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Biharmonic PNMC submanifolds in spheres

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Abstract. We obtain several rigidity results for biharmonic submanifolds in $\mathbb{S}^n$ with parallel normalized mean curvature vector fields. We classify biharmonic submanifolds in $\mathbb{S}^n$ with parallel normalized mean curvature vector fields and with at most two distinct principal curvatures. In particular, we determine all biharmonic surfaces with parallel normalized mean curvature vector fields in $\mathbb{S}^n$.

Then we investigate, for (not necessarily compact) proper-biharmonic submanifolds in $\mathbb{S}^n$, their type in the sense of B.-Y. Chen. We prove that (i) a proper-biharmonic submanifold in $\mathbb{S}^n$ is of 1-type or 2-type if and only if it has constant mean curvature $f=1$ or $f\in(0,1)$, respectively; and (ii) there are no proper-biharmonic 3-type submanifolds with parallel normalized mean curvature vector fields in $\mathbb{S}^n$.

1. Introduction

Let $\varphi\colon M\to(N,h)$ be a Riemannian immersion of a manifold M into a Riemannian manifold (N,h). We say that φ is a *biharmonic Riemannian immersion*, or M is a *biharmonic submanifold* in N, if its mean curvature vector field H satisfies the equation

$$(1)\qquad \tau_2(\varphi)=-m(\Delta^\varphi H+\operatorname{trace} R^N(d\varphi(\cdot),H)\,d\varphi(\cdot))=0,$$

where Δ^φ denotes the rough Laplacian on sections of the pull-back bundle $\varphi^{-1}(TN)$ and R^N denotes the curvature operator on (N,h). The section $\tau_2(\varphi)$ is called the *bitension field*.

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When M is compact, the biharmonic condition arises from a variational problem for maps: For an arbitrary smooth map $\varphi\colon (M,g)\to(N,h)$ we define

$$E_2(\varphi)=\frac{1}{2}\int_M |\tau(\varphi)|^2\, v_g,$$

where $\tau(\varphi)=\operatorname{trace}\nabla\, d\varphi$ is the *tension field* (see [15] for a detailed account on harmonic maps). The functional E_2 is called the *bienergy functional*. In the particular case when $\varphi\colon (M,g)\to(N,h)$ is a Riemannian immersion, the tension field has the expression $\tau(\varphi)=mH$ and (1) is equivalent to φ being a critical point of E_2.

Obviously, any minimal submanifold ($H=0$) is biharmonic. The nonharmonic biharmonic submanifolds are called *proper-biharmonic*.

The study of proper-biharmonic submanifolds is nowadays becoming a very active subject and its popularity was initiated with the challenging conjecture of B.-Y. Chen: *Any biharmonic submanifold in the Euclidean space is minimal.*

Due to some nonexistence results (see [18] and [25]) the Chen conjecture was generalized to: *Any biharmonic submanifold in a Riemannian manifold with non-positive sectional curvature is minimal*, but this was proved not to hold. Indeed, in [28] the authors constructed examples of proper-biharmonic hypersurfaces in a 5-dimensional space of nonconstant negative sectional curvature.

Yet, the conjecture is still open in its full generality for ambient spaces with constant nonpositive sectional curvature, although it was proved to be true in numerous cases when additional geometric properties for the submanifolds were assumed (see, for example, [2], [7], [10], [14] and [17]).

By way of contrast, as we shall detail in Section 2, there are several families of examples of proper-biharmonic submanifolds in the n-dimensional unit Euclidean sphere $\mathbb{S}^n$. For simplicity we shall denote these classes by B1, B2, B3 and B4. Nevertheless, a full understanding of the geometry of proper-biharmonic submanifolds in $\mathbb{S}^n$ has not been achieved. The goal of this paper is to continue the study of proper-biharmonic submanifolds in $\mathbb{S}^n$ that was initiated for the very first time in [18] and then developed in [2]–[7], [24] and [25].

In [5] the proper-biharmonic submanifolds with parallel mean curvature vector fields (PMC) in $\mathbb{S}^n$ were studied. In the first part of this paper we extend our study to biharmonic submanifolds with parallel normalized mean curvature vector fields (PNMC). We recall that there exist PNMC surfaces which are not PMC (see [8] and [21]) and, obviously, a PNMC submanifold is PMC if and only if it has constant mean curvature (CMC). We underline the fact that all known examples of proper-biharmonic submanifolds in spheres are CMC, but there is no general result concerning the constancy of the mean curvature of proper-biharmonic submanifolds in $\mathbb{S}^n$.

First, in Section 3, under some hypotheses on the mean curvature function or on the squared norm of the shape operator associated with the mean curvature vector field, we prove that compact, or complete, PNMC biharmonic submanifolds are PMC.

As we shall see in Section 4, PNMC pseudo-umbilical biharmonic submanifolds in $\mathbb{S}^n$ are of class B3. We then study the PNMC biharmonic submanifolds in $\mathbb{S}^n$ with at most two distinct principal curvatures in the direction of the mean curvature vector field, proving that they are CMC and belong to the classes B3 and B4 (Theorem 4.4).

The second part of the paper is devoted to finite-type submanifolds. These submanifolds were introduced by Chen (see, for example, [9] and [11]) in the attempt of finding the best possible estimate of the total mean curvature of a compact submanifold in the Euclidean space. Although defined in a different manner, finite-type submanifolds arise also, in a natural way, as solutions of a variational problem.

We prove that proper-biharmonic submanifolds in spheres are of 1-type or 2-type if and only if they are CMC with mean curvature $f=1$ or $f\in(0,1)$, respectively (Theorem 5.8).

Moreover, we prove that there are no 3-type PNMC biharmonic submanifolds in $\mathbb{S}^n$ (Theorem 5.10), obtaining the nonexistence of 3-type biharmonic hypersurfaces in $\mathbb{S}^n$ (Corollary 5.11).

Finally, under some extra conditions (mass-symmetric and independent) on finite k-type submanifolds in $\mathbb{S}^n$ we prove that biharmonicity implies $k=2$ (Proposition 5.12).

Conventions Throughout this paper all manifolds, metrics and maps are assumed to be smooth, i.e. C^∞. All manifolds are assumed to be connected. The following sign conventions are used

$$\Delta^\varphi V = -\operatorname{trace}\nabla^2 V \quad \text{and} \quad R^N(X,Y)=[\nabla_X,\nabla_Y]-\nabla_{[X,Y]},$$

where $V\in C(\varphi^{-1}(TN))$ and $X,Y\in C(TN)$.

By a *submanifold* M in a Riemannian manifold (N,h) we mean a Riemannian immersion $\varphi\colon M\to(N,h)$ (see, for example, [20] and [30]).

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2. Biharmonic submanifolds

2.1. Submanifolds in Riemannian manifolds

We recall here the fundamental equations of first order for a submanifold in a Riemannian manifold. These equations define the second fundamental form, the shape operator and the connection in the normal bundle.

Let $\varphi\colon M\to N$ be a Riemannian immersion. For each $p\in M$, $T_{\varphi(p)}N$ splits as an orthogonal direct sum

$$T_{\varphi(p)}N = d\varphi(T_pM)\oplus d\varphi(T_pM)^{\perp}, \tag{2}$$

and $NM=\bigcup_{p\in M} d\varphi(T_pM)^{\perp}$ is referred to as the *normal bundle* of φ, or of M in N.

Denote by ∇ and ∇^N the Levi-Civita connections on M and N, respectively, and by ∇^{φ} the induced connection in the pull-back bundle $\varphi^{-1}(TN)=\bigcup_{p\in M} T_{\varphi(p)}N$. Taking into account the decomposition in (2), one has

$$\nabla^{\varphi}_X\, d\varphi(Y) = d\varphi(\nabla_X Y)+B(X,Y), \quad X,Y\in C(TM),$$

where $B\in C(\odot^2 T^*M\otimes NM)$ is called the *second fundamental form* of M in N. The mean curvature vector field of M in N is defined by $H=(\operatorname{trace} B)/m\in C(NM)$ and the mean curvature function of M is $|H|$. All throughout this paper we shall write $f=|H|$.

Furthermore, if $\eta\in C(NM)$, then

$$\nabla^{\varphi}_X\eta = -d\varphi(A_\eta(X))+\nabla^{\perp}_X\eta, \quad X\in C(TM),$$

where $A_\eta\in C(T^*M\otimes TM)$ is called the *shape operator* of M in N in the direction of η, and $\nabla^{\perp}$ is a connection on sections of NM, called the *induced connection* in the normal bundle. Moreover, $\langle B(X,Y),\eta\rangle=\langle A_\eta(X),Y\rangle$, for all $X,Y\in C(TM)$ and $\eta\in C(NM)$.

When confusion is unlikely, locally, we identify M with its image, X with $d\varphi(X)$ and we replace $\nabla^{\varphi}_X\, d\varphi(Y)$ with $\nabla^N_X Y$. With these identifications in mind, we write

$$\nabla^N_X Y = \nabla_X Y + B(X,Y),$$

and

$$\nabla^N_X\eta = -A_\eta(X)+\nabla^{\perp}_X\eta.$$

2.2. Biharmonic submanifolds

The key ingredient in the study of biharmonic submanifolds is the splitting of the bitension field with respect to its normal and tangent components.

Theorem 2.1. *A submanifold* $\varphi\colon M^m \to N^n$ *in a Riemannian manifold* N *is biharmonic if and only if the normal and the tangent components of* $\tau_2(\varphi)$ *vanish, i.e. respectively*

$$\Delta^\perp H + \operatorname{trace} B(\,\cdot\,, A_H\,\cdot\,) + \operatorname{trace}(R^N(d\varphi(\,\cdot\,), H)\, d\varphi(\,\cdot\,))^\perp = 0, \tag{3a}$$

and

$$\frac{m}{2}\operatorname{grad} f^2 + 2\operatorname{trace} A_{\nabla^\perp_{(\,\cdot\,)} H}(\,\cdot\,) + 2\operatorname{trace}(R^N(d\varphi(\,\cdot\,), H)\, d\varphi(\,\cdot\,))^\top = 0, \tag{3b}$$

where A *denotes the shape operator,* B *the second fundamental form,* H *the mean curvature vector field,* $f=|H|$ *the mean curvature function, and* $\nabla^\perp$ *and* $\Delta^\perp$ *the connection and the Laplacian in the normal bundle of* M *in* N.

This result was obtained in [9] and [25] for submanifolds in space forms, and in [27] for general hypersurfaces (see also [22] for the normal component of τ_2). We note that the tangent part of $\tau_2(\varphi)$ vanishes if and only if the stress-energy tensor for biharmonic maps associated with φ vanishes (see [19] and [23]). In the case when the ambient space is a space form $\mathbb{E}^n(c)$ of constant sectional curvature c, (3a)–(3b) reduce to the following.

Corollary 2.2. ([9] and [25]) *A submanifold* $\varphi\colon M^m \to \mathbb{E}^n(c)$ *in the space form* $\mathbb{E}^n(c)$ *is biharmonic if and only if*

$$\begin{cases} \Delta^\perp H + \operatorname{trace} B(\,\cdot\,, A_H\,\cdot\,) - mc\,H = 0, \\ 2\operatorname{trace} A_{\nabla^\perp_{(\,\cdot\,)} H}(\,\cdot\,) + \dfrac{m}{2}\operatorname{grad} f^2 = 0. \end{cases} \tag{4}$$

Up to now there are no known examples of proper-biharmonic submanifolds in a space form $\mathbb{E}^n(c)$ with $c\le 0$, i.e. proper solutions of (4) with $c\le 0$. This fact has suggested, as we have mentioned in the introduction, the generalized Chen conjecture.

If $c=1$, the situation is rather different and we consider the following to be the *main examples* of proper-biharmonic submanifolds in the unit Euclidean sphere $\mathbb{S}^n=\mathbb{E}^n(1)$.

Example B1. The small hypersphere

$$\mathbb{S}^{n-1}\left(\frac{1}{\sqrt{2}}\right) = \left\{\left(x, \frac{1}{\sqrt{2}}\right) \in \mathbb{R}^{n+1} : |x|^2 = \frac{1}{2}\right\} \subset \mathbb{S}^n.$$

Example B2. The standard products of spheres

$$\mathbb{S}^{n_1}\left(\frac{1}{\sqrt{2}}\right)\times\mathbb{S}^{n_2}\left(\frac{1}{\sqrt{2}}\right)=\left\{(x,y)\in\mathbb{R}^{n_1+1}\times\mathbb{R}^{n_2+1}\colon |x|^2=|y|^2=\frac{1}{2}\right\}\subset\mathbb{S}^n,$$

$n_1+n_2=n-1$ and $n_1\neq n_2$.

Example B3. The submanifolds $\varphi=\imath\cdot\psi\colon M\to\mathbb{S}^n$, where $\psi\colon M\to\mathbb{S}^{n-1}(1/\sqrt{2})$ is a minimal immersion, and $\imath\colon\mathbb{S}^{n-1}(1/\sqrt{2})\to\mathbb{S}^n$ denotes the inclusion map.

Example B4. The submanifolds $\varphi=\imath\cdot(\psi_1\times\psi_2)\colon M_1\times M_2\to\mathbb{S}^n$, where the maps $\psi_i\colon M_i^{m_i}\to\mathbb{S}^{n_i}(1/\sqrt{2})$, $0<m_i\leq n_i$, $i=1,2$, are minimal immersions, $m_1\neq m_2$, $n_1+n_2=n-1$, and $\imath\colon\mathbb{S}^{n_1}(1/\sqrt{2})\times\mathbb{S}^{n_2}(1/\sqrt{2})\to\mathbb{S}^n$ denotes the inclusion map.

Example B2 was found in [18], while Example B1 was derived in [6]. The two families of examples described in Examples B3 and B4 were constructed in [7]. Moreover, Example B3 is a consequence of the following property.

Theorem 2.3. ([7]) *Let $\psi\colon M\to\mathbb{S}^{n-1}(a)$ be a minimal submanifold in a small hypersphere $\mathbb{S}^{n-1}(a)\subset\mathbb{S}^n$, of radius $a\in(0,1)$, and denote by $\imath\colon\mathbb{S}^{n-1}(a)\to\mathbb{S}^n$ the inclusion map. Then $\varphi=\imath\cdot\psi\colon M\to\mathbb{S}^n$ is proper-biharmonic if and only if $a=1/\sqrt{2}$.*

We note that the proper-biharmonic submanifolds in $\mathbb{S}^n$, obtained from minimal submanifolds of the proper-biharmonic hypersphere $\mathbb{S}^{n-1}(1/\sqrt{2})$, have constant mean curvature $f=1$.

More generally, we have the following bounds for the mean curvature of CMC proper-biharmonic submanifolds in $\mathbb{S}^n$.

Theorem 2.4. ([26]) *Let $\varphi\colon M\to\mathbb{S}^n$ be a CMC proper-biharmonic submanifold. Then $f\in(0,1]$. Moreover, if $f=1$, then φ induces a minimal immersion of M into $\mathbb{S}^{n-1}(1/\sqrt{2})\subset\mathbb{S}^n$.*

Notice also that proper-biharmonic submanifolds in $\mathbb{S}^n$ obtained from minimal submanifolds of $\mathbb{S}^{n-1}(1/\sqrt{2})$ have parallel mean curvature vector fields (PMC) and are pseudo-umbilical, i.e. $A_H=f^2\,\mathrm{Id}$. In [29] it was proved that an umbilical biharmonic surface in any 3-dimensional Riemannian manifold must be a CMC surface. This is a particular case of the following proposition.

Proposition 2.5. *Let $\varphi\colon M^m\to N$ be a submanifold in a Riemannian manifold N, $m\neq 4$. If M is pseudo-umbilical, then the tangent part of $\tau_2(\varphi)$ vanishes, i.e. (3b) is satisfied, if and only if M is CMC. In particular, if M is a biharmonic pseudo-umbilical submanifold in N, $m\neq 4$, then M is CMC.*

Proof. First, we note that, in general, (3b) is equivalent to

$$-\frac{m}{2}\operatorname{grad} f^2 + 2\operatorname{trace}(\nabla A_H)(\cdot,\cdot) = 0. \tag{5}$$

Since M is pseudo-umbilical, $A_H = f^2 \operatorname{Id}$ and we find immediately that

$$\operatorname{trace}(\nabla A_H)(\cdot,\cdot) = \operatorname{grad} f^2. \tag{6}$$

Then, (5) is equivalent to

$$(m-4)\operatorname{grad} f^2 = 0,$$

and we conclude. □

We recall that a codimension-two pseudo-umbilical submanifold $\varphi\colon M^m \to \mathbb{S}^{m+2}$, $m\neq 4$, is proper-biharmonic if and only if it is minimal in $\mathbb{S}^{m+1}(1/\sqrt{2})$ (see [2]). Now, a natural question arises: *For arbitrary codimension, is a pseudo-umbilical proper-biharmonic submanifold M^m in $\mathbb{S}^n$, $m\neq 4$, minimal in $\mathbb{S}^{n-1}(1/\sqrt{2})$?*

3. Biharmonic submanifolds with parallel normalized mean curvature vector fields in $\mathbb{S}^n$

A submanifold $\varphi\colon M \to N$ in a Riemannian manifold is said to have *parallel normalized mean curvature vector fields* (PNMC) if it has nowhere zero mean curvature and the unit vector field in the direction of the mean curvature vector field is parallel in the normal bundle, i.e.

$$\nabla^{\perp}(H/f) = 0, \tag{7}$$

where $f = |H|$ denotes the mean curvature function. Notice that, in this case, f is a smooth and positive function. In the following, for a PNMC submanifold, we shall denote by $\xi = H/f$ the normalized mean curvature vector field and by A the shape operator associated with ξ.

PNMC submanifolds generalize nonminimal PMC submanifolds. Moreover, for CMC submanifolds PNMC is equivalent to PMC. Note that, as stated in [8] and [21], it is possible to find examples of PNMC submanifolds which are not PMC.

The following characterization of PNMC biharmonic submanifolds in $\mathbb{S}^n$, which we shall use throughout this paper, is an immediate consequence of Corollary 2.2.

Theorem 3.1. *Let $\varphi\colon M^m \to \mathbb{S}^n$ be a PNMC submanifold in the n-dimensional unit Euclidean sphere $\mathbb{S}^n$. Then M is biharmonic if and only if*

$$\text{(8)}\qquad \begin{cases} \operatorname{trace} B(\,\cdot\,, A_H\,\cdot\,) = \left(m - \dfrac{1}{f}\,\Delta f\right) H, \\ A_H(\operatorname{grad} f^2) = -\dfrac{m}{2}\, f^2 \operatorname{grad} f^2, \end{cases}$$

or, equivalently,

$$\text{(9)}\qquad \begin{cases} \text{(i)}\ \ \langle A, A_\eta \rangle = 0 & \text{for all } \eta \in C(NM) \text{ with } \eta \perp \xi, \\ \text{(ii)}\ \ \Delta f = (m - |A|^2) f, \\ \text{(iii)}\ A(\operatorname{grad} f) = -\dfrac{m}{2} f \operatorname{grad} f, \end{cases}$$

where NM denotes the normal bundle of M in $\mathbb{S}^n$.

Proof. Let $p \in M$ and consider $\{E_i\}_{i=1}^m$ to be a local orthonormal frame field on M geodesic at p. Since M is PNMC, we have

$$\text{(10)}\qquad \nabla^\perp_X H = \frac{1}{f}\, X(f) H, \quad X \in C(TM).$$

From here, at p we have

$$\Delta^\perp H = -\operatorname{trace}(\nabla^\perp)^2 H = -\sum_{i=1}^m \nabla^\perp_{E_i}\left(\frac{1}{f} E_i(f) H\right) = \frac{1}{f}(\Delta f) H,$$

which implies that the first equation of (4) becomes the first equation of (8).

From (10) we obtain

$$\text{(11)}\qquad \operatorname{trace} A_{\nabla^\perp_{(\,\cdot\,)} H}(\,\cdot\,) = \sum_{i=1}^m A_{\nabla^\perp_{E_i} H}(E_i) = \frac{1}{2f^2} A_H(\operatorname{grad} f^2),$$

and the second equation of (4) becomes the second equation of (8).

Next, since $A_H = fA$, by considering the components of $\operatorname{trace} B(\,\cdot\,, A_H\,\cdot\,)$, the one parallel to ξ and the one orthogonal to ξ, one verifies immediately that (8) and (9) are equivalent. □

3.1. The compact case

Immediate consequences for compact PNMC biharmonic submanifolds follow from (9)(ii).

Corollary 3.2. *Let $\varphi\colon M\to\mathbb{S}^n$ be a compact PNMC biharmonic submanifold.*

(i) *If $|A|^2\leq m$, or $|A|^2\geq m$, on M, then M is PMC and $|A|^2=m$.*

(ii) *If $|A|$ is constant, then M is PMC and $|A|^2=m$.*

From Corollary 3.2, if M is a compact PNMC biharmonic submanifold in $\mathbb{S}^n$, then either there exists $p\in M$ such that $|A(p)|^2<m$, or $|A|^2=m$.

Moreover, as a consequence of Corollary 3.2, we shall also prove that compact PNMC biharmonic submanifolds in $\mathbb{S}^n$, with a supplementary bounding condition on the mean curvature, are PMC.

Proposition 3.3. *Let $\varphi\colon M\to\mathbb{S}^n$ be a compact PNMC biharmonic submanifold. If the mean curvature of M satisfies $f^2\geq 4(m-1)/m(m+8)$, then M is PMC. Moreover,*

(i) *if $m\in\{2,3\}$, then φ induces a minimal immersion of M into $\mathbb{S}^{n-1}(1/\sqrt{2})\subset\mathbb{S}^n$;*

(ii) *if $m=4$, then $\varphi(M)$ is the standard product $\psi_1(M_1^3)\times\mathbb{S}^1(1/\sqrt{2})$, where $\psi_1\colon M_1\to\mathbb{S}^{n-2}(1/\sqrt{2})$ is a compact minimal submanifold.*

Proof. We will show that, under the given hypotheses, we have $|A|^2\geq m$ on M. Therefore, by Corollary 3.2, M is PMC. Then, the last assertion follows from Theorem 3.11 in [5].

Let $p_0\in M$ be arbitrarily fixed. We have two cases.

Case 1. If $\operatorname{grad}_{p_0} f\neq 0$, since M is PNMC biharmonic, from (9)(iii) we have that $e_1=\operatorname{grad}_{p_0} f/|\operatorname{grad}_{p_0} f|$ is a principal direction for A with principal curvature $\lambda_1=-mf(p_0)/2$. By considering $e_k\in T_{p_0}M$, $k=2,\dots,m$, such that $\{e_i\}_{i=1}^m$ is an orthonormal basis in $T_{p_0}M$ and $A(e_k)=\lambda_k e_k$, we get at p_0 that

$$
\begin{aligned}
(12)\qquad |A|^2 &= \sum_{i=1}^m |A(e_i)|^2 = |A(e_1)|^2+\sum_{k=2}^m |A(e_k)|^2 = \frac{m^2}{4}f^2+\sum_{k=2}^m \lambda_k^2\\
&\geq \frac{m^2}{4}f^2+\frac{1}{m-1}\left(\sum_{k=2}^m \lambda_k\right)^2 = \frac{m^2(m+8)}{4(m-1)}f^2,
\end{aligned}
$$

and thus the hypothesis on the mean curvature function implies

$$|A|^2 \geq \frac{m^2(m+8)}{4(m-1)} f^2 \geq m.$$

Case 2. Consider now the case when $\operatorname{grad}_{p_0} f = 0$. If there exists an open subset $U \subset M$, $p_0 \in U$, such that $(\operatorname{grad} f)|_U = 0$, then (9)(ii) implies that $|A|^2 = m$ on U. Otherwise, p_0 is a limit point for the set $V = \{p \in M : \operatorname{grad}_p f \neq 0\}$. By case 1 we have $|A(p)| \geq m$ for all $p \in V$. Therefore, we obtain $|A(p_0)|^2 \geq m$, and the proof is completed. □

Since hypersurfaces with nowhere zero mean curvature are PNMC submanifolds, we have the following result.

Corollary 3.4. *Let* $\varphi \colon M^m \to \mathbb{S}^{m+1}$ *be a compact biharmonic hypersurface. If the mean curvature of* M *satisfies* $f^2 \geq 4(m-1)/m(m+8)$, *then* M *is CMC.*

3.2. The noncompact case

We first notice that an explicit example of a complete noncompact proper-biharmonic submanifold in $\mathbb{S}^n$ can be immediately derived from the proper-biharmonic submanifold $\mathbb{S}^{n-2}(1/\sqrt{2}) \times \mathbb{S}^1$, by considering its universal cover $\mathbb{S}^{n-2}(1/\sqrt{2}) \times \mathbb{R}$.

For the noncompact case, if $\varphi \colon M^m \to \mathbb{S}^n$ is a PNMC biharmonic submanifold such that $|A|^2 \geq m$, then f is a subharmonic function and therefore either f is constant, or f cannot attain its maximum. In the following we shall prove that, under some additional hypotheses, the latter case cannot occur.

Theorem 3.5. (Omori–Yau maximum principle, [32]) *If* M^m *is a complete Riemannian manifold with Ricci curvature bounded from below, then for any function* $u \in C^2(M)$, *bounded from above, there exists a sequence of points* $\{p_k\}_{k \in \mathbb{N}} \subset M$ *satisfying*

$$\lim_{k \to \infty} u(p_k) = \sup_M u, \quad |\operatorname{grad}_{p_k} u| < \frac{1}{k} \quad \text{and} \quad \Delta u(p_k) > -\frac{1}{k}.$$

Now we can prove our result.

Proposition 3.6. *Let* $\varphi \colon M^m \to \mathbb{S}^n$ *be a complete noncompact PNMC biharmonic submanifold with nonnegative Ricci curvature. If* $|A|$ *is constant and* $|A| \geq m$, *then* M *is PMC and* $|A|^2 = m$.

Proof. From (9)(ii), we get that $\Delta f \leq 0$ on M. On the other hand, since $f \leq |A|/\sqrt{m}$ is bounded, then by Theorem 3.5 there exists a sequence of points $\{p_k\}_{k\in\mathbb{N}} \subset M$ such that

$$\Delta f(p_k) > -\frac{1}{k} \quad \text{and} \quad \lim_{k\to\infty} f(p_k) = \sup_M f.$$

It follows that $\lim_{k\to\infty} \Delta f(p_k) = 0$, and thus $\lim_{k\to\infty} (m-|A|^2) f(p_k) = 0$.

As $\lim_{k\to\infty} f(p_k) = \sup_M f > 0$, we get $|A|^2 = m$. This, together with (9)(ii), implies that f is a harmonic function on M. Since f is also a bounded function on M, by a result of S.-T. Yau in [32], we deduce that f=constant. □

Using (12), as a direct consequence of Proposition 3.6, we obtain the following result.

Corollary 3.7. *Let $\varphi\colon M \to \mathbb{S}^n$ be a noncompact PNMC biharmonic submanifold. Assume that M is complete and has nonnegative Ricci curvature. If $|A|^2$ is constant and $f^2 \geq 4(m-1)/m(m+8)$, then M is PMC and $|A|^2 = m$.*

In the case of hypersurfaces, we recall that if $\varphi\colon M^m \to \mathbb{S}^{m+1}$ is complete and proper-biharmonic with $f=1$, then φ is an embedding and $\varphi(M) = \mathbb{S}^m(1/\sqrt{2})$, and thus M has to be compact. From here, taking also into account Theorem 3.11 in [5], we get the following consequence.

Corollary 3.8. *Let $\varphi\colon M^m \to \mathbb{S}^{m+1}$ be a noncompact biharmonic hypersurface. Assume that M is complete and has nonnegative Ricci curvature. If $|A|^2$ is constant and $f^2 \geq 4(m-1)/m(m+8)$, then M is CMC and $|A|^2 = m$. In this case, $m \geq 4$ and $f^2 \leq ((m-2)/m)^2$.*

4. PNMC biharmonic submanifolds in $\mathbb{S}^n$ with at most two distinct principal curvatures

Inspired by the case of hypersurfaces (see [2]), we intend to study PNMC biharmonic submanifolds in $\mathbb{S}^n$ by taking into account the number of distinct principal curvatures in the direction of the mean curvature vector field.

Proposition 4.1. *If $\varphi\colon M^m \to \mathbb{S}^n$, $m \geq 2$, is a pseudo-umbilical PNMC submanifold, then M is PMC. Moreover, φ induces a minimal immersion of M in $\mathbb{S}^{n-1}(a) \subset \mathbb{S}^n$ for some $a \in (0,1)$.*

Proof. By the Codazzi equation, for any submanifold in $\mathbb{S}^n$ we have

$$2\operatorname{trace}(\nabla A_H)(\cdot,\cdot) = m\operatorname{grad} f^2 + 2\operatorname{trace} A_{\nabla^\perp_{(\cdot)}H}(\cdot). \tag{13}$$

Now, taking into account (6) and (11), (13) becomes

$$(m-1)\operatorname{grad} f^2 = 0.$$

Thus M is PMC and, using a result of Chen in [9, p. 133], we conclude that φ induces a minimal immersion of M in $\mathbb{S}^{n-1}(a)\subset\mathbb{S}^n$ for some $a\in(0,1)$. □

Combining Proposition 4.1 and Theorem 2.3, one gets the following result.

Proposition 4.2. *Let $\varphi\colon M^m\to\mathbb{S}^n$, $m\geq 2$, be a PNMC pseudo-umbilical biharmonic submanifold. Then, φ induces a minimal immersion of M in $\mathbb{S}^{n-1}(1/\sqrt{2})\subset\mathbb{S}^n$.*

Thus, the next step consists in classifying the PNMC biharmonic submanifolds in $\mathbb{S}^n$ with at most two distinct principal curvatures in the direction of H. Notice that any hypersurface with nowhere zero mean curvature is PNMC, and the classification of proper-biharmonic hypersurfaces with at most two distinct principal curvatures was achieved in [2]. In order to obtain the desired general classification, we first have to prove the following result.

Theorem 4.3. *Let $\varphi\colon M^m\to\mathbb{S}^n$ be a PNMC biharmonic submanifold with at most two distinct principal curvatures in the direction of H. Then M is PMC.*

Proof. It is sufficient to prove that f, which is a positive function on M, is constant. Suppose that $f\neq$constant. Then, there exists $p\in M$ such that $\operatorname{grad}_p f\neq 0$, and hence there is a neighborhood U of p in M such that $\operatorname{grad} f\neq 0$ on U. Taking into account Proposition 4.2, U cannot be made up only of pseudo-umbilical points. We can thus assume that there exists a point $q\in U$ which is not pseudo-umbilical. Then, eventually by restricting U, we can assume that $A\neq f\operatorname{Id}$ at every point of U. Thus A has exactly two distinct principal curvatures on U. Recall that, as A has exactly two distinct principal curvatures, the multiplicities of its principal curvatures are constant and the principal curvatures are smooth (see [30]). Thus A is diagonalizable with respect to a local orthonormal frame field $\{E_1,\dots,E_m\}$. We then have $A(E_i)=\bar{k}_iE_i$, $i=1,\dots,m$, where

$$\bar{k}_1(q)=\dots=\bar{k}_{m_1}(q)=k_1(q), \quad \bar{k}_{m_1+1}(q)=\dots=\bar{k}_m(q)=k_2(q),$$

and $k_1(q)\neq k_2(q)$ for all $q\in U$. From (9)(iii) we can assume that

$$k_1=-\frac{m}{2}f \tag{14}$$

and $E_1=\operatorname{grad} f/|\operatorname{grad} f|$.

Since $\langle E_\alpha, E_1\rangle=0$, we have on U that

$$E_\alpha(f)=0, \quad \alpha=2,...,m. \tag{15}$$

We shall use the connection equations with respect to the orthonormal frame field $\{E_1,...,E_m\}$,

$$\nabla_{E_i}E_j=\omega_j^k(E_i)E_k. \tag{16}$$

Let us first prove that the multiplicity of k_1 is $m_1=1$. Suppose that $m_1\geq 2$. Then there exists $\alpha\in\{2,...,m_1\}$ such that $\bar{k}_\alpha=k_1$ on U. Since $\nabla^\perp\xi=0$, the Codazzi equation for A becomes

$$(\nabla_{E_i}A)(E_j)=(\nabla_{E_j}A)(E_i), \quad i,j=1,...,m. \tag{17}$$

By using (16), the Codazzi equation can be written as

$$E_i(\bar{k}_j)E_j+\sum_{l=1}^{m}(\bar{k}_j-\bar{k}_l)\omega_j^l(E_i)E_l=E_j(\bar{k}_i)E_i+\sum_{l=1}^{m}(\bar{k}_i-\bar{k}_l)\omega_i^l(E_j)E_l. \tag{18}$$

Putting $i=1$ and $j=\alpha$ in (18) and taking the scalar product with E_α we obtain $E_1(k_1)=0$, which, together with (14) and (15), gives $f=$constant, which is a contradiction.

Thus $\bar{k}_1=k_1$ and $\bar{k}_\alpha=k_2$, for $\alpha=2,...,m$, and since trace $A=mf$, we get

$$k_2=\frac{3}{2}\frac{m}{m-1}f. \tag{19}$$

Putting $i=1$ and $j=\alpha$ in (18) and taking the scalar product with E_α, E_β, $\beta\neq\alpha$, and E_1, respectively, one gets

$$\omega_1^\alpha(E_\alpha)=-\frac{3}{m+2}\frac{E_1(f)}{f}, \tag{20a}$$

$$\omega_1^\alpha(E_\beta)=0, \tag{20b}$$

$$\omega_1^\alpha(E_1)=0 \tag{20c}$$

for $\alpha,\beta=2,...,m$, $\alpha\neq\beta$.

Consider $\{\eta_{m+1}=\xi,\eta_{m+2},...,\eta_n\}$ to be an orthonormal frame field of the normal bundle of U in $\mathbb{S}^n$ and let $A_a=A_{\eta_a}$, $a=m+2,...,n$. As $\nabla^\perp\xi=0$, from the Ricci equation of U in $\mathbb{S}^n$, we have

$$A\cdot A_a=A_a\cdot A, \quad a=m+2,...,n.$$

Since k_1 has multiplicity 1, it follows directly that E_1 is a principal direction for A_a, for all $a=m+2,\dots,n$. Fix $a\in\{m+2,\dots,n\}$ and set $A_a(E_1)=\lambda_a E_1$ on U. From (9)(i), we have that $\sum_{i=1}^{m}\langle A(E_i),A_a(E_i)\rangle=0$ and this leads to

$$(k_1-k_2)\lambda_a+k_2\,\mathrm{trace}\,A_a=0.$$

Since $\mathrm{trace}\,A_a=m\langle H,\eta_a\rangle=0$, we conclude that $\lambda_a=0$, i.e.

$$A_a(E_1)=0,\quad a=m+2,\dots,n. \tag{21}$$

We now express the Gauss equation for U in $\mathbb{S}^n$,

$$\begin{aligned}\langle R^{\mathbb{S}^n}(X,Y)Z,W\rangle &= \langle R(X,Y)Z,W\rangle \\ &\quad +\langle B(X,Z),B(Y,W)\rangle-\langle B(X,W),B(Y,Z)\rangle,\end{aligned} \tag{22}$$

with $X=W=E_1$ and $Y=Z=E_\alpha$. Using (21) one obtains

$$B(E_1,E_\alpha)=0,\quad B(E_1,E_1)=k_1\xi\quad \text{and}\quad \langle B(E_\alpha,E_\alpha),B(E_1,E_1)\rangle=k_1k_2.$$

From (16), (20b), (20c), and using that $\omega_j^k=-\omega_k^j$, the curvature term is

$$\langle R(E_1,E_\alpha)E_\alpha,E_1\rangle=-E_1(\omega_1^\alpha(E_\alpha))-\omega_1^\alpha(E_\alpha)^2.$$

Finally, (22) and (20a) imply

$$fE_1(E_1(f))=\frac{m+2}{3}f^2-\frac{m^2(m+2)}{4(m-1)}f^4+\frac{m+5}{m+2}E_1(f)^2. \tag{23}$$

From (14) and (19), we have

$$|A|^2=k_1^2+(m-1)k_2^2=\frac{m^2(m+8)}{4(m-1)}f^2. \tag{24}$$

Moreover, using (15), (16) and (20a), the Laplacian of f becomes

$$\begin{aligned}\Delta f &= -E_1(E_1(f))-\sum_{\alpha=2}^{m}E_\alpha(E_\alpha(f))+(\nabla_{E_1}E_1)f+\sum_{\alpha=2}^{m}(\nabla_{E_\alpha}E_\alpha)f \\ &= -E_1(E_1(f))+\sum_{\alpha=2}^{m}\omega_\alpha^1(E_\alpha)E_1(f) \\ &= -E_1(E_1(f))+\frac{3(m-1)}{m+2}\frac{E_1(f)^2}{f}.\end{aligned} \tag{25}$$

From (9)(ii), by substituting (24) and (25), we get

$$fE_1(E_1(f)) = -mf^2 + \frac{m^2(m+8)}{4(m-1)}f^4 + \frac{3(m-1)}{m+2}E_1(f)^2. \tag{26}$$

Consider now $\gamma=\gamma(u)$ to be an arbitrary integral curve of E_1 in U. Along γ we have $f=f(u)$ and we set $w=E_1(f)^2=(f')^2$. Then $dw/df=2f''$, and (23) and (26) become

$$\begin{cases} \frac{1}{2}f\frac{dw}{df} = \frac{m+2}{3}f^2 - \frac{m^2(m+2)}{4(m-1)}f^4 + \frac{m+5}{m+2}w, \\ \frac{1}{2}f\frac{dw}{df} = -mf^2 + \frac{m^2(m+8)}{4(m-1)}f^4 + \frac{3(m-1)}{m+2}w. \end{cases} \tag{27}$$

By subtracting the two equations we find two cases.

If $m=4$, then

$$4f^4 - f^2 = 0,$$

and thus f is constant.

If $m\neq 4$, then

$$w = \frac{(m+2)(2m+1)}{3(m-4)}f^2 - \frac{m^2(m+2)(m+5)}{4(m-4)(m-1)}f^4.$$

Differentiating with respect to f and replacing this in the second equation of (27), we get

$$\frac{(m-1)(m+5)}{3}f^2 + \frac{3m^2(2m+1)}{4(m-1)}f^4 = 0.$$

Therefore f is constant along γ, and thus $\operatorname{grad} f=0$ along γ and we have a contradiction. $\square$

As a consequence of Theorem 4.3 we have the following rigidity result.

Theorem 4.4. *Let $\varphi\colon M^m\to\mathbb{S}^n$ be a PNMC biharmonic submanifold with at most two distinct principal curvatures in the direction of H. Then either φ induces a minimal immersion of M in $\mathbb{S}^{n-1}(1/\sqrt{2})$, or locally,*

$$\varphi(M) = M_1^{m_1}\times M_2^{m_2} \subset \mathbb{S}^{n_1}\left(\frac{1}{\sqrt{2}}\right)\times\mathbb{S}^{n_2}\left(\frac{1}{\sqrt{2}}\right)\subset\mathbb{S}^n,$$

where M_i is a minimal embedded submanifold of $\mathbb{S}^{n_i}(1/\sqrt{2})$, $i=1,2$, $m_1+m_2=m$, $m_1\neq m_2$ and $n_1+n_2=n-1$.

Proof. From Theorem 4.3 we conclude that M is PMC. Moreover, since A_H has at most two distinct principal curvatures in the direction of H, from Proposition 3.19 in [5], we get that $\nabla A_H=0$. Now, the conclusion follows by applying Theorem 3.16 in [5]. □

Moreover, as a corollary of Theorem 4.4, the following rigidity result, which generalizes Theorem 5.6 in [2], is valid.

Corollary 4.5. *Let* $\varphi\colon M^2\to\mathbb{S}^n$ *be a PNMC biharmonic surface. Then* φ *induces a minimal immersion of* M *in* $\mathbb{S}^{n-1}(1/\sqrt{2})$.

Remark 4.6. (i) In [8] it was proved that, in general, a PNMC analytic surface in $\mathbb{S}^n$ is either minimal in a small hypersphere of $\mathbb{S}^n$, and therefore it is PMC, or it lies in a 4-dimensional great sphere $\mathbb{S}^4\subset\mathbb{S}^n$. Notice that with no analyticity condition, by Corollary 4.5, the supplementary hypothesis that the surface is biharmonic leads only to the first case.

(ii) For the particular case of PNMC biharmonic surfaces in $\mathbb{S}^4$ we can give a different proof of Theorem 4.4. Indeed, if we suppose that the surface is not PMC, using the codimension reduction result of J. Erbacher in [16], we obtain that the surface lies in a great hypersphere $\mathbb{S}^3$ of $\mathbb{S}^4$. Therefore, the surface has constant mean curvature and this is a contradiction.

(iii) We can slightly relax the hypotheses of Theorem 4.4, obtaining the same result, in the following way. By the unique continuation property for biharmonic maps (see [26]), if M is a proper-biharmonic submanifold in $\mathbb{S}^n$, then H is nowhere zero on an open dense subset $W\subset M$. If we assume that $\nabla^{\perp}(H/f)=0$ on W and A_H has at most two distinct principal curvatures everywhere on W, then by Theorem 4.4 we get $\nabla^{\perp}H=0$ on W. By continuity we obtain $\nabla^{\perp}H=0$ on M.

5. On the type of biharmonic submanifolds in $\mathbb{S}^n$

Definition 5.1. ([9] and [11]) A submanifold $\phi\colon M\to\mathbb{R}^{n+1}$ is of *finite type* if it can be expressed as a finite sum of $\mathbb{R}^{n+1}$-valued eigenmaps of the Laplacian Δ of M, i.e.

$$\phi=\phi_0+\phi_{t_1}+\ldots+\phi_{t_k}, \tag{28}$$

where $\phi_0\in\mathbb{R}^{n+1}$ is a constant vector and $\phi_{t_i}\colon M\to\mathbb{R}^{n+1}$ are nonconstant maps satisfying $\Delta\phi_{t_i}=\lambda_{t_i}\phi_{t_i}$, $i=1,\ldots,k$. If, in particular, all eigenvalues λ_{t_i} are assumed to be mutually distinct, the submanifold is said to be of *k-type* and (28) is called the *spectral decomposition* of ϕ.

Remark 5.2. If M is compact, the immersion $\phi\colon M\to\mathbb{R}^{n+1}$ admits a unique spectral decomposition $\phi=\phi_0+\sum_{i=1}^{\infty}\phi_i$, where ϕ_0 is the *center of mass.* Then, it is of k-type if only k terms of $\{\phi_i\}_{i=1}^{\infty}$ are nonvanishing. In the noncompact case the spectral decomposition $\phi=\phi_0+\sum_{i=1}^{\infty}\phi_i$ is not guaranteed. Nonetheless, if Definition 5.1 is satisfied, the spectral decomposition is unique. Notice also that, in the noncompact case, the harmonic component of the spectral decomposition is not necessarily constant. Finite-type submanifolds with nonconstant harmonic component are called *null finite-type* submanifolds.

A *k-type* submanifold $\phi\colon M\to\mathbb{R}^{n+1}$ is said to be *linearly independent* if the linear subspaces

$$E_{t_i}=\operatorname{span}\{\phi_{t_i}(u):u\in M\},\quad i=1,\dots,k,$$

are linearly independent, i.e. the dimension of the subspace spanned by vectors in $\bigcup_{i=1}^{k}E_{t_i}$ is equal to $\sum_{i=1}^{k}\dim E_{t_i}$.

The following result provides us with a necessary and a sufficient condition for a submanifold to be of finite type.

Theorem 5.3. ([9] and [13]) *Let $\phi\colon M\to\mathbb{R}^{n+1}$ be a Riemannian immersion.*

(i) *If M is of finite k-type, there exist a constant vector $\phi_0\in\mathbb{R}^{n+1}$ and a monic polynomial with simple roots P of degree k with $P(\Delta)(\phi-\phi_0)=0$.*

(ii) *If there exist a constant vector $\phi_0\in\mathbb{R}^{n+1}$ and a polynomial P with simple roots such that $P(\Delta)(\phi-\phi_0)=0$, then M is of finite k-type with $k\le\operatorname{degree}(P)$.*

We shall also use the following version.

Theorem 5.4. ([9] and [13]) *Let $\phi\colon M\to\mathbb{R}^{n+1}$ be a Riemannian immersion.*

(i) *If M is of finite k-type, there exists a monic polynomial P of degree $k-1$ or k with $P(\Delta)H^0=0$.*

(ii) *If there exists a polynomial P with simple roots such that $P(\Delta)H^0=0$, then M is of infinite type or of finite k-type with $k-1\le\operatorname{degree}(P)$.*

Here H^0 denotes the mean curvature vector field of M in $\mathbb{R}^{n+1}$.

A well known result of T. Takahashi can be rewritten as the classification of 1-type submanifolds in $\mathbb{R}^{n+1}$.

Theorem 5.5. ([31]) *A submanifold $\phi\colon M\to\mathbb{R}^{n+1}$ is of 1-type if and only if either ϕ is a minimal immersion in $\mathbb{R}^{n+1}$, or ϕ induces a minimal immersion of M in a hypersphere of $\mathbb{R}^{n+1}$.*

Definition 5.6. A submanifold $\varphi\colon M\to\mathbb{S}^n$ is said to be of *finite type* if it is of finite type as a submanifold of $\mathbb{R}^{n+1}$, where $\mathbb{S}^n$ is canonically embedded in $\mathbb{R}^{n+1}$. Moreover, a nonnull finite-type submanifold in $\mathbb{S}^n$ is said to be *mass-symmetric* if the constant vector ϕ_0 of its spectral decomposition is the center of the hypersphere $\mathbb{S}^n$, i.e. $\phi_0=0$.

Remark 5.7. By Theorem 5.5, biharmonic submanifolds of class B3 are 1-type submanifolds. Indeed, the immersion $\phi\colon M\to\mathbb{R}^{n+1}$ of M in $\mathbb{R}^{n+1}$ has the spectral decomposition

$$\phi=\phi_0+\phi_p,$$

where $\phi_0=(0,1/\sqrt{2})$, $\phi_p\colon M\to\mathbb{R}^{n+1}$, $\phi_p(x)=(\psi(x),0)$ and $\Delta\phi_p=2m\phi_p$.

Moreover, biharmonic submanifolds of class B4 are mass-symmetric 2-type submanifolds. Indeed, $\phi\colon M_1\times M_2\to\mathbb{R}^{n+1}$ has the spectral decomposition

$$\phi=\phi_p+\phi_q,$$

where $\phi_p(x,y)=(\psi_1(x),0)$, $\phi_q(x,y)=(0,\psi_2(y))$, $\Delta\phi_p=2m_1\phi_p$ and $\Delta\phi_q=2m_2\phi_q$.

Let $\varphi\colon M\to\mathbb{S}^n$ be a submanifold in $\mathbb{S}^n$ and denote by $\phi=\mathbf{i}\cdot\varphi\colon M\to\mathbb{R}^{n+1}$ the immersion of M in $\mathbb{R}^{n+1}$. Denote by H the mean curvature vector field of M in $\mathbb{S}^n$ and by H^0 the mean curvature vector field of M in $\mathbb{R}^{n+1}$.

The mean curvature vector fields H^0 and H are related by $H^0=H-\phi$. Moreover, we have

$$\langle H,\phi\rangle=0,\quad \langle H^0,H\rangle=f^2\quad\text{and}\quad \langle H^0,\phi\rangle=-1. \tag{29}$$

Following [7], the bitension field of φ can be written as

$$\tau_2(\varphi)=-m\Delta H^0+2m^2H^0+m^2(2-|H^0|^2)\phi.$$

Thus, $\tau_2(\varphi)=0$ if and only if

$$\Delta H^0-2mH^0+m(f^2-1)\phi=0, \tag{30a}$$

or equivalently, since $\Delta\phi=-mH^0$,

$$\Delta^2\phi-2m\Delta\phi-m^2(f^2-1)\phi=0. \tag{30b}$$

In [2, Theorem 3.1] we proved that CMC compact proper-biharmonic submanifolds in $\mathbb{S}^n$ are of 1-type or 2-type. This can be generalized to the following result.

Theorem 5.8. *Let* $\varphi\colon M\to\mathbb{S}^n$ *be a proper-biharmonic submanifold, not necessarily compact, in the unit Euclidean sphere* $\mathbb{S}^n$. *Denote by* $\phi=\mathbf{i}\cdot\varphi\colon M\to\mathbb{R}^{n+1}$ *the immersion of* M *in* $\mathbb{R}^{n+1}$, *where* $\mathbf{i}\colon\mathbb{S}^n\to\mathbb{R}^{n+1}$ *is the canonical inclusion map. Then*

(i) M *is a* 1*-type submanifold if and only if* $f=1$. *In this case,* $\phi=\phi_0+\phi_p$, $\Delta\phi_p=2m\phi_p$, $\phi_0\in\mathbb{R}^{n+1}$ *and* $|\phi_0|=1/\sqrt{2}$.

(ii) M *is a* 2*-type submanifold if and only if* $f=\text{constant}$, $f\in(0,1)$. *In this case,* $\phi=\phi_p+\phi_q$, $\Delta\phi_p=m(1-f)\phi_p$ *and* $\Delta\phi_q=m(1+f)\phi_q$.

Proof. In order to prove (i), notice that the converse is obvious, by Theorems 5.5 and 2.4.

Let us suppose that M is a 1-type submanifold. From Theorem 5.4(i) follows that there exists $a\in\mathbb{R}$ such that

$$\Delta H^0=aH^0. \tag{31}$$

Equations (30a) and (31) imply that

$$(2m-a)H^0-m(f^2-1)\phi=0,$$

and by considering the scalar product with H and using (29), since M is proper-biharmonic, we get $a=2m$ and

$$m(f^2-1)\phi=0.$$

Thus $f=1$. Now, as the map ϕ cannot be harmonic, (30b) leads to the spectral decomposition $\phi=\phi_0+\phi_p$, where $\Delta\phi_p=2m\phi_p$. As $\Delta\phi=-mH^0$, taking into account the relation between H and H^0, we obtain $2\phi_0=\phi+H$. Since $|\phi|=1=f$, and H is orthogonal to ϕ, we conclude that $|\phi_0|=1/\sqrt{2}$.

Let us now prove (ii). The converse of (ii) follows immediately. Indeed, from (30b), if $f=\text{constant}$, $f\in(0,1)$, then choosing the constant vector $\phi_0=0$ and the polynomial with simple roots

$$P(\Delta)=\Delta^2-2m\Delta^1-m^2(f^2-1)\Delta^0,$$

we are in the hypotheses of Theorem 5.3(ii). Thus M is of finite k-type, with $k\leq 2$. Taking into account (i), since $f\in(0,1)$, this implies that M is a 2-type submanifold with

$$\phi=\phi_p+\phi_q,$$

with corresponding eigenvalues $\lambda_p=m(1-f)$ and $\lambda_q=m(1+f)$. Also, notice that

$$\phi_p=\frac{\lambda_q}{\lambda_q-\lambda_p}\phi-\frac{1}{\lambda_q-\lambda_p}\Delta\phi \quad\text{and}\quad \phi_q=-\frac{\lambda_p}{\lambda_q-\lambda_p}\phi+\frac{1}{\lambda_q-\lambda_p}\Delta\phi,$$

which are smooth nonzero maps.

Suppose now that M is a 2-type submanifold. From Theorem 5.3(i) follows that there exist a constant vector $\phi_0 \in \mathbb{R}^{n+1}$ and $a, b \in \mathbb{R}$ such that

$$\Delta H^0 = aH^0 + b(\phi - \phi_0). \tag{32}$$

Equations (30a) and (32) lead to

$$(2m - a)H^0 - (m(f^2 - 1) + b)\phi + b\phi_0 = 0. \tag{33}$$

We have to consider two cases.

Case 1. If $b = 0$, i.e. M is a null 2-type submanifold, by taking the scalar product with H in (33) and using (29), since M is proper-biharmonic, we get $a = 2m$ and $f = 1$. By (i), this leads to a contradiction.

Case 2. If $b \neq 0$, we shall prove that $\operatorname{grad} f^2 = 0$ on M, and therefore f is constant on M. Indeed, locally, by taking the scalar product with $X \in C(TU)$ in (33), we obtain $\langle \phi_0, X \rangle = 0$ for all $X \in C(TU)$, i.e. the component of ϕ_0 tangent to U vanishes

$$\phi_0^\top = 0, \tag{34}$$

where U denotes an arbitrary open set in M. Take now the scalar product with ϕ in (33) and use (29). We obtain

$$-2m + a - m(f^2 - 1) - b + b\langle \phi_0, \phi \rangle = 0,$$

and, by differentiating,

$$m \operatorname{grad} f^2 = b \operatorname{grad} \langle \phi_0, \phi \rangle. \tag{35}$$

Now, by considering $\{E_i\}_{i=1}^m$ to be a local orthonormal frame field on U, we have

$$\operatorname{grad}\langle \phi_0, \phi \rangle = \sum_{i=1}^m E_i(\langle \phi_0, \phi \rangle) E_i = \sum_{i=1}^m \langle \phi_0, \nabla^0_{E_i} \phi \rangle E_i = \sum_{i=1}^m \langle \phi_0, E_i \rangle E_i = \phi_0^\top. \tag{36}$$

This, together with (34) and (35), leads to $\operatorname{grad} f^2 = 0$ on U.

Now, as f is constant on M, using Theorem 2.4, we conclude the proof. □

Remark 5.9. The direct implication of (i) in Theorem 5.8 can be also proved in a more geometric manner (see [1]).

We are now interested in proper-biharmonic submanifolds of 3-type in spheres. In [12] it was proved that there are no CMC 3-type hypersurfaces in a hypersphere of the Euclidean space. Since the known examples of proper-biharmonic hypersurfaces in spheres are CMC, one may think that there are no such hypersurfaces of 3-type. Indeed, we have a more general result.

Theorem 5.10. *There exist no PNMC biharmonic 3-type submanifolds M^m in the unit Euclidean sphere $\mathbb{S}^n$.*

Proof. Suppose that M is a PNMC biharmonic 3-type submanifold. From Theorem 5.3 follows that there exist a constant vector $\phi_0 \in \mathbb{R}^{n+1}$ and $a, b, c \in \mathbb{R}$ such that

$$\Delta^2 H^0 = a\Delta H^0 + bH^0 + c(\phi - \phi_0). \tag{37}$$

Equations (30a) and (37) lead to

$$\Delta^2 H^0 = (2ma+b)H^0 + (c - ma(f^2-1))\phi - c\phi_0. \tag{38}$$

Now, by applying Δ to (30a) we get

$$\Delta^2 H^0 = m^2(3+f^2)H^0 - (m\Delta f^2 + 2m^2(f^2-1))\phi + 2m\, d\phi(\operatorname{grad} f^2). \tag{39}$$

By taking the scalar product with $\xi = H/f$ in (38) and (39) and by using (29), we obtain

$$-c\langle \phi_0, \xi\rangle = m^2 f^3 + (3m^2 - 2ma - b)f. \tag{40}$$

In the following, by using a local argument, we shall prove that $\operatorname{grad} f = 0$ on M. Take the scalar product with $X \in C(TU)$ in (38) and (39), where U is an arbitrary fixed open set in M. This implies that

$$-c\langle \phi_0, X\rangle = 2mX(f^2),$$

and, further, the component of $c\phi_0$ tangent to U is given by

$$-c\phi_0^\top = 2m \operatorname{grad} f^2. \tag{41}$$

Moreover, by taking the scalar product with an arbitrary vector field η normal to U in $\mathbb{S}^n$, $\eta \perp \xi$, in (38) and (39), we find that

$$-c\langle \phi_0, \eta\rangle = 0. \tag{42}$$

Equations (41) and (42) lead to

$$-c\phi_0 = 2m \operatorname{grad} f^2 - c\langle \phi_0, \xi\rangle\xi - c\langle \phi_0, \phi\rangle\phi. \tag{43}$$

Differentiating (40), one gets

$$-c\,\mathrm{grad}\langle\phi_0,\xi\rangle=(3m^2f^2+3m^2-2ma-b)\,\mathrm{grad}\,f. \tag{44}$$

By considering $\{E_i\}_{i=1}^m$ to be a local orthonormal frame field on U and using $\nabla^\perp\xi=0$, (43), (40) and (41), we have the following

$$\begin{aligned}-c\,\mathrm{grad}\langle\phi_0,\xi\rangle &= -c\sum_{i=1}^m E_i(\langle\phi_0,\xi\rangle)E_i\\ &= -c\sum_{i=1}^m(\langle\nabla^0_{E_i}\phi_0,\xi\rangle+\langle\phi_0,\nabla^0_{E_i}\xi\rangle)E_i\\ &= -c\sum_{i=1}^m\langle\phi_0,\nabla_{E_i}\xi\rangle E_i\\ &= -c\sum_{i=1}^m\langle\phi_0,\nabla^\perp_{E_i}\xi-A(E_i)\rangle E_i\\ &= c\sum_{i=1}^m\langle\phi_0^\top,A(E_i)\rangle E_i\\ &= \sum_{i=1}^m\langle A(c\phi_0^\top),E_i\rangle E_i\\ &= -2mA(\mathrm{grad}\,f^2).\end{aligned} \tag{45}$$

Equations (44) and (45) imply

$$2mA(\mathrm{grad}\,f^2)=(-3m^2f^2-3m^2+2ma+b)\,\mathrm{grad}\,f. \tag{46}$$

Since U is a PNMC biharmonic submanifold, (9)(iii), together with (46), leads to

$$(m^2f^2+3m^2-2ma-b)\,\mathrm{grad}\,f=0,$$

on U. This implies that $\mathrm{grad}\,f=0$, i.e. M is CMC. From Theorem 5.8, we have that M is a 1-type or 2-type submanifold and we get a contradiction. □

Since any hypersurface with nowhere zero mean curvature is PNMC we have the following consequence.

Corollary 5.11. *There does not exist any biharmonic 3-type hypersurface M^m in the unit Euclidean sphere $\mathbb{S}^{m+1}$.*

Proof. Suppose that M is of 3-type. Then M is not minimal in $\mathbb{S}^{m+1}$, and thus f is nowhere zero on an open dense subset $W \subset M$. Every connected component of W is PNMC and, by Theorem 5.10, it cannot be of 3-type. This leads to a contradiction. □

We note that the classes B3 and B4 of proper-biharmonic submanifolds in spheres are linearly independent (and even more, orthogonal) 2-type submanifolds. Thus it is natural to ask whether there exist proper-biharmonic independent higher finite-type submanifolds. We can prove the following result.

Proposition 5.12. *Let $\varphi\colon M \to \mathbb{S}^n$ be a proper-biharmonic submanifold. If M is of finite k-type, mass-symmetric and linearly independent, then $k=2$.*

Proof. Let M be a k-type mass-symmetric submanifold in $\mathbb{S}^n$. Then,

$$\phi = \phi_{t_1} + \phi_{t_2} + \ldots + \phi_{t_k},$$

where ϕ_{t_i} are nonharmonic maps satisfying $\Delta\phi_{t_i} = \lambda_{t_i}\phi_{t_i}$, and λ_{t_i} are mutually distinct, $i=1,\ldots,k$. This implies that

$$\Delta\phi = \sum_{i=1}^{k} \lambda_{t_i}\phi_{t_i} \quad \text{and} \quad \Delta^2\phi = \sum_{i=1}^{k} \lambda_{t_i}^2 \phi_{t_i}. \tag{47}$$

Since M is proper-biharmonic, replacing (47) in (30b), we obtain

$$\sum_{i=1}^{k} (\lambda_{t_i}^2 - 2m\lambda_{t_i} - m^2(f^2-1))\phi_{t_i} = 0.$$

Using that M is independent, we get $(\lambda_{t_i}^2 - 2m\lambda_{t_i} - m^2(f^2-1))\phi_{t_i} = 0$ on M for all $i=1,\ldots,k$. As ϕ_{t_i} is nonzero on an open dense set in M, we have

$$\lambda_{t_i}^2 - 2m\lambda_{t_i} - m^2(f^2-1) = 0 \quad \text{on } M$$

for all $i=1,\ldots,k$. This implies that f=constant. Since ϕ is mass-symmetric, by Theorem 5.8, we conclude that $k=2$. □

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