty<a<b<\infty$ *and* $U\colon[a,b[\to[0,\infty[$ *be a continuous, increasing function. Assume that there is a constant* $A<\infty$ *such that*

$$U(r)\leq A\left(\log^+\frac{1}{R-r}+\log^+ U(R)+1\right)$$

for all $r\in[a,b[$ *and all* $R\in]r,b[$. *Then there exists a constant* $B<\infty$ *depending only on* A *such that*

$$U(r)\leq B\left(\log^+\frac{1}{b-r}+1\right)\quad\text{for all } r\in[a,b[.$$

The next lemma characterizes normality of families of analytic functions in terms of the proximity functions.

Lemma 2.9. ([16, Lemma 4.4.1]) *Let $\mathcal{F}$ be a family of analytic functions in $\mathbb{D}$. If the family $\{r \mapsto m(r,f) : f \in \mathcal{F}\}$ of the corresponding proximity functions is locally (uniformly) bounded, then $\mathcal{F}$ is normal.*

Case 1 of the proof of Theorem 1.1 relies essentially on Zalcman's lemma in its original form [19].

Lemma 2.10. (Zalcman's lemma) *Let $\mathcal{F}$ be a family of functions meromorphic in $\mathbb{D}$. Then $\mathcal{F}$ is non-normal at $z_0 \in \mathbb{D}$ if and only if there are sequences $\{f_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$, $\{z_n\}_{n=1}^{\infty} \subseteq \mathbb{D}$ and $\{\varrho_n\}_{n=1}^{\infty} \subseteq]0,1[$ such that $\lim_{n\to\infty} \varrho_n = 0$, $\lim_{n\to\infty} z_n = z_0$ and such that the sequence $\{g_n\}_{n=1}^{\infty}$ defined by*

$$g_n(\zeta) := f_n(z_n + \varrho_n \zeta)$$

converges locally uniformly in $\mathbb{C}$ (with respect to the spherical metric) to a non-constant function g meromorphic in $\mathbb{C}$.

3. Proof of Theorem 1.1

For the sake of abbreviation, we set

$$c(r,\alpha) := \frac{r + |\alpha|}{r - |\alpha|}$$

for $r > |\alpha|$. Then the estimate of Lemma 2.6 can be written as

$$m_\alpha(r,f) \le c(r,\alpha) m(r,f) \text{ and } m(r,f) \le c(r,\alpha) m_\alpha(r,f) \quad \text{for } r > |\alpha|.$$

In the following, we frequently use the monotonicity of $m_\alpha(r,f)$ for analytic f (Theorem 2.4) to estimate terms like $m_\alpha(r|x|,f)$ by $m_\alpha(r,f)$. Furthermore, we use that

$$m_\alpha(r, f(xz)) = m_{x\alpha}(r|x|, f) \quad \text{and} \quad N_\alpha(r, f(xz)) = N_{x\alpha}(r|x|, f) \tag{3}$$

for functions f meromorphic in $\mathbb{D}$, $0 < |x| \le 1$ and $|\alpha| < r < 1$; these relations can be easily seen from the definitions of m_α and N_α. Finally, estimates like

$$c(r, \alpha x) \le c(r, \alpha) \le c(r, \varrho)$$

for $|\alpha|<\varrho<r<1$ and $0<|x|\leq 1$ will prove to be useful soon. By $C_1, C_2, \ldots$ we denote any absolute constants independent of the radius r and the function under consideration.

Assume that $\mathcal{F}$ is not normal at $z=0$. Then by Lemma 2.7 there exists a sequence $\{f_j\}_{j=1}^{\infty}\subseteq\mathcal{F}$, non-normal at $z=0$, such that for all $r_0\in]0,1[$ the estimate

$$m\left(r, \frac{f_j^{(k)}}{f_j}\right) \leq A\left(\log\frac{1}{R-r}+\log^+ m(R,f_j)+\log\frac{1}{r_0}+1\right) \tag{4}$$

holds for all but finitely many j and $r_0<r<R<1$. Without loss of generality we may assume that every subsequence of $\{f_j\}_{j=1}^{\infty}$ is non-normal at $z=0$ as well. We define

$$g_j(z):=f_j^{(k)}(xz) \quad \text{and} \quad \psi_j:=f_j^n g_j-1 \tag{5}$$

for $z\in\mathbb{D}$ and all j. Then all ψ_j are non-vanishing in $\mathbb{D}$ by the assumption.

Case 1. A subsequence of $\{f_j^{(k)}\}_{j=1}^{\infty}$ converges to 0 uniformly in a neighborhood of 0.

Without loss of generality we may assume that $\{f_j^{(k)}\}_{j=1}^{\infty}$ itself converges to 0 uniformly in a neighborhood of 0.

Choosing an appropriate subsequence if necessary, by Zalcman's lemma we can find sequences $\{z_j\}_{j=1}^{\infty}\subseteq\mathbb{D}$ and $\{\varrho_j\}_{j=1}^{\infty}\subseteq]0,1[$ such that $\lim_{j\to\infty} z_j=0$ and $\lim_{j\to\infty}\varrho_j=0$, and the sequence $\{\varphi_j\}_{j=1}^{\infty}$ defined by

$$\varphi_j(\zeta):=f_j(z_j+\varrho_j\zeta)$$

converges locally uniformly in $\mathbb{C}$ to a non-constant limit function φ. From

$$\varphi_j^{(k)}(\zeta)=\varrho_j^k f_j^{(k)}(z_j+\varrho_j\zeta)$$

and the assumption in case 1 we deduce that $\{\varphi_j^{(k)}\}_{j=1}^{\infty}$ converges to 0 locally uniformly in $\mathbb{C}$. On the other hand, by the Weierstraß convergence theorem $\{\varphi_j^{(k)}\}_{j=1}^{\infty}$ converges to $\varphi^{(k)}$. Therefore, $\varphi^{(k)}\equiv 0$, so φ is a polynomial of degree at most $k-1$. Since φ is non-constant, it has at least one zero z_0 in $\mathbb{C}$. From the assumption on the multiplicities of the zeros of the functions f_j and Hurwitz's theorem we conclude that z_0 is a zero of φ of multiplicity at least k. This is impossible since $\deg(\varphi)\leq k-1$.

Case 2. $\{\psi_j\}_{j=1}^{\infty}$ is normal at the origin.

Considering an appropriate subsequence, we may assume that $\{\psi_j\}_{j=1}^{\infty}$ itself converges to a limit function ψ (possibly $\psi\equiv\infty$) locally uniformly in $U_{3\delta}(0)$ for some $\delta>0$.

Case 2.1. ψ is analytic.

Then $\{\psi_j\}_{j=1}^{\infty}$ is uniformly bounded in $B_{2\delta}(0)$, so there exists an $M<\infty$ such that $|\psi_j(z)|\le M$ for all $j\in\mathbb{N}$ and $z\in B_{2\delta}(0)$. This means that

$$m_\alpha(r,\psi_j+1)\le\log(M+1)\quad\text{for all } \alpha\in U_{2\delta}(0),\ r\in]|\alpha|,2\delta[\ \text{ and } j\in\mathbb{N}.$$

We choose $\varrho\in]0,\delta[$ such that $c^2(\delta,\varrho)\le n-1$.

We may assume that case 1 does not hold. Then there exists an $\varepsilon>0$, a $j_0\in\mathbb{N}$ and a sequence $\{\alpha_j\}_{j=j_0}^{\infty}\subseteq U_\varrho(0)$ such that

$$|g_j(\alpha_j)|\ge\varepsilon\quad\text{for all } j\ge j_0.$$

By (5), we have $f_j^n=(\psi_j+1)/g_j$. Using this we obtain for $r\in]\delta,2\delta[$ and $j\ge j_0$,

$$\begin{aligned}
n m_{\alpha_j}(r,f_j)&=m_{\alpha_j}(r,f_j^n)\\
&\le m_{\alpha_j}(r,\psi_j+1)+m_{\alpha_j}\left(r,\frac{1}{g_j}\right)\\
&\le \log(M+1)+T_{\alpha_j x}(r|x|,f_j^{(k)})+\log\frac{1}{|g_j(\alpha_j)|}\\
&\le c(r,\alpha_j x)m(r,f_j^{(k)})+\log(M+1)+\log^+\frac{1}{\varepsilon}\\
&\le c^2(r,\varrho)m_{\alpha_j}(r,f_j)+c(r,\varrho)m\left(r,\frac{f_j^{(k)}}{f_j}\right)+\log(M+1)+\log^+\frac{1}{\varepsilon}.
\end{aligned}$$

In view of

$$c(r,\varrho)\le c(\delta,\varrho)\le\sqrt{n-1}\quad\text{for all } r\ge\delta$$

we deduce that

$$m_{\alpha_j}(r,f_j)\le\sqrt{n-1}\,m\left(r,\frac{f_j^{(k)}}{f_j}\right)+\log(M+1)+\log^+\frac{1}{\varepsilon}.$$

Hence

$$\begin{aligned}
(6)\qquad m(r,f_j)&\le c(r,\varrho)m_{\alpha_j}(r,f_j)\\
&\le (n-1)m\left(r,\frac{f_j^{(k)}}{f_j}\right)+\sqrt{n-1}\left(\log(M+1)+\log^+\frac{1}{\varepsilon}\right),
\end{aligned}$$

for all $r\in]\delta,2\delta[$ and all $j\ge j_0$. By inserting (4) (with $r_0:=\delta$) into (6) we conclude that

$$m(r,f_j)\le C_1\left(\log\frac{1}{R-r}+\log^+m(R,f_j)+\log\frac{1}{\delta}+1\right)$$

for $\delta<r<R<2\delta$ and all but finitely many j with some constant C_1. From Lemma 2.8 we deduce the local boundedness of $\{m(r,f_j)\}_{j=1}^{\infty}$ on the interval $]\delta,2\delta[$. Since $r\mapsto m(r,f_j)=T(r,f_j)$ is increasing, $\{m(r,f_j)\}_{j=1}^{\infty}$ is locally bounded also on $[0,2\delta[$. So from Lemma 2.9 we obtain the normality of $\{f_j\}_{j=1}^{\infty}$ in $\mathbb{D}_{2\delta}$ which contradicts our choice of $\{f_j\}_{j=1}^{\infty}$.

Case 2.2. $\psi\equiv\infty$.

Omitting finitely many j if necessary, we may assume that $|\psi_j(z)|\geq 2$ for all j and all $z\in U_{2\delta}(0)$. In particular, all f_j are non-vanishing in $U_{2\delta}(0)$ (since $f_j(z)=0$ implies $\psi_j(z)=-1$), and from $|\psi_j(z)+1|\geq 1$ for all $z\in U_{2\delta}(0)$ we see that

$$m_\beta\left(r,\frac{1}{\psi_j+1}\right)=0 \quad \text{for all } \beta\in U_{2\delta}(0),\ r\in(|\beta|,2\delta) \text{ and } j\in\mathbb{N}.$$

Again, we choose $\varrho\in]0,\delta[$ such that $c^2(\delta,\varrho)\leq n-1$.

By assumption, no subsequence of $\{f_j\}_{j=1}^{\infty}$ is normal in $U_\varrho(0)$, so by Montel's theorem we can find a sequence $\{\beta_j\}_{j=j_0}^{\infty}$ in $U_\varrho(0)$ such that $|f_j(\beta_j)|\leq 1$ for all $j\geq j_0$. Observing that by (5) we have $1/f_j^n(z)=f_j^{(k)}(xz)/(\psi_j(z)+1)$, we obtain for all $r\in]\delta,2\delta[$ and all $j\geq j_0$,

$$\begin{aligned}
nm_{\beta_j}(r,f_j) &= T_{\beta_j}\left(r,\frac{1}{f_j^n}\right)+n\log|f_j(\beta_j)| \\
&\leq m_{\beta_j}\left(r,\frac{1}{f_j^n}\right)+0 \\
&\leq m_{\beta_j}\left(r,\frac{1}{\psi_j+1}\right)+m_{\beta_j x}(r|x|,f_j^{(k)}) \\
&\leq 0+c^2(\delta,\varrho)m_{\beta_j}(r,f_j)+c(\delta,\varrho)m\left(r,\frac{f_j^{(k)}}{f_j}\right) \\
&\leq (n-1)m_{\beta_j}(r,f_j)+\sqrt{n-1}\,m\left(r,\frac{f_j^{(k)}}{f_j}\right).
\end{aligned}$$

Hence

$$m_{\beta_j}(r,f_j)\leq\sqrt{n-1}\,m\left(r,\frac{f_j^{(k)}}{f_j}\right).$$

From this, (4) and Lemma 2.8 we deduce the local boundedness of $\{m_{\beta_j}(r,f_j)\}_{j=1}^{\infty}$, and hence of $\{m(r,f_j)\}_{j=1}^{\infty}$ in $U_{2\delta}(0)$. By Lemma 2.9, $\{f_j\}_{j=1}^{\infty}$ is normal in $U_{2\delta}(0)$, once again a contradiction.

Cases 3 *and* 4. From now on, we can assume that $\{\psi_j\}_{j=1}^{\infty}$ is not normal at $z=0$. This enables us to assume (by Lemma 2.7) that

$$m\left(r,\frac{\psi_j'}{\psi_j}\right)+m\left(r,\frac{\psi_j'}{\psi_j+1}\right)\le B\left(\log\frac{1}{R-r}+\log^+ m(R,\psi_j)+1\right) \tag{7}$$

for all j and $r_0<r<R<1$ with some constant B.

Case 3. $\{\psi_j\}_{j=1}^{\infty}$ is not normal at $z=0$ and there exists a neighborhood $U_{2r_0}(0)$ of 0 and an integer $\mu\in\{n-2,n-1\}$ such that both $\{\psi_j'/\psi_j f_j^{\mu}\}_{j=1}^{\infty}$ and $\{\psi_j'/f_j^{\mu}\}_{j=1}^{\infty}$ converge to 0 uniformly in $U_{2r_0}(0)$.

We choose a $\varrho\in]0,r_0[$ such that $c^4(r_0,\varrho)<\frac{5}{4}$.

We may assume that case 1 does not hold. This provides us (after turning to a subsequence) with an $\varepsilon>0$ and a sequence $\{\alpha_j\}_{j=1}^{\infty}\subseteq U_{\varrho}(0)$ such that

$$|g_j(\alpha_j)|\ge\varepsilon\quad\text{for all } j. \tag{8}$$

In view of the non-normality assumption in case 3, by Marty's theorem and the estimate

$$\psi_j^{\#}=\frac{|\psi_j'|}{1+|\psi_j|^2}\le\frac{|\psi_j'|}{2|\psi_j|},$$

we may assume that to each j there is a $\beta_j\in U_{\varrho}(0)$ with

$$\frac{|\psi_j'(\beta_j)|}{|\psi_j(\beta_j)|}\ge 1. \tag{9}$$

From

$$\left|\frac{\psi_j'}{\psi_j}f_j^{n-\mu}g_j\right|=\left|\frac{\psi_j'}{\psi_j}\,\frac{\psi_j+1}{f_j^{\mu}}\right|\le\left|\frac{\psi_j'}{f_j^{\mu}}\right|\left(1+\frac{1}{|\psi_j|}\right)$$

and the second assumption in case 3 we see that $\{(\psi_j'/\psi_j)f_j^{n-\mu}g_j\}_{j=1}^{\infty}$ converges to 0 uniformly in $U_{2r_0}(0)$. So without loss of generality we may assume that

$$\left|\frac{\psi_j'}{\psi}(z)f_j^{n-\mu}(z)g_j(z)\right|\le 1,\quad\text{and hence}\quad |f_j^{n-\mu}(z)g_j(z)|\le\left|\frac{\psi_j}{\psi_j'}(z)\right|,$$

for all j and all $z\in U_{2r_0}(0)$. In particular, we have

$$m_{\beta}(r,f_j^{n-\mu}g_j)\le m_{\beta}\left(r,\frac{\psi_j}{\psi_j'}\right)$$

for all $\beta\in U_{2r_0}(0)$ and all $r\in]|\beta|,2r_0[$. By the first fundamental theorem, the fact that all ψ_j are non-vanishing and (9), this yields that

$$m_{\beta_j}(r,f_j^{n-\mu}g_j)\le T_{\beta_j}\left(r,\frac{\psi_j'}{\psi_j}\right)+\log\left|\frac{\psi_j}{\psi_j'}(\beta_j)\right|\le m_{\beta_j}\left(r,\frac{\psi_j'}{\psi_j}\right) \tag{10}$$

for all j and all $r\in]|\beta_j|,2r_0[$. This turns out to be the crucial step in the reasoning for case 3 because now we have managed to estimate the proximity function of $f_j^{n-\mu}g_j$ by the proximity function of ψ_j'/ψ_j which is "small" by the lemma on the logarithmic derivative.

Case 3.1. $\mu = n-2$.

Then from (10) (with $\mu = n-2$), (8), the first fundamental theorem and the analyticity of g_j, we deduce for all $r \in]r_0, 2r_0[$ and all j,

$$\begin{aligned} 2m_{\alpha_j}(r, f_j) &\leq m_{\alpha_j}(r, f_j^2 g_j) + m_{\alpha_j}\left(r, \frac{1}{g_j}\right) \\ &\leq c^2(r,\varrho) m_{\beta_j}(r, f_j^2 g_j) + T_{\alpha_j}(r, g_j) + \log \frac{1}{|g_j(\alpha_j)|} \\ &\leq c^2(r,\varrho) m_{\beta_j}\left(r, \frac{\psi_j'}{\psi_j}\right) + m_{x\alpha_j}(r|x|, f_j^{(k)}) + \log^+ \frac{1}{\varepsilon} \\ &\leq c^2(r,\varrho)\left(m_{\beta_j}\left(r, \frac{\psi_j'}{\psi_j}\right) + m_{\alpha_j}\left(r, \frac{f_j^{(k)}}{f_j}\right) + m_{\alpha_j}(r, f_j)\right) + \log^+ \frac{1}{\varepsilon}. \end{aligned}$$

Hence in view of $c^2(r_0, \varrho) < \frac{5}{4}$,

$$m_{\alpha_j}(r, f_j) \leq 2m_{\beta_j}\left(r, \frac{\psi_j'}{\psi_j}\right) + 2m_{\alpha_j}\left(r, \frac{f_j^{(k)}}{f_j}\right) + 2\log^+ \frac{1}{\varepsilon}. \tag{11}$$

From (7) we have that

$$m\left(r, \frac{\psi_j'}{\psi_j}\right) \leq B\left(\log \frac{2}{R-r} + \log^+ m\left(\frac{R+r}{2}, \psi_j\right) + 1\right)$$

for all j and $r_0 < r < R < 1$; here

$$\begin{aligned} \log^+ m\left(\frac{R+r}{2}, \psi_j\right) &\leq \log^+\left(nm(R, f_j) + m\left(\frac{R+r}{2}, g_j\right) + \log 2\right) \\ &\leq \log n + \log^+ m(R, f_j) \\ &\quad + \log^+\left(m\left(\frac{R+r}{2}, \frac{f_j^{(k)}}{f_j}\right) + m\left(\frac{R+r}{2}, f_j\right)\right) + \log 3. \end{aligned}$$

Combining this with (11), we obtain an estimate of the form

$$m_{\alpha_j}(r, f_j) \leq C_2\left(m\left(\frac{R+r}{2}, \frac{f_j^{(k)}}{f_j}\right) + \log^+ m(R, f_j) + \log^+ \frac{1}{R-r} + 1\right)$$

for all j and $r_0 < r < R < 2r_0$ with a certain constant C_2. Now inserting (4) (with $(R+r)/2$ instead of r), in the usual way we deduce the normality of $\{f_j\}_{j=1}^{\infty}$ at the origin, a contradiction.

Case 3.2. $\mu=n-1$.

In this case we mainly use the generalized second fundamental theorem (Theorem 2.5) which gives us

$$
(12)\quad \begin{aligned} &m_\gamma(r,\psi_j)+m_\gamma\left(r,\frac{1}{\psi_j}\right)+m_\gamma\left(r,\frac{1}{\psi_j+1}\right)\\ &\le 2T_\gamma(r,\psi_j)-N_\gamma\left(r,\frac{1}{\psi_j'}\right)+2m_\gamma\left(r,\frac{\psi_j'}{\psi_j}\right)+m_\gamma\left(r,\frac{\psi_j'}{\psi_j+1}\right)+\log\left|\frac{1}{\psi_j'(\gamma)}\right|+C_0\end{aligned}
$$

for all $r\in]\varrho,1[$, $j\in\mathbb{N}$ and all $\gamma\in U_\varrho(0)$ with $\psi_j'(\gamma)\neq 0$. Furthermore, the following arguments are crucial: Since the ψ_j are analytic and non-vanishing, we have

$$
N_\gamma(r,\psi_j)=N_\gamma\left(r,\frac{1}{\psi_j}\right)=0
$$

for all $r\in]\varrho,1[$, $j\in\mathbb{N}$ and all $\gamma\in U_\varrho(0)$. If z_0 is a zero of $\psi_j+1=f_j^n g_j$, then it is a zero of f_j or a zero of f_jg_j. In the first case, we conclude that it is a zero of ψ_j+1 of multiplicity at least n. Therefore

$$
(13)\quad \overline{N}_\gamma\left(r,\frac{1}{\psi_j+1}\right)\le\frac{1}{n-1}N_\gamma\left(r,\frac{1}{\psi_j+1}\right)+\frac{n-2}{n-1}\overline{N}_\gamma\left(r,\frac{1}{f_jg_j}\right)
$$

for all $r\in]\varrho,1[$, $j\in\mathbb{N}$ and all $\gamma\in U_\varrho(0)$.

We may assume that $\{\psi_j'/\psi_jf_j^{n-2}\}_{j=1}^\infty$ or $\{\psi_j'/f_j^{n-2}\}_{j=1}^\infty$ does not converge to 0 uniformly in any neighborhood of 0. (Otherwise we are in case 3.1.)

Case 3.2.1. $\{\psi_j'/\psi_jf_j^{n-2}\}_{j=1}^\infty$ does not converge to 0 uniformly in any neighborhood of 0.

Then, taking subsequences if necessary, we may assume that there exists an $\varepsilon_0>0$ and a sequence $\{\gamma_j\}_{j=1}^\infty\subseteq U_\varrho(0)$ such that

$$
(14)\quad \left|\frac{\psi_j'}{\psi_jf_j^{n-2}}(\gamma_j)\right|\ge\varepsilon_0
$$

for all j. Adding $N_\gamma(r,\psi_j)+N_\gamma(r,1/\psi_j)+N_\gamma(r,1/(\psi_j+1))$ to both sides of (12) and applying the first fundamental theorem to $T_\gamma(r,1/\psi_j)$ and to $T_\gamma(r,1/(\psi_j+1))$ gives

$$
\begin{aligned} T_\gamma(r,\psi_j+1)\le{}& N_\gamma(r,\psi_j)+N_\gamma\left(r,\frac{1}{\psi_j}\right)+N_\gamma\left(r,\frac{1}{\psi_j+1}\right)-N_\gamma\left(r,\frac{1}{\psi_j'}\right)\\ &+2m_\gamma\left(r,\frac{\psi_j'}{\psi_j}\right)+m_\gamma\left(r,\frac{\psi_j'}{\psi_j+1}\right)+\log\frac{|\psi_j(\gamma_j)|\,|\psi_j(\gamma_j)+1|}{|\psi_j'(\gamma)|}+C_0\end{aligned}
$$

for all $r\in]\varrho,1[$, all $j\in\mathbb{N}$ and all $\gamma\in U_\varrho(0)$ with $\psi_j'(\gamma)\neq 0$. Keeping in mind that $N_\gamma(r,\psi_j)=N_\gamma(r,1/\psi_j)=0$ and observing (13), we obtain for all $r\in]\varrho,1[$ and all j,

$$\begin{aligned}T_{\gamma_j}(r,\psi_j+1) &\le \overline{N}_{\gamma_j}\left(r,\frac{1}{\psi_j+1}\right)+2m_{\gamma_j}\left(r,\frac{\psi_j'}{\psi_j}\right)+m_{\gamma_j}\left(r,\frac{\psi_j'}{\psi_j+1}\right)\\ &\quad+\log\left|\frac{\psi_j(\psi_j+1)}{\psi_j'}(\gamma_j)\right|+C_0\\ &\le \frac{1}{n-1}N_{\gamma_j}\left(r,\frac{1}{\psi_j+1}\right)+\frac{n-2}{n-1}\overline{N}_{\gamma_j}\left(r,\frac{1}{f_jg_j}\right)\\ &\quad+2m_{\gamma_j}\left(r,\frac{\psi_j'}{\psi_j}\right)+m_{\gamma_j}\left(r,\frac{\psi_j'}{\psi_j+1}\right)+\log\left|\frac{\psi_j(\psi_j+1)}{\psi_j'}(\gamma_j)\right|+C_0\\ &\le \frac{1}{n-1}T_{\gamma_j}(r,\psi_j+1)+\frac{n-2}{n-1}T_{\gamma_j}(r,f_jg_j)+2m_{\gamma_j}\left(r,\frac{\psi_j'}{\psi_j}\right)\\ &\quad+m_{\gamma_j}\left(r,\frac{\psi_j'}{\psi_j+1}\right)+\log\frac{|\psi_j|\,|\psi_j+1|^{1-1/(n-1)}}{|\psi_j'|\,|f_jg_j|^{1-1/(n-1)}}(\gamma_j)+C_0.\end{aligned}\tag{15}$$

Here from $\psi_j+1=f_j^n g_j$ and (14) we get that

$$\log\frac{|\psi_j|\,|\psi_j+1|^{1-1/(n-1)}}{|\psi_j'|\,|f_jg_j|^{1-1/(n-1)}}(\gamma_j)\le\log\left|\frac{\psi_jf_j^{n-2}}{\psi_j'}(\gamma_j)\right|\le\log^+\frac{1}{\varepsilon_0},$$

so noting that $(n-1)/(n-2)\le 2$ we arrive at

$$\begin{aligned}T_{\gamma_j}(r,\psi_j+1) &\le T_{\gamma_j}(r,f_jg_j)+\frac{n-1}{n-2}\left(2m_{\gamma_j}\left(r,\frac{\psi_j'}{\psi_j}\right)+m_{\gamma_j}\left(r,\frac{\psi_j'}{\psi_j+1}\right)+\log^+\frac{1}{\varepsilon_0}+C_0\right)\\ &\le c^2(r,\varrho)m_{\beta_j}(r,f_jg_j)+4m_{\gamma_j}\left(r,\frac{\psi_j'}{\psi_j}\right)+2m_{\gamma_j}\left(r,\frac{\psi_j'}{\psi_j+1}\right)\\ &\quad+2\log^+\frac{1}{\varepsilon_0}+2C_0.\end{aligned}$$

Now inserting (10) (with $\mu=n-1$) and observing that $c^4(r_0,\varrho)<2$ we obtain

$$\begin{aligned}T_{\gamma_j}(r,\psi_j) &\le c^2(r,\varrho)m_{\beta_j}\left(r,\frac{\psi_j'}{\psi_j}\right)+4m_{\gamma_j}\left(r,\frac{\psi_j'}{\psi_j}\right)+2m_{\gamma_j}\left(r,\frac{\psi_j'}{\psi_j+1}\right)+C_3\\ &\le 6m_{\gamma_j}\left(r,\frac{\psi_j'}{\psi_j}\right)+2m_{\gamma_j}\left(r,\frac{\psi_j'}{\psi_j+1}\right)+C_3\end{aligned}$$

for all $r\in]r_0,2r_0[$ with some constant C_3. On recalling (7) we deduce the normality of $\{\psi_j\}_{j=1}^\infty$ at the origin which contradicts our assumption in case 3.

Case 3.2.2. $\{\psi_j'/f_j^{n-2}\}_{j=1}^\infty$ does not converge to 0 uniformly in any neighborhood of 0.

This case is very similar to case 3.2.1: Again, we may assume that there exists an $\varepsilon_0>0$ and a sequence $\{\gamma_j\}_{j=1}^{\infty}\subseteq U_{\varrho}(0)$ such that

$$\left|\frac{\psi_j'}{f_j^{n-2}}(\gamma_j)\right|\geq\varepsilon_0 \tag{16}$$

for all j. The new ingredient in this case is the following: By the assumption in case 3, $\{\psi_j\}_{j=1}^{\infty}$ is not normal at the origin, so on skipping finitely many j we can find $\delta_j\in U_{\varrho}(0)$ such that

$$|\psi_j(\delta_j)|\leq 1$$

for all j. Using that ψ_j is analytic and non-vanishing, this gives us

$$\begin{aligned} m_{\gamma_j}(r,\psi_j) &\leq c^2(r,\varrho)\left(m_{\delta_j}(r,\psi_j)+\log\frac{1}{|\psi_j(\delta_j)|}\right)\\ &= c^2(r,\varrho)m_{\delta_j}\left(r,\frac{1}{\psi_j}\right)\leq c^4(r,\varrho)m_{\gamma_j}\left(r,\frac{1}{\psi_j}\right). \end{aligned} \tag{17}$$

If we add $N_{\gamma}(r,\psi_j)\equiv 0$ and $N_{\gamma}(r,1/(\psi_j+1))$ to both sides of (12), apply the first fundamental theorem to $T_{\gamma}(r,1/(\psi_j+1))$ and observe (17) and (13), we obtain for all $r\in]\varrho,1[$ and all j,

$$\begin{aligned} \frac{1}{c^4(r,\varrho)}m_{\gamma_j}(r,\psi_j) &\leq m_{\gamma_j}\left(r,\frac{1}{\psi_j}\right)\\ &\leq \overline{N}_{\gamma_j}\left(r,\frac{1}{\psi_j+1}\right)+2m_{\gamma_j}\left(r,\frac{\psi_j'}{\psi_j}\right)+m_{\gamma_j}\left(r,\frac{\psi_j'}{\psi_j+1}\right)\\ &\quad+\log\left|\frac{\psi_j+1}{\psi_j'}(\gamma_j)\right|+C_0+\log 2\\ &\leq \frac{1}{n-1}N_{\gamma_j}\left(r,\frac{1}{\psi_j+1}\right)+\frac{n-2}{n-1}\overline{N}_{\gamma_j}\left(r,\frac{1}{f_jg_j}\right)\\ &\quad+2m_{\gamma_j}\left(r,\frac{\psi_j'}{\psi_j}\right)+m_{\gamma_j}\left(r,\frac{\psi_j'}{\psi_j+1}\right)+\log\left|\frac{\psi_j+1}{\psi_j'}(\gamma_j)\right|\\ &\quad+C_0+\log 2\\ &\leq \frac{1}{n-1}T_{\gamma_j}(r,\psi_j)+\frac{n-2}{n-1}T_{\gamma_j}(r,f_jg_j)+2m_{\gamma_j}\left(r,\frac{\psi_j'}{\psi_j}\right)\\ &\quad+m_{\gamma_j}\left(r,\frac{\psi_j'}{\psi_j+1}\right)+\log\frac{|\psi_j+1|^{1-1/(n-1)}}{|\psi_j'|\,|f_jg_j|^{1-1/(n-1)}}(\gamma_j)\\ &\quad+C_0+2\log 2. \end{aligned} \tag{18}$$

In view of (16) and

$$\frac{1}{c^4(r_0,\varrho)}-\frac{1}{n-1}\geq\frac{4}{5}-\frac{1}{2}>\frac{1}{4},$$

this yields

$$\begin{aligned}\frac{1}{4}T_{\gamma_j}(r,\psi_j)&\leq m_{\gamma_j}(r,f_jg_j)+2m_{\gamma_j}\left(r,\frac{\psi_j'}{\psi_j}\right)+m_{\gamma_j}\left(r,\frac{\psi_j'}{\psi_j+1}\right)+\log\left|\frac{f_j^{n-2}}{\psi_j'}(\gamma_j)\right|\\&\quad+C_0+2\\&\leq c^2(r,\varrho)m_{\beta_j}(r,f_jg_j)+2m_{\gamma_j}\left(r,\frac{\psi_j'}{\psi_j}\right)+m_{\gamma_j}\left(r,\frac{\psi_j'}{\psi_j+1}\right)\\&\quad+\log^+\frac{1}{\varepsilon_0}+C_0+2\end{aligned}$$

for all $r\in]r_0,1[$. Again recalling (10) and $c^4(r_0,\varrho)<2$, we obtain

$$\begin{aligned}T_{\gamma_j}(r,\psi_j)&\leq 4c^2(r,\varrho)m_{\beta_j}\left(r,\frac{\psi_j'}{\psi_j}\right)+8m_{\gamma_j}\left(r,\frac{\psi_j'}{\psi_j}\right)+4m_{\gamma_j}\left(r,\frac{\psi_j'}{\psi_j+1}\right)+C_4\\&\leq 16m_{\gamma_j}\left(r,\frac{\psi_j'}{\psi_j}\right)+4m_{\gamma_j}\left(r,\frac{\psi_j'}{\psi_j+1}\right)+C_4\end{aligned}$$

for all $r\in]r_0,2r_0[$ with some constant C_4.

Now we use (7) again and deduce the normality of $\{\psi_j\}_{j=1}^\infty$ at the origin, a contradiction once more.

Case 4. $\{\psi_j\}_{j=1}^\infty$ is not normal at $z=0$ and one of the sequences $\{\psi_j'/\psi_j f_j^{n-1}\}_{j=1}^\infty$ and $\{\psi_j'/f_j^{n-1}\}_{j=1}^\infty$ does not converge to 0 uniformly in any neighborhood of the origin.

This case can be treated similarly to case 3.2. Since $n\geq 3$ and $(2n-1)/(n-1)\leq\frac{5}{2}<n$ we can choose a $\varrho\in\left]0,\frac{1}{2}\right[$ such that

$$\sigma:=c\left(\tfrac{1}{2},\varrho\right)<\sqrt[4]{\tfrac{5}{4}}\quad\text{and}\quad\tau:=n-\frac{\sigma^8(2n-1)}{n-\sigma^4}>0.$$

With a similar reasoning as in (13) we deduce that

$$\overline{N}_\gamma\left(r,\frac{1}{\psi_j+1}\right)\leq\frac{1}{n}N_\gamma\left(r,\frac{1}{\psi_j+1}\right)+\frac{n-1}{n}\overline{N}_\gamma\left(r,\frac{1}{g_j}\right)\tag{19}$$

for all $r\in]\varrho,1[$, $j\in\mathbb{N}$ and all $\gamma\in U_\varrho(0)$.

We may assume that case 1 does not hold. So we may assume the existence of an $\varepsilon>0$ and of a sequence $\{\alpha_j\}_{j=1}^\infty\subseteq U_\varrho(0)$ such that

$$|g_j(\alpha_j)| \geq \varepsilon \quad \text{for all } j.$$

From this and from $f_j^n = \psi_j + 1/g_j$ (see (5)) we obtain

$$\begin{aligned}
n m_{\alpha_j}(r, f_j) &\leq m_{\alpha_j}(r, \psi_j + 1) + m_{\alpha_j}\left(r, \frac{1}{g_j}\right) \\
&\leq m_{\alpha_j}(r, \psi_j) + T_{\alpha_j}(r, g_j) + \log \frac{1}{|g_j(\alpha_j)|} + \log 2 \\
&\leq m_{\alpha_j}(r, \psi_j) + m_{x\alpha_j}(r, f_j) + m_{x\alpha_j}\left(r, \frac{f_j^{(k)}}{f_j}\right) + \log \frac{1}{\varepsilon} + 1 \qquad (20) \\
&\leq m_{\alpha_j}(r, \psi_j) + c^2(r, \varrho)\left(m_{\alpha_j}(r, f_j) + m\left(r, \frac{f_j^{(k)}}{f_j}\right)\right) + \log^+ \frac{1}{\varepsilon} + 1
\end{aligned}$$

for all $r \in]\varrho, 1[$.

Case 4.1. $\{\psi_j'/\psi_j f_j^{n-1}\}_{j=1}^{\infty}$ does not converge to 0 uniformly in any neighborhood of 0.

Then we may assume that there exists an $\varepsilon_0 > 0$ and a sequence $\{\gamma_j\}_{j=1}^{\infty} \subseteq U_\varrho(0)$ such that

$$\left|\frac{\psi_j'}{\psi_j f_j^{n-1}}(\gamma_j)\right| \geq \varepsilon_0 \quad \text{for all } j. \qquad (21)$$

As in (15), from the generalized second fundamental theorem we obtain for all $r \in]\varrho, 1[$ and all j,

$$\begin{aligned}
T_{\gamma_j}(r, \psi_j + 1) &\leq \overline{N}_{\gamma_j}\left(r, \frac{1}{\psi_j + 1}\right) + 2m_{\gamma_j}\left(r, \frac{\psi_j'}{\psi_j}\right) + m_{\gamma_j}\left(r, \frac{\psi_j'}{\psi_j + 1}\right) \\
&\quad + \log\left|\frac{\psi_j(\psi_j + 1)}{\psi_j'}(\gamma_j)\right| + C_0 \\
&\leq \frac{1}{n} N_{\gamma_j}\left(r, \frac{1}{\psi_j + 1}\right) + \frac{n-1}{n}\overline{N}_{\gamma_j}\left(r, \frac{1}{g_j}\right) \qquad (22) \\
&\quad + 2m_{\gamma_j}\left(r, \frac{\psi_j'}{\psi_j}\right) + m_{\gamma_j}\left(r, \frac{\psi_j'}{\psi_j + 1}\right) + \log\left|\frac{\psi_j(\psi_j + 1)}{\psi_j'}(\gamma_j)\right| + C_0 \\
&\leq \frac{1}{n} T_{\gamma_j}(r, \psi_j + 1) + \frac{n-1}{n} T_{\gamma_j}(r, g_j) + 2m_{\gamma_j}\left(r, \frac{\psi_j'}{\psi_j}\right) \\
&\quad + m_{\gamma_j}\left(r, \frac{\psi_j'}{\psi_j + 1}\right) + \log \frac{|\psi_j|\,|\psi_j + 1|^{1-1/n}}{|\psi_j'|\,|g_j|^{1-1/n}}(\gamma_j) + C_0.
\end{aligned}$$

Here from $\psi_j+1=f_j^n g_j$ and (21) we have

$$\log\frac{|\psi_j|\,|\psi_j+1|^{1-1/n}}{|\psi_j'|\,|g_j|^{1-1/n}}(\gamma_j)=\log\left|\frac{\psi_j f_j^{n-1}}{\psi_j'}(\gamma_j)\right|\le\log^+\frac{1}{\varepsilon_0}.$$

Noting that $n/(n-1)\le\frac{3}{2}$ and using (3) yields

$$\begin{aligned}T_{\gamma_j}(r,\psi_j+1)&\le m_{\gamma_j}(r,g_j)+\frac{n}{n-1}\left(2m_{\gamma_j}\left(r,\frac{\psi_j'}{\psi_j}\right)+m_{\gamma_j}\left(r,\frac{\psi_j'}{\psi_j+1}\right)+\log\frac{1}{\varepsilon_0}+C_0\right)\\&\le m_{x\gamma_j}(r,f_j)+m_{x\gamma_j}\left(r,\frac{f_j^{(k)}}{f_j}\right)+3m_{\gamma_j}\left(r,\frac{\psi_j'}{\psi_j}\right)\\&\quad+2m_{\gamma_j}\left(r,\frac{\psi_j'}{\psi_j+1}\right)+2\log^+\frac{1}{\varepsilon_0}+2C_0\end{aligned}\tag{23}$$

for all $r\in]\varrho,1[$ and all j. Inserting (23) into (20) we arrive at

$$\begin{aligned}&nm_{\alpha_j}(r,f_j)\\&\quad\le c^4(r,\varrho)\left(2m_{\alpha_j}(r,f_j)+2m\left(r,\frac{f_j^{(k)}}{f_j}\right)+3m\left(r,\frac{\psi_j'}{\psi_j}\right)+2m\left(r,\frac{\psi_j'}{\psi_j+1}\right)\right)+C_5\end{aligned}$$

for all $r\in]\varrho,1[$ and all j with some constant C_5. Hence in view of $2c^4(\frac{1}{2},\varrho)<\frac{5}{2}\le n-\frac{1}{2}$,

$$m_{\alpha_j}(r,f_j)\le 5m\left(r,\frac{f_j^{(k)}}{f_j}\right)+8m\left(r,\frac{\psi_j'}{\psi_j}\right)+5m\left(r,\frac{\psi_j'}{\psi_j+1}\right)+2C_5$$

for all $r\in\left]\frac{1}{2},1\right[$ and all j. Now the usual arguments (cf. the end of case 3.1, below (11)) yield the assertion in this case.

Case 4.2. $\{\psi_j'/f_j^{n-1}\}_{j=1}^{\infty}$ does not converge to 0 uniformly in any neighborhood of the origin.

Then we may assume that there exists an $\varepsilon_0>0$ and a sequence $\{\gamma_j\}_{j=1}^{\infty}\subseteq U_\varrho(0)$ such that

$$\left|\frac{\psi_j'}{f_j^{n-1}}(\gamma_j)\right|\ge\varepsilon_0\tag{24}$$

for all j. Since $\{\psi_j\}_{j=1}^{\infty}$ is not normal at the origin, skipping finitely many j we can find $\delta_j\in U_\varrho(0)$ such that

$$|\psi_j(\delta_j)|\le 1$$

for all j. As in (17) this yields

$$m_{\gamma_j}(r,\psi_j)\le c^4(r,\varrho)m_{\gamma_j}\left(r,\frac{1}{\psi_j}\right).$$

Now the same arguments as in (18) and (22) lead us to

$$\begin{aligned}\frac{1}{c^4(r,\varrho)}m_{\gamma_j}(r,\psi_j) &\le \frac{1}{n}T_{\gamma_j}(r,\psi_j)+\frac{n-1}{n}T_{\gamma_j}(r,g_j)+2m_{\gamma_j}\left(r,\frac{\psi_j'}{\psi_j}\right)\\ &\quad+m_{\gamma_j}\left(r,\frac{\psi_j'}{\psi_j+1}\right)+\log\frac{|\psi_j+1|^{1-1/n}}{|\psi_j'|\,|g_j|^{1-1/n}}(\gamma_j)+C_0+2\log 2\end{aligned}$$

for all $r\in]\varrho,1[$ and all j. Here $c(r,\varrho)\ge c\left(\frac{1}{2},\varrho\right)=\sigma$, so in view of (24) we conclude that

$$\begin{aligned}\frac{n-\sigma^4}{n\sigma^4}m_{\gamma_j}(r,\psi_j) &\le m_{\gamma_j}(r,g_j)+2m_{\gamma_j}\left(r,\frac{\psi_j'}{\psi_j}\right)+m_{\gamma_j}\left(r,\frac{\psi_j'}{\psi_j+1}\right)\\ &\quad+\log\left|\frac{f_j^{n-1}}{\psi_j'}(\gamma_j)\right|+C_0+2\\ &\le m_{x\gamma_j}(r,f_j)+m_{x\gamma_j}\left(r,\frac{f_j^{(k)}}{f_j}\right)+2m_{\gamma_j}\left(r,\frac{\psi_j'}{\psi_j}\right)\\ &\quad+m_{\gamma_j}\left(r,\frac{\psi_j'}{\psi_j+1}\right)+\log^+\frac{1}{\varepsilon_0}+C_0+2\end{aligned}\tag{25}$$

for all $r\in\left]\frac{1}{2},1\right[$ and all j. Now from (20) and (25) we obtain

$$\begin{aligned}nm_{\alpha_j}(r,f_j) &\le \sigma^2\left(m_{\gamma_j}(r,\psi_j)+m_{\alpha_j}(r,f_j)+m\left(r,\frac{f_j^{(k)}}{f_j}\right)\right)+\log^+\frac{1}{\varepsilon}+1\\ &\le\left(\frac{n\sigma^8}{n-\sigma^4}+\sigma^2\right)\left(m_{\alpha_j}(r,f_j)+m\left(r,\frac{f_j^{(k)}}{f_j}\right)\right)\\ &\quad+\frac{n\sigma^7}{n-\sigma^4}\left(2m\left(r,\frac{\psi_j'}{\psi_j}\right)+m\left(r,\frac{\psi_j'}{\psi_j+1}\right)\right)+C_6\end{aligned}$$

for all $r\in\left]\frac{1}{2},1\right[$ and all j with some constant C_6. Hence in view of

$$\frac{n\sigma^8}{n-\sigma^4}+\sigma^2\le\frac{\sigma^8(2n-1)}{n-\sigma^4}=n-\tau,$$

we have

$$\tau m_{\alpha_j}(r,f_j)\le n\left(m\left(r,\frac{f_j^{(k)}}{f_j}\right)+2m\left(r,\frac{\psi_j'}{\psi_j}\right)+m\left(r,\frac{\psi_j'}{\psi_j+1}\right)\right)+C_6$$

for all $r\in\left]\frac{1}{2},1\right[$ and all j, and we can proceed in the same way as above.

This completes the proof of Theorem 1.1.

4. Proof of Theorem 1.2

We set

$$g(z):=\prod_{j=1}^{s}f^{(k_j)}(x_jz)\quad\text{and}\quad a:=-\frac{1}{f^n}\frac{\psi'}{\psi}.$$

Then $g\not\equiv 0$ as f is transcendental. We will later (in (26)) see that a can be written as the product of g and a combination of logarithmic derivatives. This is the crucial observation in our proof. By the lemma on the logarithmic derivative we have

$$\begin{aligned}T(r,g)=m(r,g)&\le\sum_{j=1}^{s}m(r|x_j|,f^{(k_j)})\le\sum_{j=1}^{s}\left(m(r,f)+m\left(r,\frac{f^{(k_j)}}{f}\right)\right)\\&\le sm(r,f)+S(r,f).\end{aligned}$$

In particular, we have $S(r,g)\le S(r,f)$ and $S(r,\psi)\le S(r,f)$.

If ψ would be constant, then there would be a $\gamma\in\mathbb{C}\setminus\{0\}$ such that $f^n=\gamma/cg$. Hence by the first fundamental theorem

$$nT(r,f)=T\left(r,\frac{\gamma}{cg}\right)\le T(r,g)+T(r,c)+O(1)\le sm(r,f)+S(r,f),$$

and thus

$$(n-s)T(r,f)\le S(r,f),$$

which is impossible. Therefore ψ is non-constant and so $a\not\equiv 0$. Now from

$$\psi'=c'f^ng+ncf^{n-1}f'g+cf^ng'$$

we obtain

$$a=\frac{1}{f^n}\left(\psi'-(\psi+1)\frac{\psi'}{\psi}\right)=c'g+nc\frac{f'}{f}g+cg'-cg\frac{\psi'}{\psi}=cg\left(\frac{c'}{c}+n\frac{f'}{f}+\frac{g'}{g}-\frac{\psi'}{\psi}\right).\tag{26}$$

This yields

$$(27)\qquad \begin{aligned} m(r,a) &\le m(r,c)+m(r,g)+m\left(r,\frac{c'}{c}\right)+m\left(r,\frac{f'}{f}\right)\\ &\quad +m\left(r,\frac{g'}{g}\right)+m\left(r,\frac{\psi'}{\psi}\right)+\log 4+\log n\\ &\le m(r,g)+S(r,f)+S(r,c)+S(r,g)+S(r,\psi)\\ &\le sm(r,f)+S(r,f). \end{aligned}$$

Furthermore, from (26) we see that

$$N(r,a)\le N(r,c')+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{\psi}\right)\le T(r,f)+\overline{N}\left(r,\frac{1}{\psi}\right)+S(r,f).$$

Combining this with (27) yields

$$T(r,a)\le (s+1)m(r,f)+\overline{N}\left(r,\frac{1}{\psi}\right)+S(r,f).$$

Using the definition of a we deduce that

$$\begin{aligned} nT(r,f) = m(r,f^n) &\le m\left(r,\frac{1}{a}\right)+m\left(r,\frac{\psi'}{\psi}\right)\\ &\le T(r,a)+S(r,\psi)\le (s+1)m(r,f)+\overline{N}\left(r,\frac{1}{\psi}\right)+S(r,f). \end{aligned}$$

Hence

$$(n-s-1)T(r,f)\le \overline{N}\left(r,\frac{1}{\psi}\right)+S(r,f).$$

If ψ would have only finitely many zeros, then we would have $\overline{N}(r,1/\psi)=O(\log r)=S(r,f)$ since f is transcendental. So in view of $n-s-1\ge 1$ we would deduce $T(r,f)=S(r,f)$, a contradiction. This shows the assertion.

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