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Whitney coverings and the tent spaces $T^{1,q}(\gamma)$ for the Gaussian measure

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Abstract. We introduce a technique for handling Whitney decompositions in Gaussian harmonic analysis and apply it to the study of Gaussian analogues of the classical tent spaces $T^{1,q}$ of Coifman–Meyer–Stein.

1. Introduction

Much of modern harmonic analysis in Euclidean spaces depends upon the fact that the Lebesgue measure is compatible with the scalar multiplication in the sense that for any ball B in $\mathbb{R}^n$ we have $|2B|=2^n|B|$; here $2B$ is the ball with the same centre and twice the radius of B. Indeed, many results proved originally in the Euclidean setting have been extended to metric spaces endowed with a doubling measure μ, i.e., a measure satisfying $\mu(2B)\le C\mu(B)$ for some constant C depending only upon μ.

It is a simple matter to verify that the standard Gaussian measure γ on $\mathbb{R}^n$,

$$d\gamma(x)=(2\pi)^{-n/2}\exp\big(-\tfrac{1}{2}|x|^2\big)\,dx,$$

is non-doubling. In their seminal paper [5], Mauceri and Meda found a way around this by introducing the class of *admissible balls*. These are the balls $B=B(c_B,r_B)$ in $\mathbb{R}^n$ satisfying the smallness condition

$$r_B\le\min\left\{1,\frac{1}{|c_B|}\right\}.$$

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Mauceri and Meda show that admissible balls enjoy a doubling condition. Armed with this, many results from the Euclidean case can be carried over to the Gaussian case, as long as one is able to work with admissible balls only. Mauceri and Meda were thus able to define Gaussian counterparts of the spaces H^1 and BMO and extend parts of the Calderón–Zygmund theory to the Gaussian setting. Some of these results have even been extended to a more general class of locally doubling metric measure spaces in [1] and [2].

Another important tool of Euclidean harmonic analysis is the Whitney covering method. This technique allows one to cover open sets O with dyadic cubes whose sizes are proportional to the distance of the cube to the complement of O. In the Gaussian case, one runs into the problem that admissible cubes become very small at large distances from the origin. As a consequence, the distance of such a cube to the exterior of a given open set is typically much larger than the size of the cube. At first sight, this renders Whitney covering useless as a tool in the Gaussian setting. The purpose of this note is to show how Whitney covering, too, can be adapted to the Gaussian setting. To illustrate its usefulness, we use it to prove an atomic decomposition theorem and a change of aperture theorem for the Gaussian analogue of the tent space $T^{1,q}$ of Coifman–Meyer–Stein.

2. Admissible balls and cubes

Throughout this paper we fix the dimension $n\geq 1$. As usual we denote by

$$B(x,r):=\{y\in\mathbb{R}^n : |x-y|<r\}$$

the open ball in $\mathbb{R}^n$ centred at x with radius r. Following Mauceri and Meda [5] we begin by introducing the class of admissible balls.

Definition 2.1. For $\alpha>0$ we define

$$\mathscr{B}_\alpha := \bigl\{B(x,r) : x\in\mathbb{R}^n \text{ and } 0<r\leq \alpha m(x)\bigr\},$$

where

$$m(x):=\min\Bigl\{1,\frac{1}{|x|}\Bigr\},\quad x\in\mathbb{R}^n.$$

The balls in $\mathscr{B}_\alpha$ are said to be *admissible at scale* α.

It is a fundamental observation of Mauceri–Meda [5] that admissible balls enjoy a doubling property.

Lemma 2.2. (Doubling property) *Let $\alpha,\tau>0$. There exists a constant $d=d_{\alpha,\tau,n}$, depending only upon α, τ, and the dimension n, such that if $B_1=B(c_1,r_1)\in\mathscr{B}_\alpha$ and $B_2=B(c_2,r_2)$ have nonempty intersection and $r_2\le\tau r_1$, then*

$$\gamma(B_2)\le d\gamma(B_1).$$

In particular this lemma implies that for all $\alpha>0$ there exists a constant $d'=d'_{\alpha,n}$ such that for all $B(x,r)\in\mathscr{B}_\alpha$ we have

$$\gamma(B(x,2r))\le d'\gamma(B(x,r)).$$

Lemma 2.3. *Let $a,b>0$ be given.*
(i) *If $r\le am(x)$ and $|x-y|<br$, then $r\le a(1+ab)m(y)$.*
(ii) *If $|x-y|<bm(x)$, then $m(x)\le(1+b)m(y)$ and $m(y)\le(2+2b)m(x)$.*

Proof. (i) If $|y|\le1$, then $m(y)=1$ and $r\le am(x)\le a=am(y)$.
If $1<|y|\le1+ab$, then $m(y)\ge1/(1+ab)$ and

$$r\le a\le a(1+ab)m(y).$$

If $|y|>1+ab$, then $m(y)=1/|y|$ and

$$r\le\frac{a}{|x|}\le\frac{a}{|y|-br}\le\frac{a}{|y|-ab}\le\frac{a(1+ab)}{|y|}=a(1+ab)m(y).$$

(ii) Put $r'=m(x)$. Then $|x-y|<br'$ and therefore (i) (with $a=1$) implies that $r'\le(1+b)m(y)$. This gives the first estimate. To obtain the second we consider three cases. If $|x|\le1$, then $(2+2b)m(x)\ge1\ge m(y)$. If $1\le|x|\le2b$, then $(2+2b)m(x)\ge(2+2b)/2b\ge1\ge m(y)$. If $|x|\ge1$ and $|x|\ge2b$, then $|y|\ge|x|-b/|x|\ge|x|-\frac12\ge\frac12|x|$, and thus $m(y)\le2m(x)\le(2+2b)m(x)$. $\square$

For $m\in\mathbb{Z}$ let Δ_m be the set of dyadic cubes at scale m, i.e.,

$$\Delta_m=\{2^{-m}(x+[0,1)^n):x\in\mathbb{Z}^n\}.$$

In the Gaussian setting the idea is to use, at every scale, cubes whose diameter depends upon another parameter $l\ge0$, which keeps track of the distance from the cube to the origin. More precisely, define the *layers*

$$L_0=[-1,1)^n\quad\text{and}\quad L_l=[-2^l,2^l)^n\setminus[-2^{l-1},2^{l-1})^n,\ l\ge1,$$

and define, for $k\in\mathbb{Z}$ and $l\ge0$,

$$\Delta^\gamma_{k,l}=\{Q\in\Delta_{l+k}:Q\subseteq L_l\},\quad \Delta^\gamma_k=\bigcup_{l\ge0}\Delta^\gamma_{k,l}\quad\text{and}\quad\Delta^\gamma=\bigcup_{k\ge0}\Delta^\gamma_k.$$

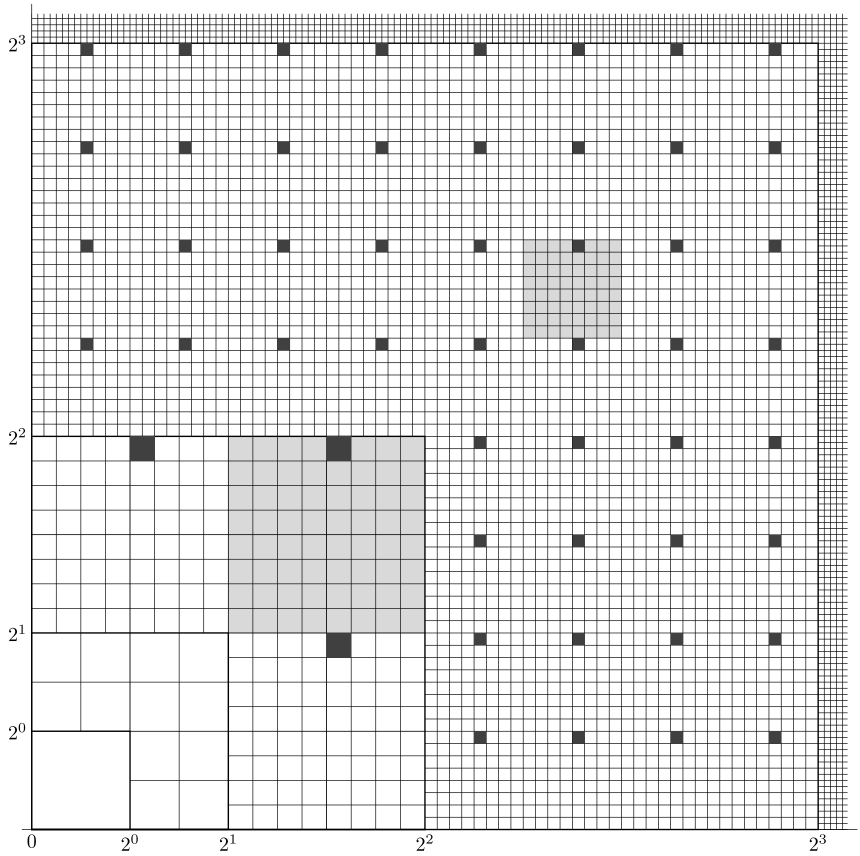

Figure 1. Subdivision of the layers $L_0, L_1, L_2, \ldots$ into cubes of $\Delta^\gamma_{0,l}$. The (5,8)-cubes in the layers L_2 and L_3 corresponding to the choice $\varkappa=3$ are *black*. A cube from $\Delta^\gamma_{-3,2}$ and a cube from $\Delta^\gamma_{-3,3}$ have been coloured *grey*.

Note that $\Delta^\gamma_{0,0}$ consists of 2^n cubes of side length 1, and that $\Delta^\gamma_{k,l}=\varnothing$ for all other $k\le -2l$. Also, if $Q\in\Delta^\gamma_{k,l}$, then Q has side-length 2^{-k-l}, diameter $2^{-k-l}\sqrt{n}$, and its centre x has norm $|x|\ge 2^{l-1}$.

Fix an integer $\varkappa\ge 1$. For each $l\ge\lceil\frac{1}{2}(\varkappa+1)\rceil$, the layer L_l is a disjoint union of cubes in $\Delta^\gamma_{-\varkappa,l}$, each of which is the disjoint union of $2^{\varkappa n}$ cubes from $\Delta^\gamma_{0,l}$. Each such cube can be labelled by a label $i=(i_1, \ldots, i_n)\in\{1, \ldots, 2^\varkappa\}^n$. See Figure 1, where $n=2$, $\varkappa=3$, and the shaded cubes are the cubes from $\Delta^\gamma_{0,l}$ with label $i=(5,8)$ for $l=2,3$.

For a set $A\subseteq\mathbb{R}^n$ we write

$$A+\mathscr{C}_\alpha=\{z\in\mathbb{R}^n : z \text{ is the centre of a ball } B\in\mathscr{B}_\alpha \text{ that intersects } A\}. \tag{1}$$

Lemma 2.4. *Let* $p \geq 0$ *and* $l \geq p+2$ *be integers, let* $Q \in \Delta^{\gamma}_{0,l}$ *be given, and consider a ball* $B = B(c_B, r_B)$ *in* $\mathscr{B}_{2^p}$ *intersecting* Q*. Then we have* $c_B \in L_{l-1} \cup L_l \cup L_{l+1}$.

Proof. Suppose first that we had $c_B \in L_{l-m}$ for some $2 \leq m \leq l$. On the one hand, $r_B \leq 2^p \leq 2^{l-2}$. On the other hand, the distance between the layers L_l and L_{l-m} is at least $2^{l-2}+2^{l-3}+...+2^{l-m} = 2^{l-1}-2^{l-m} \geq 2^{l-2}$. Since B is open, it would follow that B does not intersect $Q \subseteq L_l$.

The proof that c_B cannot be in L_{l+m} for any $m \geq 2$ is similar and requires only cruder estimates. □

Lemma 2.5. *Fix integers* $p \geq 1$ *and* $\varkappa \geq p+4$*. Let* $i \in \{1, ..., 2^{\varkappa}\}^n$ *and let* $Q_1 \in \Delta^{\gamma}_{0,l_1}$ *and* $Q_2 \in \Delta^{\gamma}_{0,l_2}$ *be two distinct cubes with the same label* i *in the layers* L_{l_1} *and* L_{l_2} *with* $l_1, l_2 \geq \max\{p+2, \lceil \frac{1}{2}(\varkappa+1) \rceil\}$*. Then*

$$d(Q_1+\mathscr{C}_{2^p}, Q_2+\mathscr{C}_{2^p}) > 0.$$

Proof. We consider the case when one of the cubes, say Q_1, lies in layer l and the other, say Q_2, lies in layer $l+1$; the case where both cubes lie in the same layer or are more than one layer apart can be handled with cruder estimates.

The centre of a ball $B = B(c_B, r_B)$ in $\mathscr{B}_{2^p}$ intersecting a layer L_l satisfies $|c_B| \geq 2^{l-1} - r_B \geq 2^{l-1} - 2^p |c_B|^{-1}$, which in view of Lemma 2.4 implies that $|c_B| \geq 2^{l-1} - 2^{p-l+2}$. Therefore $r_B \leq 2^p/(2^{l-1}-2^{p-l+2})$. For $j \in \{1, 2\}$ let $B_j = B(c_{B_j}, r_{B_j})$ be a ball in $\mathscr{B}_{2^p}$ intersecting Q_j. It follows that

$$r_{B_1} \leq \frac{1}{2^{l-p-1}-2^{-l+2}} \quad \text{and} \quad r_{B_2} \leq \frac{1}{2^{l-p}-2^{-l+1}}.$$

The cubes Q_1 and Q_2 are separated by at least $2^{\varkappa}-1$ cubes in $\Delta^{\gamma}_{0,l}$ or $\Delta^{\gamma}_{0,l+1}$, so the distance between Q_1 and Q_2 is at least $(2^{\varkappa}-1)/2^{l+1}$. Hence, using that $l \geq p+2 \geq 3$,

$$\begin{aligned} d(Q_1+\mathscr{C}_{2^p}, Q_2+\mathscr{C}_{2^p}) &\geq \frac{2^{\varkappa}-1}{2^{l+1}} - \left(\frac{1}{2^{l-p-1}-2^{-l+2}} + \frac{1}{2^{l-p}-2^{-l+1}} \right) \\ &\geq \frac{2^{\varkappa}-1}{2^{l+1}} - \left(\frac{1}{2^{l-p-1}-2^{-p}} + \frac{1}{2^{l-p}-2^{-p-1}} \right) \\ &= \frac{2^{\varkappa}-1}{2^{l+1}} - 2^p \left(\frac{2}{2^l-2} + \frac{1}{2^l-\frac{1}{2}} \right) \\ &\geq \frac{2^{\varkappa}-1}{2^{l+1}} - 2^p \frac{3}{2^l-2} \end{aligned}$$

$$\geq \frac{2^{\varkappa}-1}{2^{l+1}} - 2^p \frac{8}{2^{l+1}}$$

$$= \frac{2^{\varkappa}-2^{p+3}-1}{2^{l+1}},$$

and the right-hand side is strictly positive since $\varkappa \geq p+4$. □

In the remainder of this section we fix the integer $p \geq 2$ and take $\varkappa = p+4$. Note that all $l \geq p+2$ then satisfy the assumptions of Lemma 2.5.

Definition 2.6. The set $A_p^{(i)}$ is the union of all cubes in $\bigcup_{l \geq p+2} \Delta_{0,l}^{\gamma}$ with label $i \in \{1, ..., 2^{p+4}\}^n$.

Definition 2.7. Let $\lambda > 0$. A set $A \subseteq \mathbb{R}^n$ is said to be a *λ-admissible Whitney set* if for all $x \in A$ we have

$$d(x, \complement A) \leq \lambda m(x).$$

Clearly, subsets of admissible Whitney sets are admissible Whitney.

Theorem 2.8. *Let $p \geq 2$.*
(i) $Q + \mathscr{C}_{2^p}$ *with* $Q \in \Delta_{0,l}^{\gamma}$ *and* $l = 0, ..., p+1$ *is* $2^{2p+2}\sqrt{n}$*-admissible Whitney*;
(ii) $A_p^{(i)} + \mathscr{C}_{2^p}$ *with* $i \in \{1, ..., 2^{p+4}\}^n$ *is* $2^{p+3}\sqrt{n}$*-admissible Whitney.*

Proof. In view of Lemma 2.5 both assertions follow from the fact (proved next) that $Q + \mathscr{C}_{2^p}$ is admissible Whitney for any cube $Q \in \Delta_{0,l}^{\gamma}$, with constant $2^{2p+2}\sqrt{n}$ for $l = 0, ..., p+1$ and constant $2^{p+3}\sqrt{n}$ for $l \geq p+2$.

First let $l \in \{0, ..., p+1\}$. Let $Q \in \Delta_{0,l}^{\gamma}$, consider a ball $B = B(c_B, r_B) \in \mathscr{B}_{2^p}$ intersecting Q, and remark that $r_B \leq 2^p$. It follows that

$$Q + \mathscr{C}_{2^p} \subseteq \{z \in \mathbb{R}^n : d(z, Q) \leq 2^p\}.$$

Let $z \in Q + \mathscr{C}_{2^p}$ be given. If $z \in Q$, then the distance of z to the complement of $Q + \mathscr{C}_{2^p}$ is at most $\frac{1}{2} + 2^p$. At the same time, $m(z) \geq 1/2^{p+1}\sqrt{n}$ (since $z \in L_0 \cup ... \cup L_{p+1}$). If $z \notin Q$, then the distance of z to the complement of $Q + \mathscr{C}_{2^p}$ is at most 2^p. At the same time, $m(z) \geq 1/2^{p+2}\sqrt{n}$ (since $z \in L_0 \cup ... \cup L_{p+2}$). In both cases, the inequality in Definition 2.7 is satisfied.

Next let $l \geq p+2$. Let $Q \in \Delta_{0,l}^{\gamma}$ be given and consider a ball $B = B(c_B, r_B)$ in $\mathscr{B}_{2^p}$ intersecting Q. Using Lemma 2.4 we find that $r_B \leq 2^p |c_B|^{-1} \leq 2^{p-l+2}$. It follows that

$$Q + \mathscr{C}_{2^p} \subseteq \{z \in \mathbb{R}^n : d(z, Q) \leq 2^{p-l+2}\}.$$

Now let $z\in Q+\mathscr{C}_{2^p}$ be given. If $z\in Q$, then the distance of z to the complement of $Q+\mathscr{C}_{2^p}$ is at most $2^{-l-1}+2^{p-l+2}$. At the same time, $m(z)\geq 1/2^l\sqrt{n}$ (since $z\in L_l$). If $z\notin Q$, then the distance of z to the complement of $Q+\mathscr{C}_{2^p}$ is at most 2^{p-l+2}. At the same time, $m(z)\geq 1/2^{l+1}\sqrt{n}$ (since $z\in L_{l-1}\cup L_l\cup L_{l+1}$). In each of these cases, the inequality in Definition 2.7 is satisfied. □

Corollary 2.9. *There exists a constant N, depending only on $p\geq 2$ and the dimension n, such that every open set in $\mathbb{R}^n$ can be covered by N open $2^{2p+2}\sqrt{n}$-admissible Whitney sets.*

An explicit bound on N is obtained by counting the number of sets involved in Theorem 2.8, which can be estimated by $2^n(1+2^{(p+4)n}+...+2^{(p+1)(p+4)n})+2^{(p+4)n}$.

The next result is an immediate consequence of its Euclidean counterpart (see [7, Section VI.1] for the details). The cubes that we pick up from the Euclidean proof will automatically be admissible at a suitable scale (which depends upon n only) because we start from an admissible Whitney set.

Lemma 2.10. *Let $\lambda>0$ and suppose $O\subseteq\mathbb{R}^n$ is an open λ-admissible Whitney set. There exists a constant ρ, depending only on λ and the dimension n, a countable family of disjoint cubes $\{Q_m\}_m$ in Δ^γ, and a family of functions $\{\phi_m\}_m\subseteq C_c^\infty(\mathbb{R}^n)$ such that*

(i) $\bigcup_m Q_m=O$;

(ii) $\operatorname{diam} Q_m\leq d(Q_m,\complement O)\leq\rho\operatorname{diam} Q_m$ *for all m*;

(iii) $\operatorname{supp}\phi_m\subseteq Q_m^*$ *for all m, where Q_m^* denotes the cube with the same centre as Q_m but side length multiplied by ρ*;

(iv) $1/\rho\leq\phi_m(x)\leq 1$ *for all m and all $x\in Q_m$*;

(v) $\sum_m\phi_m(x)=1$ *for all $x\in O$.*

3. Gaussian tent spaces

Throughout this section we fix $1<q<\infty$ and let $q':=q/(q-1)$ denote its conjugate exponent. Let

$$D:=\{(x,t)\in\mathbb{R}^n\times(0,\infty):t<m(x)\}.$$

Note that a point $(x,t)\in\mathbb{R}^d\times(0,\infty)$ belongs to $\overline{D}$ if and only if $B(x,t)\in\mathscr{B}_1$.

Definition 3.1. The *Gaussian tent space* $T^{1,q}(\gamma)$ is the completion of $C_c(D)$ with respect to the norm

$$\|f\|_{T^{1,q}(\gamma)}:=\|Jf\|_{L^1(\mathbb{R}^n,d\gamma(x);L^q(D,d\gamma(y)\,dt/t))},$$

where

$$(Jf(x))(y,t) := \frac{\mathbb{1}_{B(y,t)}(x)}{\gamma(B(y,t))^{1/q}} f(y,t).$$

For a measurable set $A\subseteq\mathbb{R}^n$ and a real number $\alpha>0$ we define the *tent with aperture α over A* by

$$T_\alpha(A) := \{(y,t)\in\mathbb{R}^n\times(0,\infty) : d(y,\complement A)\geq \alpha t\}.$$

Definition 3.2. Let α be a positive real number. A function $a:D\to\mathbb{C}$ is called a $T^{1,q}(\gamma)$ *α-atom* if there exists a ball $B\in\mathscr{B}_\alpha$ such that

(i) a is supported in $T_1(B)\cap D$;

(ii) $\|a\|_{L^q(D,d\gamma\,dt/t)}\leq\gamma(B)^{-1/q'}$.

Lemma 3.3. *If a is a $T^{1,q}(\gamma)$ α-atom, then $a\in T^{1,q}(\gamma)$ and $\|a\|_{T^{1,q}(\gamma)}\leq 1$.*

Proof. Let a be a $T^{1,q}(\gamma)$ α-atom supported in $T_1(B)\cap D$ for some $B\in\mathscr{B}_\alpha$. If $(y,t)\in T_1(B)\cap D$ and $x\in B(y,t)$, then $x\in B$. First using this fact, then Hölder's inequality, then the Fubini theorem, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n}\left(\iint_D \frac{\mathbb{1}_{B(y,t)}(x)}{\gamma(B(y,t))}|a(y,t)|^q\,d\gamma(y)\frac{dt}{t}\right)^{1/q} d\gamma(x)\\
&\qquad= \int_{\mathbb{R}^n}\left(\iint_D \frac{\mathbb{1}_{B(y,t)}(x)}{\gamma(B(y,t))}|a(y,t)|^q\,d\gamma(y)\frac{dt}{t}\right)^{1/q}\mathbb{1}_B(x)\,d\gamma(x)\\
&\qquad\leq \left(\int_{\mathbb{R}^n}\iint_D \frac{\mathbb{1}_{B(y,t)}(x)}{\gamma(B(y,t))}|a(y,t)|^q\,d\gamma(y)\frac{dt}{t}\,d\gamma(x)\right)^{1/q}\gamma(B)^{1/q'}\\
&\qquad= \left(\iint_D |a(y,t)|^q\,d\gamma(y)\frac{dt}{t}\right)^{1/q}\gamma(B)^{1/q'}\\
&\qquad\leq 1. \qquad\square
\end{aligned}$$

The set D admits a locally finite cover with tents $T_1(B)$ based at balls $B\in\mathscr{B}_\alpha$ if and only if $\alpha>1$; this explains the condition $\alpha>1$ in the next theorem, which establishes an atomic decomposition of $T^{1,q}(\gamma)$. The proof follows the lines of the Euclidean counterpart in [3] (see also the expanded version in the setting of spaces of homogeneous type [6]). However, one needs to be careful not to use a doubling property for non-admissible balls; it is here where the results of the previous section come to rescue.

Theorem 3.4. (Atomic decomposition) *For all* $f\in T^{1,q}(\gamma)$ *and* $\alpha>1$, *there exist a sequence* $\{\lambda_n\}_{n\geq 1}\in\ell_1$ *and a sequence of* $T^{1,q}(\gamma)$ α*-atoms* $\{a_n\}_{n\geq 1}$ *such that*

(i) $f=\sum_{n\geq 1}\lambda_n a_n$;
(ii) $\sum_{n\geq 1}|\lambda_n|\lesssim\|f\|_{T^{1,q}(\gamma)}$.

Before we start with the proof, we need some notation and auxiliary results. Given a measurable set $A\subseteq\mathbb{R}^n$ and a real number $\alpha>0$, we define

$$R_\alpha(A)=\{(y,t)\in\mathbb{R}^d\times(0,\infty):d(y,A)<\alpha t\}=\complement T_\alpha(\complement A).$$

We also put, for any measurable set $A\subseteq\mathbb{R}^n$ and real number $\beta>0$,

$$A^{[\beta]}=\left\{x\in\mathbb{R}^n:\frac{\gamma(A\cap B)}{\gamma(B)}\geq\beta \text{ for all } B\in\mathscr{B}_{3/2} \text{ with centre } x\right\}.$$

We call $A^{[\beta]}$ the set of points of admissible β-density of A. Note that $A^{[\beta]}$ is a closed subset of $\mathbb{R}^n$ contained in $\bar{A}$.

Lemma 3.5. *For all* $\eta\in\left(\frac{1}{2},1\right)$ *there exists an* $\bar\eta\in(0,1)$ *such that, for all measurable sets* $F\subseteq\mathbb{R}^n$ *and all non-negative measurable functions* H *on* D,

$$\iint_{R_{1-\eta}(F^{[\bar\eta]})\cap D}H(y,t)\,d\gamma(y)\frac{dt}{t}\lesssim\int_F\iint_D\frac{\mathbb{1}_{B(y,t)}(x)}{\gamma(B(y,t))}H(y,t)\,d\gamma(y)\frac{dt}{t}\,d\gamma(x).$$

Proof. First let $\bar\eta\in(0,1)$ be arbitrary and fixed. Let $(y,t)\in R_{1-\eta}(F^{[\bar\eta]})\cap D$. Note that $(y,t)\in D$ implies that $B(y,t)\in\mathscr{B}_1$. There exists $x\in F^{[\bar\eta]}$ such that $|y-x|<(1-\eta)t$. Notice first that, since $t\leq m(y)$, we have $|x|<(1-\eta)t+1/t\leq\frac{1}{2}+1/t\leq 3/2t$. We thus have that $t\in\left(0,\frac{3}{2}m(x)\right)$. Moreover $B(x,\eta t)\subseteq B(y,t)\subseteq B\left(x,\frac{3}{2}t\right)$, and hence $B(y,t)\in\mathscr{B}_1$, $B(x,t)\in\mathscr{B}_{3/2}$, and $\gamma(B(x,t))\eqsim\gamma(B(y,t))$ by repeated application of the doubling property on admissible balls (Lemma 2.2). We therefore have

$$\begin{aligned}\gamma(F\cap B(y,t))&\geq\gamma(F\cap B(x,t))-\gamma(B(x,t)\cap\complement B(y,t))\\&\geq\bar\eta\gamma(B(x,t))-\gamma(B(x,t))+\gamma(B(x,t)\cap B(y,t))\\&\geq(\bar\eta-1)\gamma(B(x,t))+\gamma(B(x,\eta t)).\end{aligned}$$

Now, picking $\bar\eta$ close enough to 1 and using the doubling property, we obtain a constant $c=c(\eta,n)\in(0,1)$ such that

$$\gamma(F\cap B(y,t))\geq c\gamma(B(x,t)).$$

Therefore, there is a constant $c'=c'(\eta,n)>0$ such that $\gamma(F\cap B(y,t))\geq c'\gamma(B(y,t))$ for all $(y,t)\in R_{1-\eta}(F^{[\bar\eta]})\cap D$. Finally,

$$\int_F \iint_D \frac{\mathbb{1}_{B(y,t)}(x)}{\gamma(B(y,t)}H(y,t)\,d\gamma(y)\frac{dt}{t}\,d\gamma(x) = \iint_D \frac{\gamma(F\cap B(y,t))}{\gamma(B(y,t))}H(y,t)\,d\gamma(y)\frac{dt}{t}$$
$$\geq c'\iint_{R_{1-\eta}(F^{[\bar\eta]})\cap D} H(y,t)\,d\gamma(y)\frac{dt}{t}. \quad \square$$

Lemma 3.6. *If a function $f\in T^{1,q}(\gamma)$ admits a decomposition in terms of $T^{1,q}(\gamma)$ α-atoms for some $\alpha>1$, then it admits a decomposition in terms of $T^{1,q}(\gamma)$ α-atoms for all $\alpha>1$.*

Proof. Suppose that $f\in T^{1,q}(\gamma)$ admits a decomposition in terms of $T^{1,q}(\gamma)$ β-atoms for some $\beta>1$. We will show that f admits a decomposition in terms of $T^{1,q}(\gamma)$ α-atoms for any $\alpha>1$. This is immediate if $\alpha\geq\beta$, since in this case any $T^{1,q}(\gamma)$ β-atom is a $T^{1,q}(\gamma)$ α-atom as well.

Let us now assume that $1<\alpha<\beta$. We claim that it suffices to show that there exists an integer N, depending only upon α, β, and the dimension n, such that if $B\in\mathscr{B}_\beta$, then $T_1(B)\cap D$ can be covered by at most N tents of the form $T_1(B')$ with $B'=B(c',r')\in\mathscr{B}_\alpha$ satisfying $r'=\alpha m(c')$.

To prove the claim, it clearly suffices to consider the case that f is a $T^{1,q}(\gamma)$ β-atom having support in $T_1(B)\cap D$ for some ball $B\in\mathscr{B}_\beta$ with centre c and radius $r=\beta m(c)$. Let $\{T_1(B_1'),...,T_1(B_N')\}$ be a covering of $T_1(B)$, where each B_j', $j=1,...,N$, is a ball in $\mathscr{B}_\alpha$ with centre c_j, radius $r_j=\alpha m(c_j)$, and intersecting B. For $x\in T_1(B)\cap D$ we set $n(x):=\#\{1\leq j\leq N: x\in T_1(B_j')\}$ and $f_j(x):=n^{-1}(x)f(x)\mathbb{1}_{T_1(B_j')}(x)$. It then follows that $f=\sum_{j=1}^N f_j$. Moreover, each f_j is a $T^{1,q}(\gamma)$ α-atom, since f_j is supported in $T_1(B_j)\cap D$ and

$$\|f_j\|_{L^q(D,d\gamma\,dt/t)}\leq\|f\|_{L^q(D,d\gamma\,dt/t)}\leq\gamma(B)^{-1/q'}\lesssim\gamma(B_j')^{-1/q'}.$$

To obtain the latter estimate, we pick an arbitrary $b\in B_j'\cap B$ and use Lemma 2.3(ii) to conclude that $m(c_j)\leq(1+\alpha)m(b)\leq 2(1+\alpha)(1+\beta)m(c)$, and then we estimate

$$r_j=\alpha m(c_j)\leq 2\alpha(1+\alpha)(1+\beta)m(c)=2\alpha\beta^{-1}(1+\alpha)(1+\beta)r.$$

Combined with Lemma 2.2, we infer that $\gamma(B_j)\lesssim\gamma(B)$. It follows that $f=\sum_{j=1}^N f_j$ is a decomposition in terms of $T^{1,q}(\gamma)$ α-atoms, which proves the claim.

Fix $R>1+\beta$ so large that $\alpha(R-\beta)/(R-\beta+\alpha)>1$. The set $\{(y,t)\in D:|y|\leq R\}$ can be covered with finitely many sets—their number depending only upon R, n and α—of the form $T_1(B')$ with $B'=B(c',r')\in\mathscr{B}_\alpha$ and $r'=\alpha m(c')$.

Take a ball $B=B(c,r)\in\mathscr{B}_\beta$ with $|c|\ge R$ and choose $\delta\in(0,1)$ so small that $(1-\delta)\alpha(R-\beta)/(R-\beta+\alpha)>1$. We first remark that, if $x\in B$, then $|x|\ge R-\beta\ge 1$, and therefore $m(x)=1/|x|$. Let us then define

$$C_B:=\{(x,t)\in B\times(0,\infty):t\le m(x)\}.$$

Noting that $T_1(B)\cap D\subseteq C_B$, it remains to cover C_B with N tents $T_1(B')$ based on balls $B'\in\mathscr{B}_\alpha$, where the number N depends on α, β and n only.

To do so, let us start by picking $c'\in B$, and let $r'=\alpha m(c')=\alpha/|c'|$ and $B'=B(c',r')$. If $(x,t)\in C_B$ is such that $|x-c'|\le\delta r'$, then

$$\begin{aligned}d(x,\complement B')=d(c',\complement B')-|x-c'|&\ge(1-\delta)r'=(1-\delta)\frac{\alpha}{|c'|}\ge(1-\delta)\frac{\alpha}{|x|+|x-c'|}\\&\ge m(x)(1-\delta)\frac{\alpha|x|}{|x|+\alpha}\ge m(x)(1-\delta)\frac{\alpha(R-\beta)}{R-\beta+\alpha}\ge m(x)\ge t.\end{aligned}$$

Here we used the monotonicity of the function $t\mapsto t/(t+\alpha)$.

We have proved that a point $(x,t)\in C_B$ belongs to $T_1(B')$ whenever $|x-c'|\le\delta r'$. Using that $(|c|+\beta)r\le(|c|+\beta)\beta/|c|\le\beta+\beta^2$, we have that

$$r'=\frac{\alpha}{|c'|}\ge\frac{\alpha}{|c|+\beta}\ge\frac{\alpha}{\beta+\beta^2}r.$$

This implies that B can be covered with N balls $B'=B(c',\delta r')$ as above, with N depending only on α, β and n. The union of the N sets $T_1(B')\cap D$ will then cover C_B. The proof is complete. □

Proof of Theorem 3.4. By Lemma 3.6 it suffices to prove that each $f\in T^{1,q}(\gamma)$ admits a decomposition in terms of $T^{1,q}(\gamma)$ α-atoms for *some* $\alpha>0$.

Recall that the disjoint sets $A_p^{(i)}$ have been introduced in Definition 2.6. We shall apply Theorem 2.8 with $p=4$ (the reason for this choice is the constant $16=2^4$ produced in the argument below). Since

$$\biggl(\bigcup_{l=0}^{5}L_l\biggr)\cup\biggl(\bigcup_{i\in\{1,\dots,2^8\}^n}A_4^{(i)}\biggr)=\mathbb{R}^n$$

we may write

$$f=f\mathbb{1}_{\{\|Jf\|_q>0\}}=\sum_{l=0}^{5}\sum_{Q\in\Delta_{0,l}^{\gamma}}f\mathbb{1}_{Q\cap\{\|Jf\|_q>0\}}+\sum_{i\in\{1,\dots,2^8\}^n}f\mathbb{1}_{A_4^{(i)}\cap\{\|Jf\|_q>0\}},\tag{2}$$

where we use the notation

$$\{\|Jf\|_q>0\}:=\{x\in\mathbb{R}^n:\|Jf(x)\|_{L^q(D,d\gamma(y)\,dt/t)}>0\}$$

and

$$f\mathbb{1}_{\{\|Jf\|_q>0\}}(x,t) := f(x,t)\mathbb{1}_{\{\|Jf\|_q>0\}}(x).$$

The first equality in (2), which holds almost everywhere on D, is justified as follows. For all $x\in V:=\{\|Jf\|_q=0\}$ we have $\mathbb{1}_{B(y,t)}(x)f(y,t)=0$ for almost all $(y,t)\in D$, and therefore, by Fubini's theorem, for almost all $y\in\mathbb{R}^d$ we have $\mathbb{1}_{B(y,t)}(x)f(y,t)=0$ for almost all $t>0$. Fix $\delta>0$ arbitrary. Then for almost all $y\in B(x,\delta)$ we have $f(y,t)=0$ for almost all $t\geq\delta$. By another application of Fubini's theorem this implies that $f(y,t)=0$ for almost all $(y,t)\in(B(x,\delta)\times[\delta,\infty))\cap D$. Taking the union over all rational $\delta>0$, it follows that $f\equiv 0$ almost everywhere on $\Gamma_x:=\{(y,t)\in D:|x-y|<t\}$, the 'admissible cone' over x. If K is any compact set contained in V, then by taking the union over a countable dense set of points $x\in K$ it follows that $f(y,t)=0$ almost everywhere on the 'admissible cone' over K. Finally, by the inner regularity of the Lebesgue measure on $\mathbb{R}^n$, it follows that $f(y,t)=0$ almost everywhere on the 'admissible cone' over V. In particular this gives $f(x,t)=0$ for almost all $(x,t)\in D$ with $x\in K$. This proves the first identity in (2).

To prove the theorem it suffices to prove that each of the summands on the right-hand side of (2) has an atomic decomposition. In view of Theorem 2.8 (applied with $p=4$) it even suffices to prove that

$$g := f\mathbb{1}_{W\cap\{\|Jf\|_q>0\}}$$

has an atomic decomposition for any given measurable set W in $\mathbb{R}^n$ such that $W+\mathscr{C}_{16}$ is $2^{10}\sqrt{n}$-admissible Whitney.

Given $k\in\mathbb{Z}$, let us define

$$O_k := \{x\in\mathbb{R}^n : \|Jg(x)\|_q > 2^k\}$$

and $F_k:=\complement O_k$. Fix an arbitrary $\eta\in\left(\frac{1}{2},1\right)$ and let $\bar\eta$ be as in Lemma 3.5. With abuse of notation we let $O_k^{[\bar\eta]}:=\complement F_k^{[\bar\eta]}$, where $F_k^{[\bar\eta]}$ denotes the set of points of admissible $\bar\eta$-density of F_k, and note that $O_k\subseteq O_k^{[\bar\eta]}$. We claim that $O_k^{[\bar\eta]}$ is contained in $W+\mathscr{C}_{16}$ (see (1)).

To prove the claim we first fix $x\in O_k$ and check that $x\in W+\mathscr{C}_2$. Indeed, since $Jg(x)$ does not vanish almost everywhere on D we can find a set $D'\subseteq D$ of positive measure such that for almost all $(y,t)\in D'$ one has $\mathbb{1}_{B(y,t)}(x)g(y,t)=\mathbb{1}_{B(y,t)}(x)f(y,t)\mathbb{1}_{W\cap\{\|Jf\|_q>0\}}(y)\neq 0$. For those points we have $y\in W$, $|x-y|<t$ and $t<m(y)$, so $t<2m(x)$ by Lemma 2.3(i). Thus $B(x,t)$ belongs to $\mathscr{B}_2$ and intersects W, and so $x\in W+\mathscr{C}_2$.

Next let $x\in O_k^{[\bar\eta]}$. Then x is not a point of admissible $\bar\eta$-density of F_k, so there is a ball $B\in\mathscr{B}_{3/2}$ with centre x such that $\gamma(F_k\cap B)<\bar\eta\gamma(B)$. This is only possible if B intersects $O_k=\complement F_k$. Since O_k is contained in $W+\mathscr{C}_2$, this means that B intersects $W+\mathscr{C}_2$. Fix an arbitrary $x'\in B\cap(W+\mathscr{C}_2)$ and let $B'\in\mathcal{B}_2$ be any admissible ball centred at x' and intersecting W. From $x'\in B$ and $B\in\mathscr{B}_{3/2}$ it follows that $|x-x'|<\frac{3}{2}m(x)$. Also, since B' belongs to $\mathscr{B}_2$ and intersects W, $d(x',W)<2m(x')$. It follows that $d(x,W)<\frac{3}{2}m(x)+2m(x')$. By the second part of Lemma 2.3(ii) we have $m(x')\leq 5m(x)$ and therefore $\operatorname{dist}(x,W)\leq\frac{23}{2}m(x)$. This proves the claim (with a somewhat better constant, but that is irrelevant).

For each $N\geq 1$ define $g_N(y,t):=\mathbb{1}_{\{|y|\leq N\}}\mathbb{1}_{\{|g|\leq N\}}\mathbb{1}_{(1/N,\infty)}(t)g(y,t)$. Clearly, $g_N\in T^{1,q}(\gamma)$ and, by dominated convergence, $\lim_{N\to\infty}g_N=g$ in $T^{1,q}(\gamma)$. Defining the sets $F_{k,N}$, $O_{k,N}$, $F_{k,N}^{[\bar\eta]}$, $O_{k,N}^{[\bar\eta]}$ in the same way as above, Lemma 3.5 gives that

$$\iint_{R_{1-\eta}(F_{k,N}^{[\bar\eta]})\cap D}|g_N(y,t)|^q\,d\gamma(y)\frac{dt}{t}$$
$$\lesssim\int_{F_{k,N}}\iint_D\frac{\mathbb{1}_{B(y,t)}(x)}{\gamma(B(y,t))}|g_N(y,t)|^q\,d\gamma(y)\frac{dt}{t}\,d\gamma(x)\lesssim\|g_N\|_{T^{1,q}(\gamma)}^q.$$

As $k\to-\infty$, the middle term tends to 0 and therefore the support of g_N is contained in the union $\bigcup_{k\in\mathbb{Z}}T_{1-\eta}(O_{k,N}^{[\bar\eta]})\cap D$. Clearly, $O_{k,N}\subseteq O_k$ implies $T_{1-\eta}(O_{k,N}^{[\bar\eta]})\subseteq T_{1-\eta}(O_k^{[\bar\eta]})$, and therefore a limiting argument shows that the support of g is contained in the union $\bigcup_{k\in\mathbb{Z}}T_{1-\eta}(O_k^{[\bar\eta]})\cap D$.

Choose cubes $\{Q_k^m\}_m$ and functions $\{\phi_k^m\}_m$ as in Lemma 2.10, applied to the open sets $O_k^{[\bar\eta]}$, which are contained in $W+\mathscr{C}_{16}$. Define for $(y,t)\in D$,

$$b_k^m(y,t):=(\mathbb{1}_{T_{1-\eta}(O_k^{[\bar\eta]})}(y,t)-\mathbb{1}_{T_{1-\eta}(O_{k+1}^{[\bar\eta]})}(y,t))\phi_k^m(y)g(y,t),$$
$$\mu_k^m:=\iint_D|b_k^m(y,t)|^q\,d\gamma(y)\frac{dt}{t},$$

and put

$$\lambda_k^m:=\gamma(Q_k^m)^{1/q'}(\mu_k^m)^{1/q}\quad\text{and}\quad a_k^m(y,t):=\frac{b_k^m(y,t)}{\lambda_k^m}.$$

Then,

$$g=\sum_{k\in\mathbb{Z}}\sum_m\lambda_k^m a_k^m.$$

Let C be a constant to be determined later and denote by $(Q_k^m)^{**}$ the cube which has the same centre as Q_k^m but side-length multiplied by C. Let us further denote by ℓ_k^m and δ_k^m the side-length and the length of the diagonal of Q_k^m, respectively, and by c_k^m the centre of Q_k^m. We claim that

$$\operatorname{supp}a_k^m\subseteq T_1((Q_k^m)^{**}).$$

We have

$$\operatorname{supp} a_k^m \subseteq T_{1-\eta}(O_k^{[\bar\eta]}) \cap \{(y,t)\in D : y\in (Q_k^m)^*\},$$

where $(Q_k^m)^*$ is as in Lemma 2.10. Therefore, fixing $(y,t)\in\operatorname{supp} a_k^m$, we have

$$d(y, F_k^{[\bar\eta]}) \geq (1-\eta)t \quad\text{and}\quad y\in (Q_k^m)^*.$$

For $z\notin (Q_k^m)^{**}$ this gives

$$d(z, c_k^m) \geq \tfrac12 C\ell_k^m = \tfrac12 C d_k^m/\sqrt{n}$$

and

$$d(y,z) \geq d(z,c_k^m) - d(y,c_k^m) \geq \Big(\frac{C}{\sqrt{n}} - \rho\Big)\tfrac12 \delta_k^m, \tag{3}$$

where $\rho = \rho_{2^{10}\sqrt{n},n}$ is the constant from Lemma 2.10. Moreover, by property (ii) in Lemma 2.10,

$$d(c_k^m, F_k^{[\bar\eta]}) \leq \big(\rho + \tfrac12\big)\delta_k^m.$$

For $u\in F_k^{[\bar\eta]}$ such that $d(c_k^m,u)\leq\big(\rho+\tfrac12\big)\delta_k^m+\varepsilon$, this gives

$$(1-\eta)t \leq d(y, F_k^{[\bar\eta]}) \leq d(y,u) \leq d(y,c_k^m) + d(c_k^m,u) \leq \frac{3\rho+1}{2}\delta_k^m + \varepsilon. \tag{4}$$

Upon taking $C = 2\sqrt{n}\big(\tfrac12\rho + (3\rho+1)/2(1-\eta)\big)$, from (3) and (4) (where we let $\varepsilon\downarrow 0$) we infer that

$$d(y,z) \geq \frac{3\rho+1}{2(1-\eta)}\delta_k^m \geq t.$$

This means that $(y,t)\in T_1((Q_k^m)^{**})$, thus proving the claim.

Using the definitions of λ_k^m and a_k^m together with the doubling property for admissible balls, we also get that

$$\iint_D |a_k^m(y,t)|^q \, d\gamma(y)\frac{dt}{t} \leq \frac{1}{\gamma(Q_k^m)^{q/q'}} \lesssim \frac{1}{\gamma((Q_k^m)^{**})^{q/q'}}.$$

Up to a multiplicative constant, the a_k^m are thus $T^{1,q}(\gamma)$ α-atoms for some $\alpha = \alpha(C,n) > 0$. To get the norm estimates, we first use Lemma 3.5. Noting that $(y,t)\in\operatorname{supp}(b_k^m)$ implies that $(y,t)\notin T_{1-\eta}(O_{k+1}^{[\bar\eta]})$ and hence $(y,t)\in R_{1-\eta}(F_{k+1}^{[\bar\eta]})$, and that $(y,t)\in T_1((Q_k^m)^{**})$ and $x\in B(y,t)$ imply that $x\in (Q_k^m)^{**}$, we obtain

$$\begin{aligned}\mu_k^m &\leq \iint_{R_{1-\eta}(F_{k+1}^{[\bar\eta]})\cap D} \mathbb{1}_{T_1((Q_k^m)^{**})}(y,t)|g(y,t)|^q \, d\gamma(y)\frac{dt}{t} \\ &\lesssim \int_{F_{k+1}} \iint_D \frac{\mathbb{1}_{B(y,t)}(x)\mathbb{1}_{T_1((Q_k^m)^{**})}(y,t)}{\gamma(B(y,t))}|g(y,t)|^q \, d\gamma(y)\frac{dt}{t}\, d\gamma(x)\end{aligned}$$

$$
\begin{aligned}
&\le \int_{F_{k+1}\cap(Q_k^m)^{**}} \|Jg(x)\|^q_{L^q(D,d\gamma\,dt/t))}\,d\gamma(x)\\
&\le 2^{q(k+1)}\gamma((Q_k^m)^{**})\\
&\lesssim 2^{qk}\gamma(Q_k^m).
\end{aligned}
$$

This then gives

$$\sum_{k\in\mathbb{Z}}\sum_m \lambda_k^m = \sum_{k\in\mathbb{Z}}\sum_m (\mu_k^m)^{1/q}\gamma(Q_k^m)^{1/q'} \lesssim \sum_{k\in\mathbb{Z}} 2^k\gamma(O_k^{[\bar\eta]}).$$

Since $x\in O_k^{[\bar\eta]}$ implies $M_{3/2}(\mathbb{1}_{O_k})(x)\ge 1-\bar\eta$, the weak type 1-1 estimate for the Hardy–Littlewood maximal function $M_{3/2}$ defined by using only $\mathscr{B}_{3/2}$-balls (which is proved by copying its Euclidean counterpart, see [5, Theorem 3.1]), gives $(1-\bar\eta)\gamma(O_k^{[\bar\eta]})\lesssim \gamma(O_k)$ and thus

$$(1-\bar\eta)\sum_{k\in\mathbb{Z}}\sum_m \lambda_k^m \lesssim \sum_{k\in\mathbb{Z}} 2^k\gamma(O_k) \lesssim \int_0^\infty \gamma(x\in\mathbb{R}^n : \|Jg(x)\|_q > s)\,ds = \|g\|_{T^{1,q}(\gamma)}. \qquad \square$$

As an application of the atomic decomposition we next prove a change of aperture theorem. Our proof is different from the Euclidean proofs in [3] and [4] in that we derive the result directly from the atomic decomposition theorem.

Definition 3.7. For $\alpha>0$, the *Gaussian tent space* $T^{1,q}_\alpha(\gamma)$ with *aperture* α is the completion of $C_c(D)$ with respect to the norm

$$\|f\|_{T^{1,q}_\alpha(\gamma)} := \|J_\alpha f\|_{L^1(\mathbb{R}^n,d\gamma(x);L^q(D,d\gamma(y)\,dt/t))},$$

where

$$J_\alpha f(x,y,t) := \frac{\mathbb{1}_{B(y,\alpha t)}(x)}{\gamma(B(y,t))^{1/q}} f(y,t), \qquad f\in C_c(D).$$

Theorem 3.8. (Change of aperture) *For all* $1<\alpha_0<\alpha$ *we have* $T^{1,q}_\alpha(\gamma)= T^{1,q}_{\alpha_0}(\gamma)$ *with equivalent norms.*

Proof. It is clear that $T^{1,q}_\alpha(\gamma)\subseteq T^{1,q}_{\alpha_0}(\gamma)$, so it suffices to show that $T^{1,q}_{\alpha_0}(\gamma)\subseteq T^{1,q}_\alpha(\gamma)$. For this it suffices to show that $J_\alpha f\in L^1(\mathbb{R}^n,d\gamma(x);L^q(D,d\gamma(y)\,dt/t))$ whenever $f\in T^{1,q}_{\alpha_0}(\gamma)$. Now, by the doubling property (noting that $(y,t)\in D$ implies $B(y,t)\in\mathscr{B}_1$),

$$\|J_\alpha f\|_{L^1(\mathbb{R}^n,d\gamma(x);L^q(D,d\gamma(y)\,dt/t))} = \int_{\mathbb{R}^n}\biggl(\int_D \frac{\mathbb{1}_{B(y,\alpha t)}(x)}{\gamma(B(y,t))}|f(y,t)|^q\,d\gamma(y)\frac{dt}{t}\biggr)^{1/q} d\gamma(x)$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^n} \Big(\int_{\widetilde{D}} \frac{\mathbb{1}_{B(y,t)}(x)}{\gamma(B(y,t/\alpha))} \Big| f\Big(y, \frac{t}{\alpha}\Big)\Big|^q \, d\gamma(y) \frac{dt}{t} \Big)^{1/q} \, d\gamma(x) \\
&\lesssim \int_{\mathbb{R}^n} \Big(\int_{\widetilde{D}} \frac{\mathbb{1}_{B(y,t)}(x)}{\gamma(B(y,t))} \Big| f\Big(y, \frac{t}{\alpha}\Big)\Big|^q \, d\gamma(y) \frac{dt}{t} \Big)^{1/q} \, d\gamma(x) \\
&= \|J\tilde{f}\|_{L^1(\mathbb{R}^n, d\gamma(x); L^q(\widetilde{D}, d\gamma(y)\,dt/t))},
\end{aligned}
$$

where $\widetilde{D}:=\{(y,t)\in\mathbb{R}^n\times(0,\infty):(y,\alpha^{-1}t)\in D\}$ and $\tilde{f}(y,t):=f(y,\alpha^{-1}t)$. To prove the theorem, it thus suffices to show that

$$
\|J\tilde{f}\|_{L^1(\mathbb{R}^n, d\gamma(x); L^q(\widetilde{D}, d\gamma(y)\,dt/t))} \lesssim \|J_{\alpha_0} f\|_{L^1(\mathbb{R}^n, d\gamma(x); L^q(D, d\gamma(y)\,dt/t))} \tag{5}
$$

for $f\in T^{1,q}(\gamma)$.

Suppose a is a $T^{1,q}(\gamma)$ α_0-atom a. Then a is supported in $T_1(B)\cap D$ for some ball $B=B(c,r)\in\mathscr{B}_{\alpha_0}$. Then $\tilde{a}(y,t):=a(y,\alpha^{-1}t)$ is supported in $\widetilde{T}_1(B)\cap\widetilde{D}$, where $\widetilde{T}_1(B):=\{(y,t)\in\mathbb{R}^n\times(0,\infty):(y,t/\alpha)\in T_1(B)\}$. Using that $(y,t)\in\widetilde{T}_1(B)$ and $x\in B(y,t)$ imply that $x\in B(c,\alpha r)$, the doubling property for admissible balls gives

$$
\begin{aligned}
\int_{\mathbb{R}^n} & \Big(\iint_{\widetilde{D}} \frac{\mathbb{1}_{B(y,t)}(x)}{\gamma(B(y,t))} \Big| a\Big(y, \frac{t}{\alpha}\Big)\Big|^q d\gamma(y) \frac{dt}{t} \Big)^{1/q} d\gamma(x) \\
&\le \int_{\mathbb{R}^n} \Big(\iint_{\widetilde{D}} \frac{\mathbb{1}_{B(y,t)}(x)}{\gamma(B(y,t))} \Big| a\Big(y, \frac{t}{\alpha}\Big)\Big|^q d\gamma(y) \frac{dt}{t} \Big)^{1/q} \mathbb{1}_{B(c,\alpha r)}(x)\, d\gamma(x) \\
&\le \Big(\int_{\mathbb{R}^n} \iint_{\widetilde{D}} \frac{\mathbb{1}_{B(y,t)}(x)}{\gamma(B(y,t))} \Big| a\Big(y, \frac{t}{\alpha}\Big)\Big|^q d\gamma(y) \frac{dt}{t}\, d\gamma(x) \Big)^{1/q} \gamma(B(c,\alpha r))^{1/q'} \\
&= \Big(\iint_{\widetilde{D}} \Big| a\Big(y, \frac{t}{\alpha}\Big)\Big|^q d\gamma(y) \frac{dt}{t} \Big)^{1/q} \gamma(B(c,\alpha r))^{1/q'} \\
&\lesssim \Big(\iint_{\widetilde{D}} \Big| a\Big(y, \frac{t}{a}\Big)\Big|^q d\gamma(y) \frac{dt}{t} \Big)^{1/q} \gamma(B(c,r))^{1/q'} \\
&= \Big(\iint_{D} |a(y,t)|^q d\gamma(y) \frac{dt}{t} \Big)^{1/q} \gamma(B(c,r))^{1/q'} \\
&\le 1.
\end{aligned}
$$

This shows that $J\tilde{a}$ belongs to $L^1(\mathbb{R}^n, d\gamma(x); L^q(\widetilde{D}, d\gamma(y)\,dt/t))$ with norm $\lesssim 1$.

An appeal to Theorem 3.4 now shows that $J\tilde{f}\in L^1(\mathbb{R}^n, d\gamma(x); L^q(\widetilde{D}, d\gamma(y)\,dt/t))$ for all $f\in T^{1,q}(\gamma)$. The estimate (5) then follows from the closed graph theorem. This completes the proof. □

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