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Area-preserving isotopies of self-transverse immersions of S^1 in $\mathbb{R}^2$

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Abstract. Let C and C' be two smooth self-transverse immersions of S^1 into $\mathbb{R}^2$. Both C and C' subdivide the plane into a number of disks and one unbounded component. An isotopy of the plane which takes C to C' induces a one-to-one correspondence between the disks of C and C'. An obvious necessary condition for there to exist an area-preserving isotopy of the plane taking C to C' is that there exists an isotopy for which the area of every disk of C equals that of the corresponding disk of C'. In this paper we show that this is also a sufficient condition.

1. Introduction

Let C be a smooth self-transverse immersion of S^1 into the plane $\mathbb{R}^2$ (by Sard's theorem any immersion is self-transverse after an arbitrarily small perturbation). Then C subdivides the plane into a number of bounded connected components and one unbounded component. The bounded components are topological disks and we call them the *disks of* C. Let C' be another self-transverse immersion of S^1 into $\mathbb{R}^2$ such that there exists an isotopy of the plane taking C to C'. Then the isotopy induces a one-to-one correspondence between the disks of C and the disks of C'.

In this paper we study the existence of area-preserving isotopies of the plane taking C to C', where, if $dx\wedge dy$ denotes the standard area form on $\mathbb{R}^2$, we say that an isotopy $\phi_\tau\colon \mathbb{R}^2\to\mathbb{R}^2$, $0\le\tau\le 1$, is *area-preserving* if $\phi_\tau^*(dx\wedge dy)=dx\wedge dy$ for every $\tau\in[0,1]$. Since ϕ_τ being area-preserving implies that $\operatorname{area}(\phi_\tau(U))=\operatorname{area}(U)$ for any measurable $U\subset\mathbb{R}^2$, an obvious necessary condition for the existence of an area-preserving isotopy ϕ_τ taking C to C' is that the area of any disk D of C satisfies

$$\operatorname{area}(D)=\operatorname{area}(D'), \tag{1}$$

where D' is the disk of C' which corresponds to D under ϕ_τ. We call an isotopy which satisfies (1) *disk-area-preserving*. The main result of the paper shows that this is also a sufficient condition. More precisely, we have the following result.

Theorem 1.1. *Let C and C' be two self-transverse immersions of S^1 into $\mathbb{R}^2$ and assume that there is a disk-area-preserving isotopy ψ_τ, $0 \le \tau \le 1$, of $\mathbb{R}^2$ taking C to C' (i.e., $\psi_0 = \mathrm{id}$, $\psi_1(C) = C'$, and $\mathrm{area}(\psi_1(D)) = \mathrm{area}(D)$ for every disk D of C). Then there exists an area-preserving isotopy ϕ_τ, $0 \le \tau \le 1$, of $\mathbb{R}^2$ with $\phi_0 = \mathrm{id}$ and $\phi_1(C) = C'$.*

Theorem 1.1 is proved in Section 5. Problems related to the existence of a topological isotopy (without area condition) taking C to C' were studied by many authors, see e.g. [2], [6] and [7].

From the point of view of symplectic geometry, C is an immersed Lagrangian submanifold, and on the plane area-preserving isotopies are Hamiltonian isotopies. For related questions in higher dimensions see e.g. [3], [4] and [5].

In short outline, our proof of Theorem 1.1 is as follows. First, we construct an isotopy χ_τ which takes C to C' and such that for every disk D of C we have $\mathrm{area}(\chi_\tau(D)) = \mathrm{area}(D)$ for all τ. We call such an isotopy *semi-area-preserving*. The semi-area-preserving isotopy is constructed from the disk-area-preserving isotopy ψ_τ by first composing it with a time-dependent scaling so that the resulting isotopy γ_τ shrinks the area of each disk of C for all times. The isotopy γ_τ is then modified: we introduce a time-dependent area form ω_τ such that the area of every disk of C is constant under γ_τ with respect to ω_τ and then we use Moser's trick to find an isotopy ϕ_τ such that $\phi_\tau^* \, dx \wedge dy = \omega_\tau$, and hence the isotopy $\phi_\tau \circ \gamma_\tau$ is semi-area-preserving, see Section 3. Second, we subdivide the semi-area-preserving isotopy into small time steps and use a cohomological argument to show the existence of an area-preserving isotopy, see Section 4.

For simpler notation below, we assume that all maps are smooth and that all immersions are self-transverse.

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2. Background

In this section we introduce notation and discuss standard background material on Hamiltonian vector fields on surfaces.

Let M be a surface and let $v\colon M \to TM$ be a vector field with compact support. We write $\Phi_v^t\colon M \to M$ for the time-t flow of v.

Let ω be a symplectic form on M and write $I\colon T^*M\to TM$ for the isomorphism defined through the equation

$$\alpha(\eta)=\omega(\eta, I(\alpha)) \quad \text{for all } \alpha\in T^*M \text{ and } \eta\in T_xM.$$

Let $H\colon M\to\mathbb{R}$ be a smooth function with compact support. The vector field $X_H=I(dH)$ is the *Hamiltonian vector field* of H and its flow is area-preserving.

Let C be an immersion of S^1 into the plane and let $\varphi\colon S^1\to\mathbb{R}^2$ be a parameterization of C. Write $e(s)$ for the unit vector field along C such that $(d\varphi/ds(s), e(s))$ is a positively oriented basis of $\mathbb{R}^2$ for all $s\in S^1$. Then for all sufficiently small $\varepsilon>0$ the map $\Phi\colon S^1\times(-\varepsilon,\varepsilon)\to\mathbb{R}^2$, given by

$$\Phi(s,t)=\varphi(s)+te(s), \tag{2}$$

parameterizes a neighborhood C^ε of C. Notice that if C has double points then this parameterization is not one-to-one.

Let $dx\wedge dy$ be the standard symplectic form on $\mathbb{R}^2$ and consider coordinates (s,t) on $S^1\times\mathbb{R}=(\mathbb{R}/2\pi\mathbb{Z})\times\mathbb{R}$ with the corresponding symplectic form $ds\wedge dt$. The following lemma is a special case of Moser's lemma, see e.g. [1] for a proof.

Lemma 2.1. *Let C be an immersion of S^1 in $\mathbb{R}^2$ and let Φ be as in (2). Then there exists a $\delta>0$ and a diffeomorphism $\vartheta\colon S^1\times\mathbb{R}\to S^1\times\mathbb{R}$ with $\vartheta(s,0)=(s,0)$ such that*

$$(\Phi\circ\vartheta)^*\, dx\wedge dy=ds\wedge dt$$

for all $|t|<\delta$.

Below we will often combine Lemma 2.1 with a Hamiltonian isotopy of $S^1\times\mathbb{R}$. In the following lemma we use this argument to construct area-preserving isotopies between nearby curves C and C' which agree near double points. We will use the following terminology: For an immersed circle $C\subset\mathbb{R}^2$, we call an arc $A\subset C$ *maximal smooth* of C if $A\cap\{x_i\}_{i=1}^n=\{x_i,x_j\}=\partial A$, where $\{x_i\}_{i=1}^n\subset C$ are the double points of C.

Lemma 2.2. *Let C be an immersion of S^1 into $\mathbb{R}^2$ and let $\xi\colon S^1\times(-\varepsilon,\varepsilon)\to\mathbb{R}^2$ be an area-preserving parameterization of a neighborhood C^ε of C as in Lemma 2.1. Assume that C' is an immersion of S^1 into $\mathbb{R}^2$ which coincides with C in a neighborhood U_x of every double point x of C and such that there is a function $g\colon S^1\to(-\varepsilon,\varepsilon)$ with $C'=\xi(\Gamma)$, where Γ is the graph of g. If there exists a disk-area-preserving isotopy taking C to C' then there exists an area-preserving isotopy of the plane taking C' to C.*

Proof. Shrink C^ε so that we still have $C \cup C' \subset C^\varepsilon$, but so that the parameterization is one-to-one outside $\bigcup U_x$, where the union is taken over all double points of C. In other words, we let C^ε be so small so that $\overline{C^\varepsilon - \bigcup U_x}$ consists of a number of simply connected components V_A, where each component corresponds to a maximal smooth arc A of C. Let W be an open neighborhood of $C \cup C'$ so that $\overline{W} \subset C^\varepsilon$ and so that $V_A \cap W$ and $U_x \cap W$ are simply connected for all V_A and U_x. Let $G\colon S^1 \to \mathbb{R}$ be defined by $G(s) = \int_0^s g(s')\,ds'$, and let $\widetilde{G}\colon \mathbb{R}^2 \to \mathbb{R}$ be a function satisfying

$$\widetilde{G}(x) = \begin{cases} G((\xi^{-1})^1(x)) & \text{for } x \in W \cap V_A, \\ G((\xi^{-1})^1(x')) & \text{for } x \in W \cap U_{x'}, \\ 0 & \text{for } x \notin C^\varepsilon, \end{cases}$$

where $\xi^{-1} = ((\xi^{-1})^1, (\xi^{-1})^2)$. Then $\widetilde{G}$ is a well-defined function: Suppose that $\xi(s_1, t_1) = \xi(s_2, t_2)$ for $s_1 \neq s_2$. Then $\xi(s_1, t_1) \subset U_x$ for some x, and since $\widetilde{G}$ is constant in $U_x \cap W$ we can assume that $\xi(s_1, t_1) = x$. But clearly $\xi((s_1, s_2) \times \{0\})$ is a 1-chain, so it bounds a number of disks of C. Since every disk of C' has the same area as the corresponding disk of C we thus have $\int_{s_1}^{s_2} g(s)\,ds = 0$, so $G(s_1) = G(s_2)$.

The Hamiltonian vector field of $\widetilde{G}$ in the parameterization of C^ε is $X_{\widetilde{G}} = -g(s)\partial/\partial t$ for $(s,t) \in W \cap V_A$ and $X_{\widetilde{G}} = 0$ in $U_x \cap W$. Hence its time-1 flow takes $(s, g(s))$ to $(s, 0)$ for all s and we get an area-preserving isotopy of the plane taking C' to C. □

3. Construction of semi-area-preserving isotopies

In this section we construct a semi-area-preserving isotopy from a disk-area-preserving isotopy.

Let C and C' be two immersions of S^1 into $\mathbb{R}^2$ such that there exists a disk-area-preserving isotopy ϕ_t taking C to C'. Without loss of generality we can assume that ϕ_t has support in some B_r, where B_r denotes the open disk of radius r centered at 0. Let $\gamma_t\colon \mathbb{R}^2 \to \mathbb{R}^2$, $t \in [0,1]$, $\gamma_0 = \mathrm{id}$, be an isotopy of the plane with support in B_{r+1}, acting as follows: First let γ_t shrink B_r to some $B_{\varepsilon r}$ radially, where ε is small and depends on the area of the disks of C. Next we let γ_t take the shrunken curve C to the shrunken curve C' by using $\varepsilon\phi_t(\varepsilon x)$, and then finally we let γ_t enlarge $B_{\varepsilon r}$ to B_r again, so that we get $\gamma_1(C) = C'$. By choosing ε small enough we thus get an isotopy γ_t of the plane taking C to C' such that $\mathrm{area}(\gamma_t(D)) < \mathrm{area}(D)$ for every disk D of C, and for all $t \in (0,1)$.

Next we use Moser's trick to find an isotopy $\psi_t\colon \mathbb{R}^2 \to \mathbb{R}^2$, $t \in [0,1]$, $\psi_0 = \mathrm{id}$, such that $\chi_t = \psi_t \circ \gamma_t$ is semi-area-preserving with respect to C. So if we then can take

$\psi_1\gamma_1(C)$ to C' with a semi-area-preserving isotopy we get a semi-area-preserving isotopy taking C completely to C'. We start with the following lemma.

Lemma 3.1. *Let γ_t, C and C' be as above. Then there is an isotopy ψ_t: $\mathbb{R}^2\to\mathbb{R}^2$, $t\in[0,1]$, $\psi_0=\mathrm{id}$, such that $\int_{\psi_t\gamma_t(D)} dx\wedge dy=\int_D dx\wedge dy$ for every disk D of C. Moreover, ψ_t can be chosen so that $\psi_1^* dx\wedge dy=dx\wedge dy$.*

Proof. Let $D_1, D_2, ..., D_n$ be the disks of C. For each D_i choose a point $\xi_i\in D_i$, and let $r_i(t)$: $[0,1]\to(0,\infty)$ be such that $B_{r_i(t),\gamma_t(\xi_i)}\subset\gamma_t(D_i)$ for all $0\le t\le 1$, where $B_{\rho,p}$ is the open disk of radius ρ centered at p.

For each disk D_i let σ_t^i: $[0,\infty)\to(0,\infty)$ be a smooth one-parameter family of functions such that for each $t\in[0,1]$ we have, if (ρ,θ) are polar coordinates centered at $\gamma_t(\xi_i)=(x_i(t),y_i(t))$, that $\omega_t^i=d\big(\tfrac{1}{2}\sigma_t^i(\rho^2)\,d\theta\big)$ is nondegenerate and satisfies

$$\int_{\gamma_t(D_i)}\omega_t^i=\int_{D_i}dx\wedge dy-\frac{n-1}{n}\int_{\gamma_t(D_i)}dx\wedge dy. \tag{3}$$

Also choose σ_t^i so that

$$\omega_t^i=\frac{1}{n}\,dx\wedge dy \qquad \text{in } B_r\setminus B_{r_i(t),\gamma_t(\xi_i)}, \tag{4}$$

$$\omega_0^i=\frac{1}{n}\,dx\wedge dy=\omega_1^i \tag{5}$$

and so that $\sigma_t^i(s)=s/n$ outside some $B_{r'}$, where $r'>r$ is chosen big enough to be independent of t and D_i.

We can find such a σ_t^i due to the fact that we want ω_t^i to satisfy

$$\int_{\gamma_t(D_i)}\omega_t^i>\frac{1}{n}\int_{\gamma_t(D_i)}dx\wedge dy.$$

So even if the disk $B_{r_i(t),\gamma_t(\xi_i)}$ is small we can let $d\sigma_t^i(s)/ds$ be large in this disk to obtain (3), which need not have been the case if the area of $\gamma_t(D_i)$ was greater than the area of D_i for some t. We use the space between B_r and $B_{r'}$ to decrease $d\sigma_t^i(s)/ds>0$ so that we get $\sigma_t^i(s)=s/n$ outside $B_{r'}$.

Now let $\omega_t=\sum_{i=1}^n\omega_t^i$. Then by (3) and (4) we have

$$\begin{aligned}\int_{\gamma_t(D_i)}\omega_t&=\int_{\gamma_t(D_i)}\omega_t^i+\sum_{\substack{j=1\\ j\ne i}}^{n}\int_{\gamma_t(D_i)}\omega_t^j\\ &=\int_{D_i}dx\wedge dy-\frac{n-1}{n}\int_{\gamma_t(D_i)}dx\wedge dy+\frac{n-1}{n}\int_{\gamma_t(D_i)}dx\wedge dy\\ &=\int_{D_i}dx\wedge dy.\end{aligned}$$

So if we can find an isotopy ψ_t satisfying $\omega_t = \psi_t^* \omega_0$ for all t then $\psi_t \circ \gamma_t$ will be semi-area-preserving with respect to C.

To do this we use Moser's trick. Namely, for each disk D_i and for each t let μ_t^i be the 1-form

$$\mu_t^i = \frac{d}{dt}\left(\frac{1}{2}\sigma_t^i(\rho^2)\, d\theta\right),$$

and let v_t be the vector field defined by $\iota_{v_t}(\omega_t) + \sum_{i=1}^n \mu_t^i = 0$, where $\iota_{v_t}(\omega_t)$ is the 1-form satisfying $\iota_{v_t}(\omega_t)(\eta) = \omega_t(v_t, \eta)$ for all $\eta \in T_x\mathbb{R}^2$. Then we get that

$$v_t = \sum_{i=1}^n \frac{dx_i}{dt}\frac{\partial}{\partial x} - \frac{dy_i}{dt}\frac{\partial}{\partial y}$$

outside $B_{r'}$, since here we have that

$$\sigma_t^i(\rho^2)\, d\theta = \frac{1}{n}\rho^2\, d\theta = \frac{1}{n}((x - x_i(t))\, dy - (y - y_i(t))\, dx)$$

so

$$\omega_t^i = \frac{1}{n}\, dx \wedge dy$$

and

$$\mu_t^i = \frac{1}{n}\left(\frac{dy_i}{dt}\, dx - \frac{dx_i}{dt}\, dy\right)$$

here. Thus v_t satisfies a Lipschitz condition with the same Lipschitz constant L for all $x \in \mathbb{R}^2$ and for all $t \in [0,1]$, and hence we can find an isotopy $\chi_t : \mathbb{R}^2 \to \mathbb{R}^2, 0 \le t \le 1$, such that $\chi_0 = \mathrm{id}$ and $d\chi_t/dt = v_t \chi_t$. Now we get

$$\frac{d}{dt}(\chi_t^* \omega_t) = \chi_t^*\left(d\iota_{v_t}(\omega_t) + \sum_{i=1}^n d\mu_t^i\right) = 0,$$

so $\chi_t^* \omega_t = \chi_0^* \omega_0 = dx \wedge dy$ for all $t \in [0,1]$. Letting ψ_t be the inverse of χ_t for each $0 \le t \le 1$ we get that $\omega_t = \psi_t^*\, dx \wedge dy$ and hence that $\psi_t \circ \gamma_t$ is a semi-area-preserving isotopy with respect to C, and by (5) we have $\psi_1^*\, dx \wedge dy = dx \wedge dy$. $\square$

Now by finding an area-preserving isotopy taking $\psi_1 \gamma_1(C)$ to $\gamma_1(C) = C'$ we can prove the main lemma of this section.

Lemma 3.2. *If C and C' are immersions of S^1 into $\mathbb{R}^2$ such that there exists a disk-area-preserving isotopy taking C to C', then there exists a semi-area-preserving isotopy taking C to C'.*

Proof. Let γ_t and ψ_t be constructed as above, and let $F_t\colon \mathbb{R}^2 \to \mathbb{R}^2$, $t\in[0,1]$, be defined as

$$F_t(x) = \begin{cases} \dfrac{\psi_1(tx)}{t}, & t\neq 0, \\ d\psi_1(0)x, & t=0. \end{cases}$$

Then $dF_t(x)=d\psi_1(tx)$ for all t and since $\psi_1^* dx\wedge dy=dx\wedge dy$ we get that F_{1-t}, $t\in[0,1]$, is an area-preserving isotopy taking $\psi_1\gamma_1(C)$ to $d\psi_1(0)(\gamma_1(C))$. Moreover, since $\det(d\psi_1(0))=1$ there is a one-parameter family of linear diffeomorphisms $A_t\in\mathrm{SO}(2)$ such that $A_0=d\psi_1(0)$ and $A_1=\mathrm{id}$, and hence we can find an area-preserving isotopy of the plane taking $\psi_1\gamma_1(C)$ to $\gamma_1(C)=C'$. Since $\psi_t\circ\gamma_t$ is semi-area-preserving with respect to C we thus get a semi-area-preserving isotopy of the plane taking C to C'. $\square$

4. Area-preserving isotopies between nearby curves

In this section we show that if C and C' are two immersed circles in the plane such that there exists a disk-area-preserving isotopy taking C to C', and if C' lies sufficiently close to C, then there exists an area-preserving isotopy taking C to C'. This implies that if we have two immersions C and C', not necessary close to each other, and a semi-area-preserving isotopy ψ_τ taking C to C', then we can find an area-preserving isotopy taking C to $\psi_{\tau_0}(C)$ for τ_0 sufficiently small. Thus, by compactness arguments, we can find an area-preserving isotopy taking C completely to C'.

We begin by finding a suitable parameterization of a neighborhood of C, and then we define what we mean by C' being "sufficiently close" to C.

So given C, let $\nu>0$ be so small that $\overline{B}_{\nu,x_1}\cap\overline{B}_{\nu,x_2}=\varnothing$ for any double points $x_1\neq x_2$ of C. Let $\xi\colon S^1\times(-\varepsilon,\varepsilon)\to\mathbb{R}^2$ be an area-preserving parameterization of a neighborhood C^ε of C as in Lemma 2.1. Then at each double point x of C we get a double point of ξ, i.e. a subset $U_x\subset C^\varepsilon$ where C^ε overlaps itself. Let ε be so small that U_x is a disk contained in $B_{\nu,x}$ and so that $\overline{C\cap U_x}$ consists of two smooth arcs L_s and L_t intersecting at x. Suppose that $x=\xi(0,0)$ and that $L_s=\xi([-s_1,s_1]\times\{0\})$. Since L_s intersects L_t transversely at x there is a $t_1>0$ so that $L_t\cap((-s_1,s_1)\times(-t_1,t_1))$ coincides with the graph of a function $g\colon(-t_1,t_1)\to(-s_1,s_1)$ over the t-axis in the parameterization of C^ε. Let $S=(-s_1,s_1)\times(-t_1,t_1)$ and let $\vartheta\colon S\to\mathbb{R}^2$ be defined by

$$\vartheta(s,t)=(s-g(t),t)=(\mu(s,t),\eta(s,t)).$$

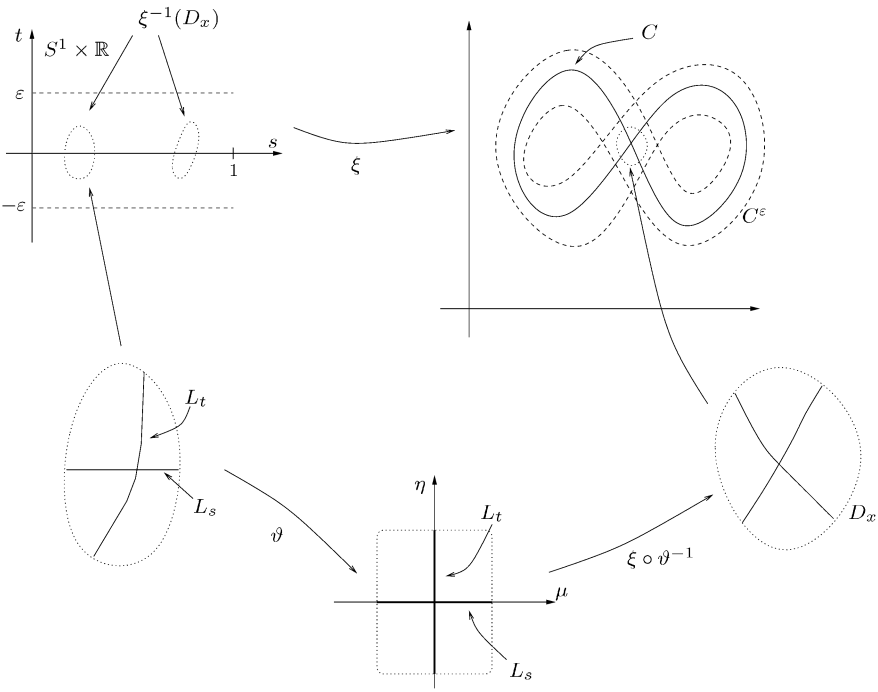

Figure 1. An example of a regular neighborhood.

Then $\vartheta^* \, d\mu \wedge d\eta = ds \wedge dt$, and ϑ maps $L_s \cap S$ to the μ-axis and $L_t \cap S$ to the η-axis. Let

$$D_x = \xi\vartheta^{-1}((-\tilde{s}_1, \tilde{s}_1) \times (-\tilde{t}_1, \tilde{t}_1)),$$

where $\tilde{s}_1, \tilde{t}_1 > 0$ are so small that $\overline{\vartheta^{-1}((-\tilde{s}_1, \tilde{s}_1) \times (-\tilde{t}_1, \tilde{t}_1))} \subset S$.

Definition 1. We call the data $\{C^\varepsilon, D_x\}$ a *regular neighborhood* of C.

This means that a regular neighborhood of C consists of an immersed annulus $C^\varepsilon = \xi(S^1 \times (-\varepsilon, \varepsilon))$, and also a parameterization of a neighborhood of each double point of C, so that in this parameterization C coincides with the coordinate axes of $\mathbb{R}^2$, see Figure 1.

Now let $C' \subset C^\varepsilon$ be an immersion such that there exists a disk-area-preserving isotopy taking C to C'. Let $Q_{r,p}$ be the open square with sides of length $2r$ centered at p and $Q_r = Q_{r,0}$. Let $\delta > 0$ be so small that for every double point $x \in C$ the square $Q_{\delta,x}$ is contained in the parameterization of D_x. Further, for each double point $x \in C$, let x' be the corresponding double point of C', and let $L'_s, L'_t \subset C'$ be the arcs corresponding to L_s respectively L_t in C. Assume that $C' \cap D_x \subset L'_s \cup L'_t$ and that $x' \in Q_{\delta,x}$ in the parameterization of D_x. Also assume that $L'_s \cap D_x$, respectively $L'_t \cap D_x$, is a graph of a function g_μ, respectively g_η, over the μ-axis, respectively η-axis, in the parameterization of D_x, satisfying $|g_\mu|, |g_\eta|, |dg_\mu/d\mu|, |dg_\eta/d\eta| < \delta$. If this holds for all double points of C, and if C' is a graph of a function $g\colon S^1 \to (-\delta, \delta)$ in the parameterization of C^ε satisfying $|dg/ds| < \delta$, we say that C' is *δ-close to C in* $\{C^\varepsilon, D_x\}$.

The following result shows that if C' is sufficiently close to C in the above sense, then there is an area-preserving isotopy taking C' to C.

Lemma 4.1. *Let C be an immersion of S^1 in $\mathbb{R}^2$ and let $\{C^\varepsilon, D_x\}$ be a regular neighborhood of C. Then there exists a $\delta > 0$ such that for every immersion C' which is δ-close to C in $\{C^\varepsilon, D_x\}$ there is an area-preserving isotopy taking C' to C.*

Proof. Let $\sigma > 0$ be so small so that in each parameterized disk D_x we can find a square $Q_\sigma = Q_{\sigma,x}$, where x corresponds to $(0,0)$ in the parameterization. Let $\delta > 0$ be sufficiently small so that $\sigma > \delta^{1/2}$ and let $\psi\colon \mathbb{R} \to \mathbb{R}$ be a smooth cut-off function satisfying

$$\psi(y) = \begin{cases} 1 & \text{for } y \in (-\delta, \delta), \\ 0 & \text{for } y \notin (-\sigma, \sigma) \end{cases}$$

with

$$|\psi| \le 1, \quad \left|\frac{d\psi}{dy}\right| < \frac{a}{\delta^{1/2} - \delta} \quad \text{and} \quad \left|\frac{d^2\psi}{dy^2}\right| < \frac{b}{(\delta^{1/2} - \delta)^2}$$

for some constants a and b, i.e.

$$\frac{d\psi}{dy} = O(\delta^{-1/2}) \quad \text{and} \quad \frac{d^2\psi}{dy^2} = O(\delta^{-1})$$

as $\delta \to 0$.

Now let C' be an immersion which is δ-close to C in $\{C^\varepsilon, D_x\}$, and let $x \in C$ be a double point. We start with showing that if δ is sufficiently small then there is a neighborhood U of x and an area-preserving isotopy ϕ_τ, $0 \le \tau \le 1$, with support in D_x so that $\phi_1(C') \cap U$ coincides with $C \cap U$ and so that $\phi_1(C')$ is a graph over S^1 in C^ε. By finding one such isotopy for each double point of C and then use Lemma 2.2 we get an area-preserving isotopy taking C' completely to C.

So given a double point $x \in C$, first consider the arc $L'_s \subset C'$, defined as above. Since C' is δ-close to C in $\{C^\varepsilon, D_x\}$ we see that L'_s coincides with the graph of a function $g_\mu\colon (-\sigma, \sigma) \to (-\delta, \delta)$ in Q_σ. Let δ be so small that we can find an exact function $\tilde{g}\colon \mathbb{R} \to (-\delta, \delta)$ with support in $(-\delta^{1/2}, \delta^{1/2})$ whose graph coincides with L'_s

in Q_δ and which satisfies $|d\tilde{g}/d\mu|=O(\delta^{1/2})$. Let $G(\mu)=\int_{-\sigma}^{\mu}\tilde{g}(\mu')\,d\mu'$, and consider the Hamiltonian $H(\mu,\eta)=-G(\mu)\psi(\eta)$ with corresponding vector field

$$X_H=G(\mu)\frac{d\psi}{d\eta}(\eta)\frac{\partial}{\partial\mu}-\tilde{g}(\mu)\psi(\eta)\frac{\partial}{\partial\eta}.$$

Then the Hamiltonian isotopy $\Phi_{X_H}^{\tau}=(\chi_\tau^1,\chi_\tau^2)=\chi_\tau$, $0\le\tau\le1$, takes L'_s to the μ-axis in Q_δ, and has support in Q_σ.

Next we want to take $\chi_1(L'_t)$ to the η-axis in a neighborhood of 0 in such a way that the image of $\chi_1(L'_s)$ still coincides with the μ-axis here. But first, to make sure that $\chi_1(C')$ is still a graph over S^1 in the parameterization of C^ε we find an estimate for the derivative $d\chi_1$ of χ_1. Divide $[0,1]$ into N intervals of length $1/N$. By Taylor expansion we have, for $\tau\le1/N$, that

$$\begin{aligned}\frac{\partial\chi_\tau^1}{\partial\mu}&=\frac{\partial\chi_0^1}{\partial\mu}+\tau\frac{d}{d\tau}\frac{\partial\chi_0^1}{\partial\mu}+O(\tau^2)=1+\tau\frac{\partial}{\partial\mu}\left(G(\mu)\frac{d\psi}{d\eta}(\eta)\right)+O(\tau^2)\\&=1+\tau\tilde{g}(\mu)\frac{d\psi}{d\eta}(\eta)+O\left(\frac{1}{N^2}\right)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial\chi_{1/N+\tau}^1}{\partial\mu}&=\frac{\partial\chi_{1/N}^1}{\partial\mu}+\tau\frac{d}{d\tau}\frac{\partial\chi_{1/N}^1}{\partial\mu}+O(\tau^2)\\&=\left(1+\frac{1}{N}\tilde{g}(\mu)\frac{d\psi}{d\eta}(\eta)+O\left(\frac{1}{N^2}\right)\right)+\tau\tilde{g}(\mu)\frac{d\psi}{d\eta}(\eta)+O\left(\frac{1}{N^2}\right).\end{aligned}$$

If we continue like this we get

$$\begin{aligned}\frac{\partial\chi_1^1}{\partial\mu}&=1+\sum_{n=0}^{N-1}\frac{1}{N}\tilde{g}\left(\mu\left(\frac{n}{N}\right)\right)\frac{d\psi}{d\eta}\left(\eta\left(\frac{n}{N}\right)\right)+NO\left(\frac{1}{N^2}\right)\\&=1+O(\delta^{1/2})+O\left(\frac{1}{N}\right)\end{aligned}$$

since $|\tilde{g}|<\delta$ and $|d\psi/d\eta|=O(\delta^{-1/2})$. Hence for N big enough, depending on C', we get $\partial\chi_1^1/\partial\mu=1+O(\delta^{1/2})$, where the $O(\delta^{1/2})$-term depends on C, C^ε and D_x. Similarly we have

$$\frac{\partial\chi_1^1}{\partial\eta}=0+\sum_{n=0}^{N-1}\frac{1}{N}G\left(\mu\left(\frac{n}{N}\right)\right)\frac{d^2\psi}{d\eta^2}\left(\eta\left(\frac{n}{N}\right)\right)+O\left(\frac{1}{N}\right)=O(\delta^{1/2})$$

since $|G|<\delta^{1/2}\delta$, $|d^2\psi/d\eta^2|=O(\delta^{-1})$, and

$$\frac{\partial\chi_1^2}{\partial\mu}=0-\sum_{n=0}^{N-1}\frac{1}{N}\frac{d\tilde{g}}{d\mu}\Big(\mu\Big(\frac{n}{N}\Big)\Big)\psi\Big(\eta\Big(\frac{n}{N}\Big)\Big)+O\bigg(\frac{1}{N}\bigg)=O(\delta^{1/2}),$$

$$\frac{\partial\chi_1^2}{\partial\eta}=1-\sum_{n=0}^{N-1}\frac{1}{N}\tilde{g}\Big(\mu\Big(\frac{n}{N}\Big)\Big)\frac{d\psi}{d\eta}\Big(\eta\Big(\frac{n}{N}\Big)\Big)+O\bigg(\frac{1}{N}\bigg)=1+O(\delta^{1/2}).$$

Thus we get that

$$d\chi_1=E+O(\delta^{1/2}), \tag{6}$$

where E is the 2×2 unit matrix and $O(\delta^{1/2})$ denotes a 2×2 matrix with entries of size $O(\delta^{1/2})$.

Let next $\vartheta=(\vartheta^1,\vartheta^2)\colon D_x\to D_x$ be a change of coordinates from (μ,η) to $(s,t)\subset C^\varepsilon$. In (s,t)-coordinates by assumption we have that $L'_s\cap D_x=\{(s,g(s))\}$ for $s\in(\sigma_1,\sigma_2)$, say, and g satisfies $|g|,|dg/ds|<\delta$. By (6) we have

$$\frac{d}{ds}\vartheta^1(\chi_1\vartheta^{-1}(s,g(s)))=1+O(\delta^{1/2})$$

for all $s\in(\sigma_1,\sigma_2)$, so $\chi_1(L'_s)$ is a graph of a function $\alpha\colon S^1\to\mathbb{R}$ in the parameterization of C^ε if we let δ be small enough. Furthermore, for the slope of α we get that

$$\bigg|\frac{d\alpha}{ds}\bigg|=\left|\frac{\frac{d}{ds}\vartheta^2(\chi_1\vartheta^{-1}(s,g(s)))}{\frac{d}{ds}\vartheta^1(\chi_1\vartheta^{-1}(s,g(s)))}\right|=\frac{O(\delta^{1/2})}{1+O(\delta^{1/2})}=O(\delta^{1/2}).$$

Similar calculations show that $\chi_1(L'_t)$ is a subset of both a graph over S^1 in the parameterization of C^ε and a graph over the η-axis in the parameterization of D_x for δ sufficiently small. Moreover, the slope of these graphs are of order $\delta^{1/2}$.

Now we find an isotopy $\widetilde{\chi}_\tau$, $0\le\tau\le 1$, taking $\chi_1(L'_t)$ to L_t in a neighborhood of x, and so that $\widetilde{\chi}_1(\chi_1(L'_s))$ still coincides with L_s here. Since by assumption we had $x'\in Q_\delta\subset D_x$, where $x'\in C'$ is the double point corresponding to x, we have $\chi_1(x')\in(-\delta,\delta)\times\{0\}$. Hence we can find a $0<\delta'<\delta$ so that $\chi_1(L'_t)$ coincides with the graph of an exact function $f\colon\mathbb{R}\to(-\delta,\delta)$ in $(-\delta,\delta)\times(-\delta',\delta')$, that is,

$$\chi_1(L'_t)\cap((-\delta,\delta)\times(-\delta',\delta'))=\{(f(\eta),\eta)\}.$$

In addition we can choose f so that $|df/d\eta|=O(\delta^{1/2})$ for all $\eta\in\mathbb{R}$ and so that $f(\eta)=0$ for $|\eta|>\delta^{1/2}$. Let $F(\eta)=\int_{-\sigma}^{\eta}f(\eta')\,d\eta'$. Then the isotopy $\Phi^\tau_{X_H}=\widetilde{\chi}_\tau$, $0\le\tau\le 1$, obtained from the Hamiltonian $H(\mu,\eta)=\psi(\mu)F(\eta)$ takes $\chi_1(L'_t)$ to the η-axis in

$(-\delta,\delta)\times(-\delta',\delta')$, and we have that $\widetilde{\chi}_1\chi_1(L'_s)$ still coincides with the μ-axis in a neighborhood of $(0,0)=\widetilde{\chi}_1\chi_1(x')$.

As before we get that

$$d\widetilde{\chi}_1=E+\begin{pmatrix}\frac{d\psi}{d\mu}f & \psi\frac{df}{d\eta}\\ \frac{d^2\psi}{d\mu^2}F & \frac{d\psi}{d\mu}f\end{pmatrix}+O\Big(\frac{1}{N}\Big)=E+O(\delta^{1/2})$$

for N large. So for $\widetilde{\chi}_1\chi_1(L'_s)$ in $C^\varepsilon\cap D_x$ we have, with $\chi_1(L'_s)=\{(s,\alpha(s))\}$, that

$$\frac{d}{ds}\vartheta\widetilde{\chi}_1\vartheta^{-1}(s,\alpha(s))=\begin{pmatrix}1 & 0\\ 0 & \frac{d\alpha}{ds}\end{pmatrix}+O(\delta^{1/2}).$$

Hence $\widetilde{\chi}_1\chi_1(L'_s)$ will be a subset of a graph over S^1 for δ small enough, and similarly we get that $\widetilde{\chi}_1\chi_1(L'_t)$ is a subset of a graph over S^1 in the parameterization of C^ε too.

By doing the same thing at all double points of C we get an area-preserving isotopy taking C' to C in a neighborhood of every double point of C, and so that the time-1 image of C' is still a graph over S^1 in C^ε. So by Lemma 2.2 there is an area-preserving isotopy taking C' completely to C. $\square$

5. Proof of Theorem 1.1

Now if we combine Lemma 3.2 with Lemma 4.1 we can prove our theorem.

Proof of Theorem 1.1. By Lemma 3.2 there is a semi-area-preserving isotopy ϕ_τ, $0\le\tau\le1$, with respect to C taking C to C'. Let $C_\tau=\phi_\tau(C)$ for $\tau\in[0,1]$, and for each $\tau_0\in[0,1]$ let $\{C^\varepsilon_{\tau_0},D^{\tau_0}_x\}$ be a regular neighborhood of C_{τ_0}. By Lemma 4.1 we can find a $\delta_{\tau_0}>0$ so that for every C_τ which is δ_{τ_0}-close to C_{τ_0} there exists an area-preserving isotopy taking C_τ to C_{τ_0}, and by the continuity of ϕ_τ there is a $\nu_{\tau_0}>0$ so that C_τ is δ_{τ_0}-close to C_{τ_0} for all $0\le\tau-\tau_0<\nu_{\tau_0}$.

Let $\nu=\min_{\tau_0\in I}\nu_{\tau_0}$ and let

$$0=\tau_1<...<\tau_n=1$$

be a partition of $[0,1]$ so that $\tau_{i+1}-\tau_i<\nu$ for $1\le i<n$. Then by Lemma 4.1 there is an area-preserving isotopy taking $C_{\tau_{i+1}}$ to C_{τ_i} for $i=1,...,n-1$. Composing the inverses of these isotopies we thus get an area-preserving isotopy taking C to C'. $\square$

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