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An essay on Bergman completeness

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Abstract. We give first of all a new criterion for Bergman completeness in terms of the pluricomplex Green function. Among several applications, we prove in particular that every Stein subvariety in a complex manifold admits a Bergman complete Stein neighborhood basis, which improves a theorem of Siu. Secondly, we give for hyperbolic Riemann surfaces a sufficient condition for when the Bergman and Poincaré metrics are quasi-isometric. A consequence is an equivalent characterization of uniformly perfect planar domains in terms of growth rates of the Bergman kernel and metric. Finally, we provide a noncompact Bergman complete pseudoconvex manifold without nonconstant negative plurisubharmonic functions.

1. Introduction

In one complex variable, the Bergman kernel and the classical Green function are two important conformal invariants which are closely related (e.g., the Bergman–Schiffer formula). An analogue of the Green function in several complex variables is given by the following definition.

Definition. Let X be a complex manifold and $y\in X$. The *pluricomplex Green function* $g_X(x,y)$ of X with logarithmic pole at y is defined as

$$g_X(x,y)=\sup u(x),$$

where the supremum is taken over all negative plurisubharmonic (psh) functions on X such that $u(z)\leq\log|z|+O(1)$ in a coordinate patch at y. We also denote by $\mathrm{PSH}(X)$ the set of psh functions on X (we allow for $-\infty\in\mathrm{PSH}(X)$).

Recently, there has been a lot of activity in studying the Bergman metric using the pluricomplex Green function (cf. [3], [4], [7]–[9], [17], [18] and [25], see also [12]

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for an implicit use of the pluricomplex Green function). A principle achievement in this direction, which was obtained independently by Błocki and Pflug [4] and Herbort [17], says that every bounded hyperconvex domain is Bergman complete (see also Chen [8] for a generalization to hyperconvex manifolds). In this paper, we continue to study the Bergman metric along the same line.

First of all we give a criterion for Bergman completeness as follows.

Theorem 1.1. *If a Stein manifold X possesses the Bergman metric, then it is Bergman complete provided the following condition is satisfied*:

(E) *For any infinite sequence of points $\{y_k\}_{k=1}^{\infty}$ in X without adherent point in X, there are a subsequence $\{y_{k_j}\}_{j=1}^{\infty}$, a number $a>0$ and a continuous volume form dV on X such that for any compact subset K of X, one has*

$$\int_{\{z\in K: g_X(z,y_{k_j})\le -a\}} dV \to 0 \quad \text{as } j\to\infty.$$

As an application, we improve a famous result of Siu [28] as follows (see also Demailly [11]).

Theorem 1.2. *Every Stein subvariety Y in a complex manifold X admits a fundamental family of* Bergman complete *Stein neighborhoods of Y in X.*

It should be mentioned that each totally real submanifold of a complex manifold also admits a fundamental family of Bergman complete Stein neighborhoods (see Proposition 5.2).

Proposition 1.3. *Let X be a Stein manifold satisfying condition* (E). *If X admits a negative continuous strictly psh function, then every Galois covering of X is Bergman complete.*

Proposition 1.4. *Let X be a locally trivial holomorphic fiber bundle. Suppose that the fiber is Bergman complete and that the base is a Stein manifold which admits a negative continuous strictly psh function and satisfies condition* (E). *Then X is Bergman complete.*

Proposition 1.3 is connected with a problem of Ohsawa (private communication) on whether every Galois covering of a Bergman complete manifold is Bergman complete.

Even for open Riemann surfaces, the Bergman completeness is not fully characterized, although there exist some nice works for bounded planar domains (cf. Zwonek [35] and Pflug and Zwonek [24]). As far as we know, the following fundamental problems are still open.

Question 1.5. Is Bergman completeness a quasi-conformal invariant of Riemann surfaces?([1])

Question 1.6. Which are the relationships between the Bergman metric and the Poincaré (hyperbolic) metric of constant negative curvature -1?

Based on a precise estimate of the Green function, we shall show the next result.

Theorem 1.7. *Let X be a hyperbolic Riemann surface with positive injectivity radius and positive isoperimetric constant with respect to the Poincaré metric. Then the Bergman kernel form, the Bergman metric and the Poincaré metric of X are quasi-isometric.*

Recall that two Riemannian metrics g_1 and g_2 are said to be *quasi-isometric* if $\text{const}_1\, g_1 \le g_2 \le \text{const}_2\, g_1$. As an application, we give the following new characterization of uniformly perfect planar domains.

Theorem 1.8. *A hyperbolic domain $\Omega\subset\mathbb{C}$ is uniformly perfect if and only if the Bergman kernel K_Ω and the Bergman metric $b_\Omega|dz|$ enjoy the properties*

$$K_\Omega \ge \frac{\text{const}}{\delta_\Omega^2} \quad \textit{and} \quad b_\Omega \ge \frac{\text{const}}{\delta_\Omega},$$

where δ_Ω denotes the Euclidean boundary distance.

Finally, we give several examples for which the technique of the pluricomplex Green function is not valid. The following result is also known by Tsuji (private communication).

Proposition 1.9. *The universal covering $\widetilde{S}$ of a smooth ample divisor S in an Abelian variety is Bergman complete.*

According to a result of Lyons and Sullivan [23], $\widetilde{S}$ does not admit any non-constant negative harmonic function with respect to the Laplace operator since the fundamental group $\pi_1(S)$ of S is Abelian. In particular, every bounded holomorphic

([1]) Recently, a counterexample was found by Wang [34].

function on $\widetilde{S}$ is constant. Yet it is still unknown whether there exists a nonconstant negative psh function on $\widetilde{S}$. On the other hand, we have the following result.

Theorem 1.10. *There exists a two-dimensional noncompact Bergman complete pseudoconvex manifold which does not possess any nonconstant negative psh function.*

By a pseudoconvex manifold we mean a complex manifold which admits a C^∞ psh exhaustion function.

The paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we prove Theorem 1.1. In Section 4, we prove Propositions 1.3 and 1.4. In Section 5, we prove Theorem 1.2 and present some related results. In Section 6, we prove Theorems 1.7 and 1.8. In Section 7, we prove Proposition 1.9 and Theorem 1.10.

2. Preliminaries

Let X be a complex manifold of dimension n. Following Kobayashi [22], let $\mathcal{H}(X)$ denote the Hilbert space of holomorphic n-forms f on X satisfying $|\int_X f\wedge\bar{f}|<\infty$. The *Bergman kernel form* is defined by

$$K_X(x,y)=\sum_{j=1}^{\infty} h_j(x)\wedge\overline{h_j(y)},$$

where $\{h_j\}_{j=1}^{\infty}$ is a complete orthonormal basis of $\mathcal{H}(X)$. If $K_X(x,x)\neq 0$ for every $x\in X$, then we set

$$ds_X^2=\sum_{\alpha,\beta=1}^{n}\frac{\partial^2\log K_X^*}{\partial z_\alpha\partial\bar{z}_\beta}\,dz_\alpha\,d\bar{z}_\beta,$$

where $K_X(z,z)=K_X^*(z,z)\,dz_1\wedge...\wedge dz_n\wedge d\bar{z}_1\wedge...\wedge d\bar{z}_n$ in local coordinate systems of X. If ds_X^2 is positive definite, then it is called the *Bergman metric* of X.

In his pioneer paper [22], Kobayashi proved the following theorem.

Theorem 2.1. (1) *A complex manifold X possesses the Bergman metric ds_X^2 provided the following two conditions are satisfied*:

(A1) *For every $y\in X$, there exists an n-form $f\in\mathcal{H}(X)$ such that $f(y)\neq 0$.*

(A2) *For every $y\in X$, there are n-forms $f_1,...,f_n$ in $\mathcal{H}(X)$ satisfying $f_\alpha(y)=0$ and $\partial f_\alpha^*/\partial z_\beta(y)=\delta_{\alpha\beta}$ (the Kronecker delta) for $1\le\alpha,\beta\le n$. Here f_α^*, $1\le\alpha\le n$, are local representations of f.*

(2) *ds_X^2 is complete provided furthermore the following condition is satisfied*:

(A3) *There is a dense subset S of $\mathcal{H}(X)$ such that for every $f\in S$ and for any infinite sequence $\{y_k\}_{k=1}^{\infty}$ of points in X without adherent point in X, there is a subsequence $\{y_{k_j}\}_{j=1}^{\infty}$ such that*

$$\frac{f(y_{k_j})\wedge\overline{f(y_{k_j})}}{K_X(y_{k_j},y_{k_j})}\to 0, \quad \textit{as } j\to\infty.$$

Let $g_X(x,y)$ denote the pluricomplex Green function of X and set $A_X(y,-a)=\{x\in X:g_X(x,y)\le -a\}$ for each $a>0$. We also recall a result slightly weaker than Theorem 1.1.

Theorem 2.2. ([9]) (1) *A Stein manifold X possesses the Bergman metric provided the following condition is satisfied*:

(B1) *For any $y\in X$ there is a positive number $a>0$ such that $A_X(y,-a)$ is relatively compact in X.*

(2) *If a Stein manifold X possesses the Bergman metric, then it is Bergman complete provided the following condition is satisfied*:

(B2) *For any infinite sequence $\{y_k\}_{k=1}^{\infty}$ of points in X without adherent point in X there exist a subsequence $\{y_{k_j}\}_{j=1}^{\infty}$ and a number $a>0$ such that for any compact subset K of X one has $A_X(y_{k_j},-a)\subset X\setminus K$ for all sufficiently large j.*

3. Proof of Theorem 1.1

The underlying idea is essentially implicit in [8]. Fix $y\in X$ for a moment. Let $a>0$ be a constant and let $\{X_j\}_{j=1}^{\infty}$ be a sequence of relatively compact strongly pseudoconvex domains in X with $\overline{X}_j\subset X_{j+1}$ and $X=\bigcup_{j=1}^{\infty}X_j$. For each fixed j, we may take a sequence of C^{∞} strictly psh functions $\psi_{j,k}<a/2$ on X_j, $k=1,2,...$, such that $\psi_{j,k}$ decreases to $g_X(\cdot,y)$ as $k\to\infty$. Let $\chi\colon\mathbb{R}\to[0,1]$ be a smooth function such that $\chi|_{(-\infty,-1]}=1$, $\chi|_{[0,\infty)}=0$ and $\sup|\chi'|\le 2$. If $f\in\mathcal{H}(X)$, let

$$\begin{aligned}\eta_{j,k}&=\chi\,(-\log(-\psi_{j,k}+a)+\log 2a)\,f,\\ \eta&=\chi\,(-\log(-g_X(\cdot,y)+a)+\log 2a)\,f,\\ \varphi_{j,k}&=2n\psi_{j,k}-\log(-\psi_{j,k}+a),\\ \varphi&=2ng_X(\cdot,y)-\log(-g_X(\cdot,y)+a).\end{aligned}$$

To proceed with the proof, we need the following L^2-estimate of $\bar\partial$ due to Hörmander [19], Andreotti and Vesentini [1], and Demailly [10].

Theorem 3.1. *Let M be a complete Kähler manifold of dimension n. Let ϕ be a psh function on M such that $i\partial\bar\partial\phi\geq\omega$ in the sense of distributions for a Kähler metric ω on M. Then for every C^∞ $\bar\partial$-closed $(n,1)$-form v on M, there exists a C^∞ $(n,0)$-form on M such that $\bar\partial u=v$ and*

$$\int_M |u|^2_\omega e^{-\phi}\frac{\omega^n}{n!}\leq\int_M |v|^2_\omega e^{-\phi}\frac{\omega^n}{n!}$$

provided that the right-hand side is finite. Here $|\cdot|^2_\omega$ denotes the pointwise length with respect to ω.

Applying Theorem 3.1 with $M=X_j$, $\phi=\varphi_{j,k}$ and $v=\bar\partial\eta_{j,k}$, we get a C^∞ $(n,0)$-form $u_{j,k}$ on X_j such that $\bar\partial u_{j,k}=\bar\partial\eta_{j,k}$ and

$$\left|\int_{X_j} u_{j,k}\wedge\bar u_{j,k}e^{-\varphi_{j,k}}\right|\leq\left|\int_{X_j}|\bar\partial\widetilde\chi|^2_{i\partial\bar\partial\varphi_{j,k}}f\wedge\bar f e^{-\varphi_{j,k}}\right|,$$

where $\widetilde\chi=\chi\circ(-\log(-\psi_{j,k}+a)+\log 2a)$. Now

$$\bar\partial\widetilde\chi=\chi'\bar\partial(-\log(-\psi_{j,k}+a))$$

and

$$i\partial\bar\partial\varphi_{j,k}\geq i\partial(-\log(-\psi_{j,k}+a))\wedge\bar\partial(-\log(-\psi_{j,k}+a)).$$

We get $|\bar\partial\widetilde\chi|^2_{i\partial\bar\partial\varphi_{j,k}}\leq\sup|\chi'|^2$ and

$$\operatorname{supp}\bar\partial\widetilde\chi\subset\{x\in X_j:-(2e-1)a\leq\psi_{j,k}(x)\leq -a\}\subset A_X(y,-a)$$

(notice that $\psi_{j,k}\geq g_X(\,\cdot\,,y)$ on X_j). Thus we obtain

$$\left|\int_{X_j} u_{j,k}\wedge\bar u_{j,k}e^{-\varphi_{j,k}}\right|\leq\text{const}_{n,a}\left|\int_{A_X(y,-a)} f\wedge\bar f\right|.$$

Set $f_{j,k}:=\eta_{j,k}-u_{j,k}$. Then $f_{j,k}\in\mathcal H(X_j)$ and

$$\left|\int_{X_j} f_{j,k}\wedge\bar f_{j,k}\right|\leq\text{const}_{n,a}\left|\int_{A_X(y,-a)} f\wedge\bar f\right|$$

(we use one symbol $\text{const}_{n,a}$ to denote all positive constants depending only on n,a).

Let u_j and $\tilde f_j$ be weak limits of $u_{j,k}$ and $f_{j,k}$ as $k\to\infty$ and let u and $\tilde f$ be weak limits of u_j and f_j as $j\to\infty$. Then we have

$$\tilde f=\eta-u\in\mathcal H(X),\quad \left|\int_X\tilde f\wedge\bar{\tilde f}\right|\leq\text{const}_{n,a}\left|\int_{A_X(y,-a)} f\wedge\bar f\right|$$

and

$$\left|\int_X u\wedge\bar{u}e^{-\varphi}\right|<\infty.$$

Since u is holomorphic in a neighborhood of y (with $u=f-\tilde{f}$ therein) and $g_X(\cdot,y)$ has at least a logarithmic singularity at y, we get $u(y)=0$, i.e., $\tilde{f}(y)=f(y)$.

Now let $\{y_k\}_{k=1}^\infty$ be an infinite sequence of points in X without adherent point in X. Let $f\in\mathcal{H}(X)$. For any $\varepsilon>0$, there is a compact subset K of X such that $|\int_{X\setminus K} f\wedge\bar{f}|<\varepsilon$. By the above argument we have for every k an $(n,0)$-form $f_k\in\mathcal{H}(X)$ such that $f_k(y_k)=f(y_k)$ and

$$\begin{aligned}\left|\int_X f_k\wedge\bar{f}_k\right| &\le \mathrm{const}_{n,a}\left|\int_{A_X(y_k,-a)} f\wedge\bar{f}\right| \\ &\le \mathrm{const}_{n,a}\left(\left|\int_{X\setminus K} f\wedge\bar{f}\right|+\left|\int_{K\cap A_X(y_k,-a)} f\wedge\bar{f}\right|\right) \\ &\le \mathrm{const}_{n,a}\left(\varepsilon+\sup_K(-1)^{n^2/2}(f\wedge\bar{f})\otimes(dV)^{-1}\int_{K\cap A_X(y_k,-a)} dV\right).\end{aligned}$$

Take $\{y_{k_j}\}_{j=1}^\infty$ and a as in condition (E). Then we have $|\int_X f_{k_j}\wedge\bar{f}_{k_j}|\le\mathrm{const}_{n,a}\,\varepsilon$ provided j is large enough. By the extreme property of the Bergman kernel, we obtain

$$\frac{f(y_{k_j})\wedge\overline{f(y_{k_j})}}{K_X(y_{k_j},y_{k_j})}\le\left|\int_X f_{k_j}\wedge\bar{f}_{k_j}\right|\le\mathrm{const}_{n,a}\,\varepsilon,$$

or in other words, Kobayashi's criterion holds.

4. Proofs of Propositions 1.3 and 1.4

Proof of Proposition 1.3. Let $\pi\colon\widetilde{X}\to X$ be a Galois covering. By Stein's theorem [29], $\widetilde{X}$ is also Stein. Let $\psi<0$ be a continuous strictly psh function on X. By Richberg's theorem [27], we may assume that ψ is C^∞. Let $\tilde{\psi}$ be the lift of ψ to $\widetilde{X}$. Since $\tilde{\psi}<0$ is strictly psh on $\widetilde{X}$, it is easy to verify that condition (B1) holds. In particular, $\widetilde{X}$ possesses the Bergman metric. It suffices to verify that $\widetilde{X}$ satisfies condition (E). Let $\{y_k\}_{k=1}^\infty\subset\widetilde{X}$ be an infinite sequence of points without adherent point in $\widetilde{X}$. Let $x_k=\pi(y_k)$. We divide into the following two cases.

Case 1. $\{x_k\}_{k=1}^\infty$ *has an adherent point* x_0 *in* X. Then $\widetilde{X}$ has to be an infinite covering of X because $\{y_k\}_{k=1}^\infty$ has no adherent point in $\widetilde{X}$. Thus we may take a small coordinate ball $B(x_0,r_0)$ at x_0 such that $\pi^{-1}(B(x_0,r_0))$ is an infinite union of mutually disjoint coordinate patches $\widetilde{U}_k$ in $\widetilde{X}$. Since $\{y_k\}_{k=1}^\infty$ has no adherent point in $\widetilde{X}$, we may choose a subsequence $\{y_{k_j}\}_{j=1}^{\infty}$ such that $y_{k_j}\in\widetilde{U}_{k_j}\cap\pi^{-1}(B(x_0,r_0/4))$. As $\widetilde{U}_{k_j}$ is biholomorphic to $B(x_0,r_0)$,

$$i\partial\bar\partial\tilde\psi(z)\geq \mathrm{const}\, i\partial\bar\partial|z|^2,\quad z\in\widetilde{U}_{k_j},\ j\geq 1.$$

Thus there is a sufficiently large constant $C>0$ such that

$$C\max\Big\{\tilde\psi,\min_{B(x_0,r_0)}\psi\Big\}+\varkappa_j\log\frac{|z|}{2r_0}$$

is psh on $\widetilde{X}$ for any j, where $\varkappa_j\geq 0$ is a smooth function on $\widetilde{X}$ such that $\varkappa_j=1$ on $\{z:|z|\leq r_0/2\}$ and $\varkappa_j=0$ on $\widetilde{X}\setminus\widetilde{U}_{k_j}$. Using this function as a candidate for the extreme property of $g_{\widetilde{X}}(\,\cdot\,,y_{k_j})$, we get

$$A_{\widetilde{X}}(y_{k_j},-a)\subset\widetilde{U}_{k_j},\quad \text{where } a=-C\min_{B(x_0,r_0)}\psi+1.$$

Thus for any compact subset K of $\widetilde{X}$ we have $K\cap A_{\widetilde{X}}(y_{k_j},-a)=\varnothing$ provided j is large enough.

Case 2. *$\{x_k\}_{k=1}^{\infty}$ has no adherent point in X.* Since X satisfies condition (E), there exist a subsequence $\{x_{k_j}\}_{j=1}^{\infty}$, a positive number a and a continuous volume form dV on X such that for any compact subset K' of X,

$$\int_{K'\cap A_X(x_{k_j},-a)}dV\to 0,\quad j\to\infty.$$

Let K be any compact subset of $\widetilde{X}$. It is easy to see from the definition that

$$g_{\widetilde{X}}(x,y_{k_j})\geq g_X(\pi(x),\pi(y_{k_j}))=g_X(\pi(x),x_{k_j}).$$

Thus $\pi(A_{\widetilde{X}}(y_{k_j},-a))\subset A_X(x_{k_j},-a)$. Since $K'=\pi(K)$ is a compact subset of X and the sheet of the covering $K\to K'$ is finite, we conclude that

$$\int_{K\cap A_{\widetilde{X}}(y_{k_j},-a)}d\widetilde{V}\to 0,\quad j\to\infty,$$

where $d\widetilde{V}$ is the lift of dV to $\widetilde{X}$. $\square$

Proof of Proposition 1.4. Let $\pi:X\to M$ be a locally trivial holomorphic fiber bundle over a complex manifold M with typical fiber Y such that Y is Bergman complete and M is a Stein manifold which admits a negative C^{∞} strictly psh function ψ. Let $\{(U_\alpha,\theta_\alpha)\}_\alpha$ be a local trivialization of X. For the sake of simplicity, we identify $\pi^{-1}(U_\alpha)$ with $U_\alpha\times Y$ and denote the pullbacks of variants on M to X by the same characters.

Localization Principle. *X possesses the Bergman metric such that for each pair of domains $V \Subset U \Subset M$ there exists a constant $C>0$ so that*

$$C\, ds^2_{\pi^{-1}(U)} \geq ds^2_X \geq C^{-1}\, ds^2_{\pi^{-1}(U)}$$

on $\pi^{-1}(V)$.

Proof. Fix a complete Kähler metric ω_M on M. Since the Bergman metric ds^2_Y of Y is invariant under holomorphic automorphisms, we may define a complete Kähler metric on X simply by $\omega_M \oplus ds^2_Y$. Take a finite number of coordinate neighborhoods $U_{\alpha_1}, ..., U_{\alpha_m}$ which cover $\overline{U}$ and let $V_{\alpha_j} \Subset U_{\alpha_j} \cap U$ be a cover of $\overline{V}$. Let $p \in V$. Without loss of generality, we assume that $p \in V_1$. Let $\widetilde{\chi}$ be a real smooth function with compact support in $U_1 \cap U$ which is equal to 1 in a neighborhood of $\overline{V}_1$. Let z denote the local holomorphic coordinate on U_1. Then there is a constant $C>0$ such that

$$i\partial\bar{\partial}(C\psi + 2(n+1)\widetilde{\chi} \log|z-z(p)|) \geq \omega_M \quad \text{on } U,$$

where n is the dimension of M. Let $f \in \mathcal{H}(\pi^{-1}(U))$. By Theorem 3.1, we can solve the equation $\bar{\partial}u = \bar{\partial}\widetilde{\chi} \wedge f$ on X such that

$$\begin{aligned}\left|\int_X u \wedge \bar{u} e^{-C\psi-2(n+1)\widetilde{\chi}\log|z-z(p)|}\right| &\leq \left|\int_X |\bar{\partial}\widetilde{\chi}|^2_{\omega_M} f \wedge \bar{f} e^{-C\psi-2(n+1)\widetilde{\chi}\log|z-z(p)|}\right| \\ &\leq \text{const}\left|\int_{\pi^{-1}(U)} f \wedge \bar{f}\right|.\end{aligned}$$

Let $\tilde{f} = \widetilde{\chi} f - u$. It is easy to see that $\tilde{f} \in \mathcal{H}(X)$ is such that $\tilde{f} = f$, $\nabla_{\partial/\partial z}\tilde{f} = \nabla_{\partial/\partial z} f$ on $\{p\} \times Y$, and $\|\tilde{f}\|_{L^2(X)} \leq \text{const}\, \|f\|_{L^2(\pi^{-1}(U))}$. The assertion follows immediately from this fact and the extreme properties of the Bergman kernel and the Bergman metric. □

We proceed with the proof of Proposition 1.4. Let $\{y_k\}_{k=1}^\infty$ be a sequence of points without adherent point in X. If $\{\pi(y_k)\}_{k=1}^\infty$ is contained in a compact subset of M, it follows from the localization principle that $\{y_k\}_{k=1}^\infty$ is also a discrete sequence with respect to the Bergman distance. Thus we may assume that $\{\pi(y_k)\}_{k=1}^\infty$ contains a discrete subsequence $\{p_j := \pi(y_{k_j})\}_{j=1}^\infty$. For simplicity, we still denote by $\{p_j\}_{j=1}^\infty$ the subsequence appearing in condition (E) which holds on M. Let $f \in \mathcal{H}(X)$. Replacing $g_X(\cdot, y_k)$ by $g_M(\cdot, p_j)$ in the argument in the proof of Theorem 1.1, we get for every j a form $f_j \in \mathcal{H}(X)$ such that $f_j = f$ on $\{p_j\} \times Y$ and

$|\int_X f_j\wedge\bar{f}_j|\le \mathrm{const}_{n,a}\,|\int_{\pi^{-1}(A_M(p_j,-a))} f\wedge\bar{f}|$. For any $\varepsilon>0$, there is a compact subset K of X such that $|\int_{X\setminus K} f\wedge\bar{f}|<\varepsilon$. Then we have

$$\begin{aligned}\left|\int_X f_j\wedge\bar{f}_j\right| &\le \mathrm{const}_{n,a}\left(\left|\int_{X\setminus K} f\wedge\bar{f}\right|+\left|\int_{K\cap\pi^{-1}(A_M(p_j,-a))} f\wedge\bar{f}\right|\right)\\ &\le \mathrm{const}_{n,a}\left(\varepsilon+\sup_K(-1)^{n^2/2}(f\wedge\bar{f})\otimes(dV)^{-1}\int_{K\cap\pi^{-1}(A_M(p_j,-a))} dV\right)\\ &\le \mathrm{const}_{n,a}\,\varepsilon\end{aligned}$$

provided j is large enough. Thus

$$\frac{f(y_{k_j})\wedge\overline{f(y_{k_j})}}{K_X(y_{k_j},y_{k_j})}\le\left|\int_X f_j\wedge\bar{f}_j\right|\le \mathrm{const}_{n,a}\,\varepsilon,$$

in other words, Kobayashi's criterion holds. □

Remark. It becomes more difficult when the quotient manifold X in Proposition 1.3 or the base manifold M in Proposition 1.4 is compact.

For Riemann surfaces we have the following result.

Proposition 4.1. *Let X be a hyperbolic Riemann surface, i.e., g_X is not identically equal to $-\infty$. Suppose that X can be exhausted by a sequence of relatively compact domains $\{X_j\}_{j=1}^{\infty}$ with $\overline{X}_j\subset X_{j+1}$ such that*

$$b:=\inf_j \liminf_{x\to\partial X}\inf_{y\in\partial X_j} g_X(x,y)>-\infty.$$

Then X is Bergman complete. Here ∂X denotes the ideal boundary of X.

Proof. Notice that in the case of the Riemann surface the fact that g_X is not identically $-\infty$ implies that $g_X(x,y)>-\infty$ for all $x\neq y$. Thus by Theorem 2.2, every hyperbolic Riemann surface possesses the Bergman metric. Let K be any compact subset of X. We take j_0 such that $K\subset X_{j_0}$. Thus

$$\inf_{y\in\partial X_{j_0}} g_X(x,y)>b-1,\quad \text{as } x\to\partial X.$$

Since $g_X(x,y)=g_X(y,x)$, we have

$$\inf_{x\in\partial X_{j_0}} g_X(x,y)>b-1,\quad \text{as } y\to\partial X.$$

By the maximum principle for harmonic functions, we get

$$\inf_{x\in X_{j_0}} g_X(x,y) > b-1, \quad \text{as } y\to\partial X.$$

In other words, $K\cap A_X(y,b-1)=\varnothing$ for y sufficiently close to ∂X. By Theorem 2.2 (or Theorem 1.1), the assertion follows. □

Remark. Unfortunately, the author does not know how to construct a non-hyperconvex Riemann surface satisfying the assumption of the above proposition.

5. Bergman completeness of Stein neighborhoods of a submanifold

Proof of Theorem 1.2. Demailly's simplified proof of Siu's theorem contains the following two main ingredients (cf. [11], pp. 411–413):

(1) *For every C^∞ strictly psh function ψ on Y and any continuous function $\delta>0$ on X, there exists a C^∞ strictly psh function on a neighborhood V of Y such that $\psi\le\varphi|_Y\le\psi+\delta$ on Y.*

(2) *There exists an almost psh function ρ on X such that $\rho|_Y=-\infty$ with logarithmic poles and $\rho\in C^\infty(X\backslash Y)$.*

By an almost psh function on a complex manifold we mean a function that is locally equal to the sum of a psh function and a smooth function.

Fix a C^∞ strictly psh exhaustion function $\psi>0$ on Y. Applying (1) with $\delta=1$, we get a C^∞ strictly psh function $\varphi>0$ on a neighborhood W_0 of Y such that $\varphi|_Y$ is an exhaustion function. Let W_1 be a neighborhood of Y such that $\overline{W}_1\subset W_0$ and $\varphi|_{\overline{W}_1}$ be an exhaustion function, i.e., $\{x\in\overline{W}_1:\varphi(x)\le C\}$ is a compact subset of X for each $C>0$. We may choose W_1 as follows: Let

$$K_j:=\{x\in Y: j-1\le\varphi(x)\le j+1\},\quad j=1,2,\dots.$$

The we have $Y=\bigcup_{j=1}^\infty K_j$. Since K_j is a compact set in Y, there is a neighborhood $U_j\Subset W_0$ of K_j such that

$$j-2\le\varphi(x)\le j+2 \quad \text{for all } x\in\overline{U}_j.$$

It suffices to take $W_1=\bigcup_{j=1}^\infty U_j$.

Next we are going to show that every neighborhood $W\subset W_1$ of Y contains a Bergman complete Stein neighborhood V of Y. Let ρ be the function in (2). Shrinking W if necessary, we may assume that $\rho<-1$ on $\overline{W}$. We set

$$\tilde\rho=-\log(-\rho)+\lambda\circ\varphi \quad \text{on } \overline{W},$$

where $\lambda\colon \mathbb{R}\to[0,\infty)$ is a smooth convex increasing function. For every $y\in\overline{W}$, there is a local coordinate ball $B(y,r_y)$, at y with radius $0<r_y\le 1$, such that r_y^{-1} is locally bounded as a function of $y\in\overline{W}$. If λ grows fast enough, we get

(i) $\tilde\rho>0$ on ∂W;

(ii) for every $y\in\overline{W}$, $i\partial\bar\partial\tilde\rho\ge(C+\varphi(y))/r_y^2 i\partial\bar\partial|z|^2$ on $B(y,r_y)$, where $C>0$ is a universal constant to be determined later;

(iii)

$$\int_{B(y,r_y)} e^{-\tilde\rho}\,dV_z=\int_{B(y,r_y)}(-\rho)e^{-\lambda\circ\varphi}\,dV_z\le 1.$$

(Notice that ρ has only logarithmic pole at Y, and thus $-\rho$ is locally integrable on $\overline{W}$.)

By (i), the open set $V=\{x\in W:\tilde\rho(x)<0\}$ is a Stein neighborhood of Y such that $\bar V\subset W$. The rest of the proof is divided into two steps:

Step 1. *Existence of the Bergman metric.* For any $y\in V$, we set

$$\eta_0=\varkappa_y(z)\,dz_1\wedge\ldots\wedge dz_n,\quad \eta_j=\frac{z_j}{r_y}\varkappa_y(z)\,dz_1\wedge\ldots\wedge dz_n,\ 1\le j\le n,$$

and

$$\phi=\tilde\rho+(n+1)\varkappa_y(z)\log\frac{|z|^2}{r_y^2},$$

where $\varkappa_y\ge 0$ is a smooth function on V such that $\varkappa_y=1$ on $\{z:|z|\le r_y/2\}$ and $\varkappa_y=0$ on $V\setminus\{z:|z|\le r_y\}$. Furthermore,

$$i\partial\bar\partial\left(\varkappa_y(z)\log\frac{|z|^2}{r_y^2}\right)\ge-\frac{C'}{r_y^2}i\partial\bar\partial|z|^2$$

for a suitable universal constant $C'>0$. If $C=(n+1)C'+1$ in (ii), then we have $i\partial\bar\partial\phi\ge i/r_y^2\partial\bar\partial|z|^2$ on $B(y,r_y)$. By Theorem 3.1, there is, for every $0\le j\le n$, a C^∞ $(n,0)$-form u_j on V such that $\bar\partial u_j=\bar\partial\eta_j$ and (by (iii))

$$\begin{aligned}\left|\int_V u_j\wedge\bar u_j e^{-\phi}\right| &\le\int_{B(y,r_y)}|\bar\partial\varkappa_y|^2_{i/r_y^2\partial\bar\partial|z|^2}e^{-\phi}\,dV_z\\ &\le \mathrm{const}_n\int_{B(y,r_y)}e^{-\tilde\rho}\,dV_z\le\mathrm{const}_n\,.\end{aligned}$$

Let $f_j=\eta_j-u_j$. Then $f_j\in\mathcal{H}(V)$, $f_0(y)=1$, $f_j(y)=0$, $1\le j\le n$, and $\partial f_j^*/\partial z_k(y)=\delta_{jk}/r_y$. By Theorem 2.1, we conclude that the Bergman metric exists on V.

Step 2. *Completeness of the Bergman metric.* Let $\{y_k\}_{k=1}^\infty$ be an infinite sequence of points in V without adherent point in V. If $\{y_k\}_{k=1}^\infty$ has an adherent point on ∂V, then a standard argument shows that Kobayashi's criterion holds because each point on ∂V is a strongly pseudoconvex point. Thus we may assume that $\varphi(y_k)\to\infty$ as $k\to\infty$.

Let K be any compact subset of V and let dV be any continuous volume form of V. For any $\tau>0$, there exists a positive integer k_0 such that for any $k>k_0$ we have $B(y_k, r_{y_k})\subset V\setminus K$ and

$$u_{\tau,k} := \tau\tilde{\rho} + \varkappa_{y_k}(z)\log\frac{|z|}{r_{y_k}}$$

is a negative psh function on V, according to (ii) (notice that $\varphi(y_k)\to\infty$). Since $u_{\tau,k}$ has an at least logarithmic singularity at y_k, we have

$$g_X(x, y_k) \geq u_{\tau,k}(x) = \tau\tilde{\rho} \quad \text{for } x\in K.$$

Thus

$$K\cap A_V(y_k, -a) \subset \left\{x\in K : \tilde{\rho}(x) < -\frac{a}{\tau}\right\},$$

which implies that

$$\int_{K\cap A_V(y_k,-a)} dV < \varepsilon, \quad \text{if } \tau \ll 1.$$

By Theorem 1.1, the proof is finished. □

It is interesting to point out that the role of $A_X(y,-a)$ is almost optimal for Bergman completeness (cf. Pflug and Zwonek [25]). Here is a simple example.

Proposition 5.1. *Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$ and let $\varphi>0$ be a continuous psh function on D satisfying*

$$\liminf_{z\to\partial D}\frac{\varphi(z)}{\log(1/\delta_D(z))} = \infty.$$

Then the Hartogs domain $\Omega := \{(z,w)\in D\times\mathbb{C} : |w| < e^{-\varphi(z)}\}$ is Bergman complete.

It is easy to verify that the above condition is *sharp*, e.g., let D be a punctured disc and $\varphi(z)$ be psh on D satisfying $\varphi(z)\sim N\log(1/|z|)$ as $z\to 0$, where N is a positive integer, then Ω is not Bergman complete.

Proof. The underlying idea is essentially due to Błocki. Let $\{y_k=(z_k,w_k)\}_{k=1}^{\infty}$ be an infinite sequence of points in Ω without adherent point in Ω. Since Ω is locally hyperconvex at each $p\in\partial\Omega\setminus(\partial D\times\{0\})$ (that is, there is a ball $B(p,r_p)$ such that $\Omega\cap B(p,r_p)$ is hyperconvex), we may assume that $y_k\to\partial D\times\{0\}$, in particular, $\varphi(z_k)\to\infty$ as $k\to\infty$. Let $R=\operatorname{diam} D$. If $x=(z,w)$, then

$$g_\Omega(x,y_k)\geq\log\frac{|z-z_k|}{R}\geq\log\frac{\delta_D(z_k)}{R},\quad \text{if } \delta_D(z)\geq 2\delta_D(z_k).$$

Let

$$N_k:=\inf\{\varphi(z):\delta_D(z)=2\delta_D(z_k)\}.$$

Then $N_k/\log(1/2\delta_D(z_k))\to\infty$, as $k\to\infty$, and $\log|w|<-\varphi(z)\leq -N_k$ for $(z,w)\in\Omega$ with $\delta_D(z)=2\delta_D(z_k)$. Thus

$$g_\Omega(x,y_k)\geq\log\frac{\delta_D(z_k)}{R}\frac{\log|w|}{-N_k},\quad \text{if } \delta_D(z)=2\delta_D(z_k).$$

On the other hand, since $g_\Omega(x,y_k)=0$ for any $x\in\partial\Omega$ with $\delta_D(z)\geq 2\delta_D(z_k)$ (notice that Ω is locally hyperconvex at x), the same inequality holds on this part of boundary. As the pluricomplex Green function $g_\Omega(\,\cdot\,,y_k)$ is maximal on $\Omega\backslash\{y_k\}$ (cf. [21]), we conclude that the above inequality holds for all $x\in\Omega$ with $\delta_D(z)\geq 2\delta_D(z_k)$. Thus

$$\{x\in\Omega:g_\Omega(x,y_k)\leq -1\}\subset\{x\in\Omega:\delta_D(z)<2\delta_D(z_k)\}\cup\left\{x\in\Omega:\delta_D(z)\geq 2\delta_D(z_k),\ \log|w|\leq -\frac{N_k}{\log(R/\delta_D(z_k))}\right\}.$$

By Theorem 1.1, the assertion follows. □

In contrast to complex submanifolds, totally real submanifolds are of independent interest. A C^1-differentiable submanifold Y of a complex manifold X is called *totally real* if the tangent space to Y at any point does not contain any complex lines. According to Theorem 2.2 of Harvey and Wells [16], Y admits a fundamental neighborhood system of Stein manifolds in X. Indeed, their proof even implies that these Stein manifolds are hyperconvex. Thus by [8], we obtain the following result.

Proposition 5.2. *Every totally real submanifold of a complex manifold admits a fundamental neighborhood system of Bergman complete manifolds.*

6. Comparison of the Bergman and Poincaré metrics

Due to a result of Myrberg, every hyperbolic Riemann surface may be written as Δ/Γ, where Δ is the unit disc and Γ is a (torsion-free) Fuchsian group satisfying $\sum_{\gamma\in\Gamma}(1-|\gamma(0)|)<\infty$ (cf. Tsuji [32]). There are several important isometric invariants related to the Poincaré metric listed as follows.

(1) The *critical exponent* of Poincaré series defined as

$$\delta(\Gamma)=\inf\left\{s\geq 0:\sum_{\gamma\in\Gamma}(1-|\gamma(0)|)^s<\infty\right\}.$$

(2) The *injectivity radius* of Δ/Γ defined as

$$r(\Gamma)=\frac{1}{2}\inf_{z\in\Delta} r_z, \quad \text{with } r_z=\inf_{\gamma\in\Gamma\setminus\{1\}}\rho(z,\gamma z),$$

where ρ denotes the Poincaré distance and r_z is called the *injectivity radius* at z.

(3) The *bottom* of the spectrum with respect to the Laplace–Beltrami operator defined as

$$\lambda(\Gamma)=\inf\left\{\frac{\int_{\Delta/\Gamma}|\operatorname{grad}\phi|^2\,dV}{\int_{\Delta/\Gamma}|\phi|^2\,dV}:\phi\in C_0^\infty(\Delta/\Gamma)\right\}.$$

(4) The *isoperimetric constant* defined as

$$h(\Gamma)=\inf\frac{\operatorname{vol}\partial\Omega}{\operatorname{vol}\Omega},$$

where Ω runs over all relatively compact open sets in Δ/Γ with smooth boundaries.

There are also some well-known relationships.

(i) Elstrodt, Patterson and Sullivan [31]:

$$\lambda(\Gamma)=\begin{cases}\frac{1}{4}, & \text{if } 0\leq\delta(\Gamma)\leq\frac{1}{2},\\ \delta(\Gamma)(1-\delta(\Gamma)), & \text{if } \frac{1}{2}\leq\delta(\Gamma)\leq 1.\end{cases}$$

It is also known that $\delta(\Gamma)\leq 1$ always holds (cf. Tsuji [32]).

(ii) Cheeger [6] and Buser [5]: $\lambda(\Gamma)>0$ if and only if $h(\Gamma)>0$.

The above two facts imply the following result:

(iii) $h(\Gamma)>0$ if and only if $\delta(\Gamma)<1$.

Proof of Theorem 1.7. Write $X=\Delta/\Gamma$, where Γ is a Fuchsian group. Let Ω be an open set with smooth boundary in Δ/Γ. For a relatively compact open set $U\subset\Omega$, we define its capacity by

$$\operatorname{Cap}(U,\Omega)=\inf\int_\Omega|\operatorname{grad}\phi|^2\,dV,$$

where ϕ runs over all locally Lipschitz functions with compact support in Ω such that $0\leq\phi\leq 1$ and $\phi|_{\overline{U}}=1$. The relationship between the Green function g_Ω and the capacity is as follows:

$$a:=\inf_{x\in\partial U}(-g_\Omega(x,y))\leq\frac{1}{\operatorname{Cap}(U,\Omega)}\leq\sup_{x\in\partial U}(-g_\Omega(x,y))=:b \tag{1}$$

and

$$\{x\in\Omega:-g_\Omega(x,y)\geq a\}\supset\overline{U}\supset\{x\in\Omega:-g_\Omega(x,y)\geq b\} \tag{2}$$

(cf. Grigor′yan [15], p. 18). As $h(\Gamma)>0$, by Theorem 8.1 of Grigor′yan [15] we have

$$\frac{1}{\operatorname{Cap}(U,\Omega)}\leq\int_{\operatorname{vol}U}^{\operatorname{vol}\Omega}\frac{dv}{h(\Gamma)^2v^2}=\frac{1}{h(\Gamma)^2}\left(\frac{1}{\operatorname{vol}U}-\frac{1}{\operatorname{vol}\Omega}\right).$$

Now fix any $y\in\Delta/\Gamma$. We take U to be the geodesic ball $B(y,r(\Gamma))$. Let $\{\Omega_j\}_{j=1}^\infty$ be an exhaustion sequence of smooth relatively compact open sets in Δ/Γ. Since $\operatorname{vol}U=4\pi\sinh^2\frac{1}{2}r(\Gamma)$, we conclude that $\operatorname{Cap}(U,\Omega_j)^{-1}\leq\operatorname{const}(\Gamma)$ provided j is sufficiently large. Let $\varpi\colon\Delta\to\Delta/\Gamma$ be the natural projection. By a Möbius transformation, we may assume that $\varpi(0)=y$. As $\rho(0,z)=\log((1+|z|)/(1-|z|))$, we may identify U with the coordinate disc $\Delta(\Gamma):=\{z\in\mathbb{C}:|z|<\tanh\frac{1}{2}r(\Gamma)\}$ such that the Poincaré metric is uniformly quasi-isometric to the Euclidean metric on U. By Harnack's inequality, we have $a\geq\operatorname{const}(\Gamma)b$. Thus by (1) and (2), there is a constant $c=c(\Gamma)>0$ such that

$$A_{\Omega_j}(y,-c)=\{x\in\Omega_j:-g_{\Omega_j}(x,y)\geq c\}\subset\overline{U}.$$

Letting $j\to\infty$, we get the crucial estimate

$$A_{\Delta/\Gamma}(y,-c)\subset\overline{U}. \tag{3}$$

The argument in the proof of Theorem 1.1 shows that for every holomorphic 1-form f on $A_{\Delta/\Gamma}(y,-c)$ there is a holomorphic 1-form F on Δ/Γ such that $F(y)=f(y)$ and $\|F\|\leq\operatorname{const}(\Gamma)\|f\|_{L^2(A_{\Delta/\Gamma}(y,-c))}$. In particular, for $f=dz$ there exists a holomorphic 1-form F on Δ/Γ such that $F(y)=dz$ and by (3),

$$\|F\|\leq\operatorname{const}(\Gamma)\left|\int_U dz\wedge d\bar z\right|\leq\operatorname{const}(\Gamma).$$

Write $K_{\Delta/\Gamma}=K^*_{\Delta/\Gamma}\,dz\wedge d\bar z$ and $F=F^*\,dz\wedge d\bar z$ in local coordinates at y. Then

$$K^*_{\Delta/\Gamma}(y)\geq\frac{|F^*(y)|^2}{\|F\|^2}\geq\operatorname{const}(\Gamma).$$

On the other hand, Cauchy's integral implies

$$|f^*(y)|^2\leq\frac{1}{\operatorname{vol}(\Delta(\Gamma))}\left|\int_{\Delta(\Gamma)}f\wedge\bar f\right|\leq\operatorname{const}(\Gamma)\|f\|^2\quad\text{for any } f\in\mathcal{H}(\Delta/\Gamma),$$

and hence

$$K^*_{\Delta/\Gamma}(y)\leq\operatorname{const}(\Gamma).$$

Since y is arbitrarily chosen, we conclude that the Bergman kernel form is quasi-isometric to the Poincaré metric. The case of the Bergman metric is similar. □

Conjecture. The hypothesis $\lambda(\Gamma)>0$ in Theorem 1.7 is unnecessary for hyperbolic Riemann surfaces.

Definition. (Cf. [2] and [26]) A hyperbolic domain $\Omega\subset\mathbb{C}$ is said to be *uniformly perfect* if there exists a constant $c>0$ such that for any boundary point $p\in\partial\Omega$ and $0<r<\operatorname{diam}\partial\Omega$ there is a point $q\in\partial\Omega$ such that $cr\le|q-p|\le r$.

It turns out that the uniform perfectness of Ω is equivalent to each of the following conditions (cf. Sugawa [30]):

(α) The Poincaré metric $\rho_\Omega|dz|$ satisfies $\rho_\Omega\ge\text{const}\,\delta_\Omega^{-1}$.

(β) The injectivity radius of Ω with respect to the Poincaré metric is positive.

(γ) The logarithmic capacity

$$\operatorname{LogCap}(\partial\Omega\cap\Delta(p,r))\ge\text{const}\,r\quad\text{for all } p\in\partial\Omega \text{ and } 0<r<\operatorname{diam}\Omega.$$

We also mention the following result of Fernández (cf. [13], see also [30]).

(δ) Uniform perfectness of Ω implies that the bottom $\lambda(\Omega)$ of the spectrum of the Laplace–Beltrami operator with respect to the Poincaré metric is positive.

Proof of Theorem 1.8. The only if part follows directly from (ii), (α), (β), (δ) and Theorem 1.7. It suffices to verify the if part. For simplicity, we denote the Bergman kernel function of Ω by K_Ω. Fix any point $z_0\in\Omega$ and set $\delta_0=\delta_\Omega(z_0)$. Since

$$K_\Omega(z_0)\le K_{\Delta(z_0,\delta_0)}(z_0)\le\text{const}\,\delta_0^{-2},$$

we have $K_\Omega(z_0)\asymp\delta_0^{-2}$. Viewing $K_\Omega(z,\bar w)$ as a holomorphic function on $\Delta(z_0,\delta_0/2)\times\Delta(z_0,\delta_0/2)$, we infer from Cauchy's integral that

$$\begin{aligned}\left|\frac{\partial}{\partial z}K_\Omega(z_0)\right| &\le 2\delta_0^{-1}\max\bigl\{|K_\Omega(z_0+z,z_0+\bar w)|: z,w\in\Delta\bigl(0,\tfrac12\delta_0\bigr)\bigr\}\\ &\le 2\delta_0^{-1}\max\bigl\{|K_\Omega(z_0+z)|: z\in\Delta\bigl(0,\tfrac12\delta_0\bigr)\bigr\}\le\text{const}\,\delta_0^{-3},\end{aligned}$$

where the second inequality follows from $|K_\Omega(z,\bar w)|^2\le K_\Omega(z)K_\Omega(w)$. Thus

$$\left|\frac{\partial\log K_\Omega}{\partial z}\right|\le\text{const}\,\delta_\Omega^{-1}.$$

On the other hand, we know that the Bergman metric $b_\Omega|dz|$ satisfies $b_\Omega\ge\text{const}\,\delta_\Omega^{-1}$. Thus there is a constant $\varepsilon_0>0$ such that

$$i\partial\bar\partial\log K_\Omega\ge\varepsilon_0 i\partial\log K_\Omega\wedge\bar\partial\log K_\Omega.$$

Let $\psi=-e^{-\varepsilon_0/2\log K_\Omega}$. Notice that

$$i\partial\bar\partial\psi = ie^{-\varepsilon_0/2\log K_\Omega}\left(\frac{\varepsilon_0}{2}\partial\bar\partial\log K_\Omega-\frac{\varepsilon_0^2}{4}\partial\log K_\Omega\wedge\bar\partial\log K_\Omega\right)$$
$$\geq \frac{\varepsilon_0}{4}e^{-\varepsilon_0/2\log K_\Omega}i\partial\bar\partial\log K_\Omega\geq\frac{\text{const}}{\delta_\Omega^{2-\varepsilon_0}}i\,dz\wedge d\bar z$$

because $K_\Omega\asymp\delta_\Omega^{-2}$. Thus there is a constant $C>0$ such that

$$\varphi=C\delta_0^{-\varepsilon_0}\psi+\chi\left(\frac{2|z-z_0|}{\delta_0}\right)\log\frac{|z-z_0|}{\delta_0}$$

becomes a negative subharmonic function on Ω with a logarithmic pole at z_0, where $\chi\colon\mathbb{R}\to[0,1]$ is a C^∞ function such that $\chi|_{(1,\infty)}=0$ and $\chi|_{(-\infty,1/2)}=1$. It follows that the Azukawa metric

$$a_\Omega(z_0,\zeta):=\lim_{|t|\to0+}\frac{1}{|t|}e^{g_\Omega(z_0,z_0+t\zeta)}\geq\lim_{|t|\to0+}\frac{1}{|t|}e^{\varphi(z_0+t\zeta)}$$
$$=\exp(-C\delta_0^{-\varepsilon_0}K_\Omega(z_0)^{-\varepsilon_0/2})\frac{|\zeta|}{\delta_0}\geq\text{const}\,\frac{|\zeta|}{\delta_0},\quad\zeta\in\mathbb{C},$$

because $K_\Omega(z_0)\asymp\delta_0^{-2}$. Since the Poincaré metric coincides with the Kobayashi–Royden metric on Ω, which is no less than a_Ω (cf. Jarnicki and Pflug [20]), the assertion follows immediately from (α). □

Remark. In fact, one can also show that $b_\Omega\leq\text{const}\,\delta_\Omega^{-1}$ for uniformly perfect domains since Theorem 1.7 implies that $b_\Omega|dz|$ is quasi-isometric to the Poincaré metric $\rho_\Omega|dz|$ while $\rho_\Omega\leq\text{const}\,\delta_\Omega^{-1}$ always holds.

7. Bergman completeness of complex submanifolds

Proof of Proposition 1.9. Consider an Abelian variety A, whose universal covering is $\mathbb{C}^{n+1}$ with $n\geq2$. Let S be a smooth ample divisor of A. According to the Lefschetz hyperplane theorem (cf. [14], p. 156), we have $\pi_1(A)\cong\pi_1(S)$ between fundamental groups of A and S. In particular, the closed submanifold $\widetilde S=\pi^{-1}(S)$ of $\mathbb{C}^{n+1}$ is the universal covering of S, where $\pi\colon\mathbb{C}^{n+1}\to A$ is the natural projection.

Since $\widetilde S$ covers a compact complex manifold, it suffices to verify the existence of the Bergman metric. Let L be a positive holomorphic line bundle on A corresponding to S and let $\tilde L$ be the lift of L to $\mathbb{C}^{n+1}$. Let h be a smooth positive Hermitian metric of L whose local representation is $e^{-\varphi}$, where φ is C^∞ strictly psh. Thus $\tilde{h}=e^{-\widetilde{\varphi}}$ and $\widetilde{\varphi}=\pi^*\varphi$, gives a smooth positive Hermitian metric of $\tilde{L}$ such that

$$\mathrm{const}\, i\partial\bar\partial|z|^2 \le i\partial\bar\partial\widetilde{\varphi} \le \mathrm{const}^{-1}\, i\partial\bar\partial|z|^2.$$

Let $\mathcal{H}(\mathbb{C}^{n+1},\tilde{L})$ be the set of holomorphic $\tilde{L}$-valued $(n+1)$-forms f on $\mathbb{C}^{n+1}$ satisfying

$$\left|\int_{\mathbb{C}^{n+1}} f\wedge\bar{f}e^{-\widetilde{\varphi}}\right| < \infty.$$

We claim that

(C1) *for all* $z^0\in\mathbb{C}^{n+1}$, *there exists* $f_0\in\mathcal{H}(\mathbb{C}^{n+1},\tilde{L})$ *such that* $f_0(z^0)\neq 0$;

(C2) *for all* $z^0\in\mathbb{C}^{n+1}$, *there are* $f_1,\dots,f_{n+1}\in\mathcal{H}(\mathbb{C}^{n+1},\tilde{L})$ *satisfying* $f_\alpha(z^0)=0$ *and* $\partial f^*_\alpha/\partial z_\beta(z^0)=\delta_{\alpha\beta}$ *for* $1\le\alpha,\beta\le n+1$.

Indeed, a direct calculation shows that

$$i\partial\bar\partial\left(\varkappa\left(\frac{|z-z^0|^2}{R^2}\right)\log\frac{|z-z^0|^2}{R^2}\right) \ge -\frac{\mathrm{const}}{R^2} i\partial\bar\partial|z|^2,$$

where $\varkappa\colon \mathbb{R}\to[0,1]$ is a cut-off function satisfying $\varkappa|_{(-\infty,1/2)}=1$ and $\varkappa|_{(1,\infty)}=0$. Thus we may choose R sufficiently large such that

$$i\partial\bar\partial\left(\widetilde{\varphi}(z)+(n+1)\varkappa\left(\frac{|z-z^0|^2}{R^2}\right)\log\frac{|z-z^0|^2}{R^2}\right) \ge \frac{\mathrm{const}}{R^2} i\partial\bar\partial|z|^2.$$

A standard application of L^2-estimates of $\bar\partial$ yields solutions u_j to the equations $\bar\partial u=\bar\partial\eta_j$, where

$$\eta_0 = \varkappa\left(\frac{|z-z^0|^2}{R^2}\right) dz_1\wedge\dots\wedge dz_{n+1}\otimes\xi,$$

$$\eta_j = (z_j-z_j^0)\varkappa\left(\frac{|z-z^0|^2}{R^2}\right) dz_1\wedge\dots\wedge dz_{n+1}\otimes\xi, \quad 1\le j\le n+1,$$

and ξ is a frame of $\tilde{L}$, such that

$$\left|\int_{\mathbb{C}^{n+1}} u_j\wedge\bar{u}_j e^{-\widetilde{\varphi}-(n+1)\varkappa(|z-z^0|^2/R^2)\log(|z-z^0|^2/R^2)}\right| < \infty.$$

It is easy to see that $f_j:=\eta_j-u_j$, $0\le j\le n+1$, satisfy (C1) and (C2). Thus it suffices to verify the following L^2 *adjunction formula*:

$$\mathcal{H}(\mathbb{C}^{n+1},\tilde{L})|_{\widetilde{S}} \subset \mathcal{H}(\widetilde{S}). \quad \square$$

Proof of the adjunction formula. By the standard adjunction formula (cf. [14], p. 147), we conclude that every restriction $f|_{\widetilde{S}}$ of any $f\in\mathcal{H}(\mathbb{C}^{n+1},\tilde{L})$ to $\widetilde{S}$ is a holomorphic n-form on $\widetilde{S}$. It suffices to verify that $f|_{\widetilde{S}}$ is square-integrable on $\widetilde{S}$. We cover S by finite coordinate patches $(U_1, z^1), ..., (U_m, z^m)$ on A such that $S\cap U_k = \{z^k \in U_k : z^k_{n+1}=0\}$. Without loss of generality, we assume that U_k are unit polydiscs and that slightly smaller polydiscs $\{V_k\}_{k=1}^m$ still cover S. Let Γ be a discrete group of translations of $\mathbb{C}^{n+1}$ such that $A=\mathbb{C}^{n+1}/\Gamma$. Clearly, Γ also acts on $\widetilde{S}$ such that $S=\widetilde{S}/\Gamma$. Let $\{\widetilde{U}_\nu\}_{\nu=1}^m$ and $\{\widetilde{V}_\nu\}_{\nu=1}^m$ be lifts of $\{U_k\}_{k=1}^m$ and $\{V_k\}_{k=1}^m$ to $\mathbb{C}^{n+1}$. Then $\{\widetilde{V}_\nu\}_{\nu=1}^m$ covers $\widetilde{S}$. By Cauchy's inequality, we have

$$|f_\nu^*| \leq \text{const} \int_{\widetilde{U}_\nu} |f_\nu^*|^2 \, dV \quad \text{on } \widetilde{V}_\nu.$$

Thus,

$$\int_{\widetilde{V}_\nu \cap \widetilde{S}} |f_\nu^*|^2 \, dV \leq \text{const} \int_{\widetilde{U}_\nu} |f_\nu^*|^2 \, dV \leq \text{const} \left| \int_{\widetilde{U}_\nu} f \wedge \bar{f} e^{-\widetilde{\varphi}} \right|$$

since $\widetilde{U}_\nu$ and $\widetilde{\varphi}$ are lifts of some U_k and a smooth function φ on $\overline{U}_k$. Notice that there are m coordinate patches, say, $\widetilde{U}_1, ..., \widetilde{U}_m$, which intersect a fixed fundamental domain of Γ in $\widetilde{S}$. Thus

$$\sum_{\nu=1}^m \left| \int_{\widetilde{U}_\nu} f \wedge \bar{f} e^{-\widetilde{\varphi}} \right| = \sum_{k=1}^m \sum_{\gamma \in \Gamma} \left| \int_{\gamma(\widetilde{U}_k)} f \wedge \bar{f} e^{-\widetilde{\varphi}} \right| \leq m \left| \int_{\mathbb{C}^{n+1}} f \wedge \bar{f} e^{-\widetilde{\varphi}} \right|,$$

from which we get

$$\left| \int_{\widetilde{S}} f \wedge \bar{f} \right| \leq \sum_{\nu=1}^m \int_{\widetilde{V}_\nu \cap \widetilde{S}} |f_\nu^*|^2 \, dV < \infty. \quad \square$$

Proof of Theorem 1.10. Let Γ be the discrete group of translations of $\mathbb{C}^3$ which is generated by the six translations

$$\gamma_1(z_1, z_2, z_3) = (z_1+1, z_2, z_3), \quad \gamma_2(z_1, z_2, z_3) = (z_1+i, z_2, z_3),$$
$$\gamma_3(z_1, z_2, z_3) = (z_1, z_2+1, z_3), \quad \gamma_4(z_1, z_2, z_3) = (z_1, z_2+i, z_3),$$
$$\gamma_5(z_1, z_2, z_3) = (z_1, z_2, z_3+1), \quad \gamma_6(z_1, z_2, z_3) = (z_1, z_2, z_3+i).$$

Then $A := \mathbb{C}^3/\Gamma$ is the product of three one-dimensional tori, and hence is projective algebraic. Let Γ' be the normal subgroup of Γ which is generated by γ_1, γ_2, γ_3 and γ_4. Then $A' := \mathbb{C}^3/\Gamma'$ is a (noncompact) pseudoconvex Galois covering of A (notice that A' is the product of a two-dimensional torus and $\mathbb{C}$). Let S be a smooth ample divisor of A and let $\widetilde{S} \subset \mathbb{C}^3$ be the universal covering of S. We set $S' = \widetilde{S}/\Gamma'$. In other words, S' is the lift of S to A'. Thus it is also pseudoconvex. By a similar argument as in the proof of Proposition 1.9, we conclude that S' is

Bergman complete provided S is sufficiently ample. Furthermore, we claim that S' does not possess any nonconstant negative psh function. Let us recall the following result.

Varopoulos' theorem. (Cf. [33], see also [15]) *Let a noncompact complete Riemannian manifold $\widetilde{X}$ be a Galois covering of a compact manifold X with a deck transformation group Γ. Then $\widetilde{X}$ is parabolic if and only if Γ contains a finite index subgroup isomorphic with $\mathbb{Z}$ or $\mathbb{Z}^2$.*

By a parabolic Riemannian manifold we mean a manifold which does not possess any nonconstant negative subharmonic function. In our case, S' is a noncompact complete Kähler manifold. Since plurisubharmonic functions are always subharmonic on Kähler manifolds and the deck transformation group of S' is isomorphic with $\mathbb{Z}^2$, our claim follows immediately from Varopoulos' theorem. □

Remark. It is not known whether there exists a two-dimensional Bergman complete Stein manifold (or unbounded domain in $\mathbb{C}^2$) which does not possess any nonconstant negative psh function.

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