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Encomplexed Brown invariant of real algebraic surfaces in $\mathbb{R}P^3$

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Abstract. We construct an invariant of parametrized generic real algebraic surfaces in $\mathbb{R}P^3$ which generalizes the Brown invariant of immersed surfaces from smooth topology. The invariant is constructed using self-intersections, which are real algebraic curves with points of three local characters: the intersection of two real sheets, the intersection of two complex conjugate sheets or a Whitney umbrella. In Kirby and Melvin (Local surgery formulas for quantum invariants and the Arf invariant, in *Proceedings of the Casson Fest*, Geom. Topol. Monogr. **7**, pp. 213–233, Geom. Topol. Publ., Coventry, 2004) the Brown invariant was expressed through a self-linking number of the self-intersection. We extend the definition of this self-linking number to the case of parametrized generic real algebraic surfaces.

1. Introduction

Following the philosophy of Viro in [4] we are interested in encomplexing invariants from smooth topology to construct invariants in real algebraic geometry. In [4] the writhe (which can also be considered as a self-linking number with the normal vector field chosen along the direction of projection) of a curve was encomplexed. While the writhe of a projected curve depends on the projection, the encomplexed writhe of a real algebraic curve does not. Another example, immersions of curves in the plane, has been extensively studied. Two immersions of curves lie in the same component of the space of immersions if and only if they have the same Whitney index. The Whitney index can be calculated from the self-intersections in the case of a generic immersed curve. It turns out that this notion survives to (parametrized) real algebraic curves of type I, where a corresponding encomplexed Whitney index can be calculated from self-intersections (both solitary and nonsolitary), as proved by Viro [5].

In this paper we study a similar situation concerning the space of generically immersed oriented surfaces in $\mathbb{R}^3$. The Brown invariant is an invariant up to regular homotopy of immersed surfaces in $\mathbb{R}^3$. The Brown invariant of an immersed surface can be defined using the self-intersection of the surface, as has been shown by Kirby and Melvin [3]. In their article they express the Brown invariant by constructing a certain curve called the "pushoff" close to the self-intersection. It has a natural projection to the self-intersection, with four points in the preimage of each point in the self-intersection. The linking number between this pushoff and the self-intersection is shown to give the Brown invariant.

In this paper we will encomplex the Brown invariant, using the interpretation in [3] as a self-linking number of the self-intersection.

Let M_S denote the space of real algebraic mappings from some smooth projective real algebraic surface S into $\mathbb{R}P^3$. We also define two discriminants, σ and γ. The discriminant σ consists of those points in M_S such that the corresponding parametrized surface in $\mathbb{R}P^3$ has topologically unstable singularities. The discriminant γ consists of those points in M_S such that the corresponding parametrized surface has points in its self-intersection where the corresponding quadratic form in the normal bundle of the self-intersection curve has a matrix which is a nonzero multiple of the identity matrix. The discriminant γ is dependent on the metrics chosen for $\mathbb{R}P^3$ and $\mathbb{C}P^3$. We construct an invariant called the fourfold pushoff invariant, defined on points of $M_S \backslash (\sigma \cup \gamma)$, in Section 3.

Theorem 1.1. *The fourfold pushoff invariant is constant on connected components of* $M_S \backslash (\sigma \cup \gamma)$.

In Remark 3.4 we explain that in the case of the real algebraic surface being an immersed surface without solitary self-intersections the Brown invariant coincides with the fourfold pushoff invariant.

Theorem 1.2. *Counted* $\bmod 8$, *the fourfold pushoff invariant is constant on connected components of* $M_S \backslash \gamma$.

The corresponding smooth situation, immersed surfaces up to regular homotopy, has been studied by Goryunov in [1], where he describes the space of Vassiliev invariants for this situation.

For proofs, see Section 3.

2. Preliminaries

Let S be a smooth projective real algebraic surface together with its complexification $\mathbb{C}S$. We consider the space M_S^d of real algebraic maps of S into $\mathbb{R}P^3$ of degree d. We also equip $\mathbb{R}P^3$ and $\mathbb{C}P^3$ with two Riemannian metrics. The metrics are the metric inherited from S^3 and the Fubini–Study metric, respectively. In Section 2.1 we examine the singularities of the parametrized surfaces closer. In Section 2.2 we examine the self-intersections of the parametrized surfaces and show how to use the quadratic forms in the normal bundle of such a self-intersection curve to construct the pushoff, a curve lying very close to the self-intersection with a fourfold projection to the self-intersection.

2.1. Singularities in the space M_S^d

Lemma 2.1. *The space M_S^d is diffeomorphic to the complement C_S^d of a codimension-2 subspace of the space of projections from P^m to P^3 for some m.*

Proof. Since S is a projective variety, we naturally have that $S\subset P^n$ for some n. The ring of regular functions of degree d on S is then generated by the monomials of degree d on P^n. Consider the Veronese embedding v_d from P^n into P^m. It is clear that any map f of degree d from S to P^3 can be uniquely factorized as $f=\pi_h\circ v_d$, where π_h is the projection to some 3-space h, since they can be represented as linear combinations of monomials from the ring of regular functions on P^n in each coordinate. □

Remark 2.2. Locally, the space of projections from P^m to P^3 is naturally diffeomorphic to the projections from A^m to A^3. Thus, a generic path in M_S^d can, by a compactness argument, be reduced to pieces which can be considered to be paths in the space of projections from A^m to A^3, that is, paths in $G(3,m)$.

Remark 2.3. Not all of the projections do correspond to points on M_S^d since we cannot let the $(m-4)$-plane we project from intersect $S\subset P^m$. However, since S is of dimension 2, this is at most a codimension-2 condition, so any generic path of projections can be assumed to not intersect this subvariety.

Theorem 2.4. *The space M_S^d contains a subvariety σ of codimension 1 such that outside of this codimension-1 hypersurface, the only possible topological local configurations of self-intersections in the image of $f\in M_S^d\setminus\sigma$ are:*
- *two real sheets intersecting transversally;*
- *three real sheets intersecting pairwise transversally;*
- *two complex conjugate sheets intersecting transversally;*
- *two complex conjugate and one real sheet intersecting pairwise transversally;*
- *a Whitney umbrella.*

Proof. We begin by noting that the allowed types of intersections are topologically stable, and so cannot be avoided. Following Theorem 2.1 we can study the space of projections from some P^m to P^3. We examine the affine real projections from $\mathbb{C}^m$ to $\mathbb{C}^3$ which form the real Grassmannian $G(3,m)$ having dimension $3m-9$. The hypersurface σ consists of the subvariety of projections which result in deeper singularities than the mentioned stable singularities.

We will calculate the dimension of σ by examining the singularities that appear in its top strata and their dimensions separately. Note that the stable cases mentioned concern up to three distinct points ending up at the same point under the projection, such that their tangent planes were in general position, and the Whitney umbrella, arising from a projection which destroyed the tangent plane for a single point.

The higher singularities we need to consider lie in the closure of the following sets in M_S^d: projections with quadruple points; projections with a triple point such that the common intersection of two of the tangent planes lie tangent to the third; projections with a double point such that the tangent planes are tangent to each other; projections with a double point such that one point lacks a tangent plane and one has a tangent plane (Whitney umbrella plus plane); and projections with a single singularity where two Whitney umbrellas collide, arising from projecting along a tangent vector with an odd intersection number with the surface (under a generic projection). By Lemmas 2.5–2.9 below, we know that these singularities have codimension at least 1. Any more complicated singularity will necessarily have even higher codimension. It is also clear that this collection gives all the codimension-1 singularities, since any singularity must necessarily have some point in the preimage and we have examined the different singularities for $1-4$ points (and any singularity involving more than four points or deeper tangency would have a higher codimension than the ones constructed from four points). □

We need to understand the higher singularities in M_S^d. The strategy is to examine the space of projections admitting at least one such singularity by calculating the dimension of the space of points chosen to end up at the singularity. We then examine which vectors are prescribed to project along and finally how to complete these prescribed vectors to a projection. We finish by comparing the dimension to $3m-9$. To shorten the proof we omit the detailed calculations concerning genericity of points (since we are on a Veronese embedding) and the defining equations for the subvariety. In the case of two points with coinciding tangent planes we demonstrate these calculations as well. In some of the situations we should also examine the case of two of the involved sheets being complex conjugate sheets from the complexification of S. However, this does not change dimensions since choosing one point in the complexification is a 4-dimensional choice while the other point is by necessity its complex conjugate. This gives the same dimension as choosing two real points. We use Goryunov's notation of the singularities from [1].

Lemma 2.5. *The points of M_S^d corresponding to singularities of type Q, that is, singularities with a quadruple point, are of codimension at least* 1.

Proof. Consider the variety of projections with at least one quadruple point (regardless of if they are arising from real or complex parts of the surface). For the moment we assume that the points are real.

For a quadruple point in the image, we need four points on the surface in P^m mapped to the same point by a projection. The surface is 2-dimensional, so the configuration space of four points on the surface has dimension 8. The surface S intersects a given $(m-2)$-hyperplane in a discrete number of points generically, so choosing points in a line/plane would decrease the dimension by at least 6 and 4, respectively. To choose a projection from P^m to P^3 we need to complete the 3-dimensional space generated by the four points to an $(m-3)$-dimensional space, i.e. we need to choose a point from $G(m-6, m-3)$ which has dimension $3m-18$. If the points chosen were situated on the same line/plane we would only be proscribed to use one and two vectors, respectively, giving an increase in dimension. However, the original choice of points had a lower dimension, compensating for this increase. The dimension of the space of projections with at least one quadruple point is then at most $3m-10$ and so the space of projections admitting a quadruple point has codimension 1 in the space of projections. The dimension count for quadruple points with two and four points arising from complex parts of the surface proceeds in the same manner. □

Lemma 2.6. *The points of M_S^d corresponding to singularities of type T, that is, singularities with a triple point such that the three sheets intersect with a common tangent line, are of codimension at least* 1.

Proof. The choice of three points is 6-dimensional. To give the three tangent planes a common line after the projection we need to choose one line from each tangent plane which we want to be the common line after the projection. This choice is 3-dimensional. We need to choose a plane from the space spanned by these three lines to project along. This is a 2-dimensional choice. We now have a 4-space which we need to project along (two dimensions for ensuring that the three points ended up at the same points, and two to ensure that the tangent planes had a common line). We complete it by choosing a point from $G(m-7, m-4)$ which has dimension $3m-21$. The total dimension is then $3m-10$. □

Lemma 2.7. *The points of M_S^d corresponding to singularities of type H and E, that is, singularities with a double point such that the two sheets are tangent (with hyperbolic tangency for type H and elliptic tangency for type E), are of codimension at least* 1.

Proof. To see that projections which result in two sheets tangent to each other form a variety we observe that (in affine coordinates with the point of tangency being the origin) the leading term of the polynomial defining such a degenerate surface must be $(ax+by+cz)^2+O(\bar{x}^3)$, where $\bar{x}^3$ stands for terms of order 3. The matrix corresponding to the quadratic form is thus degenerate and of rank 1. This can be expressed in terms of the 2×2 minorants of the matrix being zero. This defines a subvariety of $G(3,m)\times S$ which after projection to $G(3,m)$ gives the projections which result in these singularities.

The dimension of the space of projections with two points having the same tangent plane in the image will now be calculated. We assume that the points are real. Two distinct points in the Veronese embedding v_d of P^n do not have a tangent in common as long as $d>1$, and thus in particular the points on our surface do not have a tangent in common. In the case of $d=1$ and $S\subset P^n, n>3$, it is easy to see that there are only a finite number of pairs of points with common tangent plane. Thus, the choice of such points would contribute zero dimensions and completing the projection would result in dimension $3m-12$ resulting in codimension 3. We can ignore such points henceforth and assume that the tangent planes of the points chosen have no nonzero vectors in common. Choosing the two points contributes four dimensions. Ensuring that they end up at the same point under the projections gives one vector along which we need to project. The tangent planes will form a 4-dimensional linear subspace from which we need to choose a 2-dimensional subspace to project along to give a common tangent plane. Since $\dim(G(2,4))=4$ we get four additional dimensions. Again we wish to complete the three vectors by choosing a point from $G(m-6,m-3)$. Giving a total dimension of at most $3m-10$. Either the tangency is elliptic (case E) or hyperbolic (case H) (described by $x^2+y^2=\varepsilon$ and $xy=\varepsilon$, respectively, in the common tangent plane). □

Lemma 2.8. *The points of M_S^d corresponding to singularities of type C, that is, singularities with a double point such that a Whitney umbrella intersects a sheet, are of codimension at least* 1.

Proof. For the case of point and Whitney umbrella, we have a 4-dimensional choice of points. To ensure that we get a Whitney umbrella, we need to destroy the tangent plane of one of the points. We need to choose one vector in the tangent plane of one of the points to project along. This is a 1-dimensional choice. This vector together with the vector along which we project to ensure that the two points end up at the same space gives us two vectors. To complete them we need to choose a point from $G(m-5, m-2)$, which has dimension $3m-15$. Again we have a total dimension of $3m-10$. □

Lemma 2.9. *The points of M_S^d corresponding to singularities of type B and K, that is, singularities with a single point such that two Whitney umbrellas collide, are of codimension at least* 1.

Proof. For the final case, of a single point, we needed to destroy the tangent plane to the point and so we have to choose one vector in the tangent plane. This tangent vector is by definition tangent to the surface. The surface has some curvature in this point. The surface has zero curvature only in finitely many points (choosing such a point would decrease the dimension by 2, leading to codimension 2). For a point with nonzero curvature only finitely many tangent lines intersect the surface with an odd intersection number (two in the case of negative curvature, zero in the case of positive curvature). Projecting along such a special tangent line will yield a different picture, namely the result of two umbrellas colliding (along either a real or a solitary self-intersection). Choosing our point is a 2-dimensional choice, and choosing our tangent line is of dimension 0 since the choice was discrete. Completing to a projection necessitates a choice of a point from $G(m-4, m-1)$ which is of dimension $3m-12$. The dimension of projections resulting in at least one point with deeper singularity than the umbrella is then of dimension $3m-10$. □

Remark 2.10. For a closer description of these singularities (and others of higher codimension) see Hobbs and Kirk [2], more specifically Table 1. There B and K correspond to $S_k^{\pm}$; H and E correspond to $A_0^2|A_k^{\pm}$; C correspond to $(A_0S_0)_k$; T to $A_0^3|A_k$; and Q to A_0^4.

2.2. Constructing the pushoff from the self-intersection

Around generic points, the self-intersection C_S of a parametrized surface associated with a point in $M_S^d \backslash \sigma$ arises from either two real sheets intersecting along a real line or two complex conjugate sheets intersecting in a real line. A piece of the self-intersection arising from two complex conjugate sheets is called a *solitary* self-intersection. Around isolated points the self-intersection can also be either a Whitney umbrella separating a real line appearing from two complex conjugate sheets from a real line appearing from two real sheets, or three real sheets intersecting, giving rise to three intersecting real lines or two complex conjugate planes and a real plane, resulting in a single real line (and two complex conjugate lines which are ignored). The set of self-intersection points C_S will then be a collection of immersed circles. Let the immersed circles C_{S_i} be indexed by i. Since the circles are immersed, each circle has an associated normal bundle in $\mathbb{R}P^3$ while their complexification has an associated normal bundle in $\mathbb{C}P^3$. Given a point p in C_{S_i} we can examine the defining polynomial of the surface in $\mathbb{C}P^3$ and examine its restriction to the normal plane p_n. This will associate a quadratic form to p by taking the terms of at most order 2 from the polynomial (we do not have any linear terms since we are in the self-intersection). This assigns a continuous family of quadratic forms to C_S. We can then associate a continuous family of eigenvectors as long as the matrix associated with the quadratic form has two different eigenvalues. For those points in C_{S_i} which are not triple points or Whitney umbrellas we can assume that no such quadratic form has only one eigenvalue of multiplicity two (i.e. the associated matrix is a multiple of the identity matrix). The condition of having only one eigenvalue is of codimension 1 in M_S^d as described by the following lemma.

Lemma 2.11. *The points in* $M_S^d \backslash \sigma$ *which correspond to parametrizations which have a point in the self-intersection assigned a corresponding* 2*-form in the normal bundle which has just one eigenvalue* (*i.e. the corresponding matrix is a multiple of the identity matrix*) *are of codimension* 1.

Proof. Since the matrix depends on the metric, we can disturb the parametrization by some linear transformation of $\mathbb{R}P^3$ which is close to identity to gain a different parametrization which do not have a corresponding matrix which is a multiple of the identity along the self-intersection. Since the set of points in the image with such associated matrices is a discrete set, it is enough to show this for one point. We assume that the associated 2-form is $x^2+tx^2+y^2$ in local coordinates around the point at the self-intersection $x=y=0$. By letting $L(x,y)=(x,y+\varepsilon x)$ we disturb the intersection enough to remove the singular case.

The matrices which are a multiple of the identity matrix have codimension 2 in the space of real symmetric matrices. The self-intersection itself form a 1-dimensional space, and so the singular maps cannot be of higher codimension than 1. $\square$

Definition 2.12. We let γ denote the parametrizations in M_S which has points in the corresponding self-intersection such that the matrix associated with the quadratic form is a multiple of the identity matrix.

Remark 2.13. Note that changing the metric on the space would change γ. In this article we assume that the metric is the natural Fubini–Study metric on $\mathbb{C}P^3$.

Associated with each such form are then four (real) eigenvectors of unit length. After scaling them by some small ε these four vectors will be four nonzero sections in the normal bundle of the immersed circle and thus form a fiber bundle B_{S_i} over it.

We now push off our original curve as follows. Take our original immersed circle C_{S_i} and include the bundle B_{S_i} wherever it is defined (that is, over points which come from exactly two sheets intersecting transversally). This will locally give us four curves lying at the boundary of an ε-tubular neighborhood of C_{S_i}. At the special points where the self-intersection does not look like two intersecting sheets we may extend our construction as shown by the following lemma.

Lemma 2.14. *The fourfold pushoff can be extended in a continuous way to nongeneric points along an immersed circle* C_{S_i}.

Proof. To see that this can be defined in a natural way we examine each situation, when necessary expressed in local coordinates x, y, z.

– Two complex conjugate sheets and one real sheet intersecting in a triple point: we ignore the real sheet and extend by continuity.

– A Whitney umbrella, defined by $y^2-zx^2=0$: the eigenvectors are kept, one eigenvalue shifts sign.

– Three real sheets intersecting at a triple point: Defined by $xyz=0$. The pushoff (following one of the intersection curves) will clearly be extendable by continuity, furthermore the pushoff and the self-intersections will be disjoint. □

Definition 2.15. Given such an immersed circle C_{S_i} the associated extended fourfold pushoff will be denoted by $P(C_{S_i})$ and called the *pushoff* of C_{S_i}.

Lemma 2.16. *The pushoff* $P(C_{S_i})$ *has either one*, *two or four components.*

Proof. This is obvious, see Section 4 for some examples. □

Lemma 2.17. *If the immersed surface S is orientable and if generic points of* C_{S_i} *come from the intersection of two real sheets*, *then* $P(C_{S_i})$ *will consist of four connected components.*

Proof. Choose an orientation for the surface. At each generic point of this real self-intersection the corresponding normal vectors will distinguish one eigenvector.

Since this distinction does not depend on any local choice, all four components are separate. □

3. The fourfold pushoff invariant

In Section 3.1 we define the fourfold pushoff invariant and show that it coincides with the Brown invariant for generic immersed surfaces without solitary self-intersection. In Section 3.2 we restate and prove Theorem 1.1 from the introduction. In Section 3.3 we restate and prove Theorem 1.2 from the introduction.

3.1. Defining the fourfold pushoff invariant

We wish to define a self-linking number of an immersed circle C_{S_i} in the self-intersection using its pushoff $P(C_{S_i})$. In general, the linking number between two chains is not well defined unless one of the chains is zero homologous. While C_{S_i} is not necessarily zero homologous, we know that $H_1(\mathbb{R}P^3)=\mathbb{Z}_2$ and we know that the pushoff $P(C_{S_i})$ is four times C_{S_i} in the homology (since we have a natural fourfold projection to C_{S_i}) and thus zero homologous. Thus, the linking number $L(P(C_{S_i}), C_{S_i})$ is well defined.

Definition 3.1. We define the *fourfold pushoff linking number*, $T(C_{S_i})$ of an immersed circle C_{S_i} in the self-intersection as the linking number between C_{S_i} and its pushoff $P(C_{S_i})$.

The definition does not depend on the orientation of the immersed circle, since changing the orientation of C_{S_i} would change the orientation of $P(C_{S_i})$ as well.

Definition 3.2. We define the *fourfold pushoff invariant* $T(S')$ of a surface S', corresponding to a point $p\in M_S\setminus(\sigma\cup\gamma)$, as $\sum_{i\in I} T(C_{S_i})$, where S_i are immersed circles in the self-intersection of S' indexed by some set I.

Remark 3.3. Note that the fourfold pushoff invariant takes the value zero on embeddings.

Remark 3.4. In the affine situation with no solitary self-intersections the fourfold pushoff coincides with Kirby's and Melvin's definition of Brown's invariant in [3].

While Kirby's and Melvin's pushoff is defined by using the sheets intersected with the boundary of a tubular neighborhood of the self-intersection, it is easy

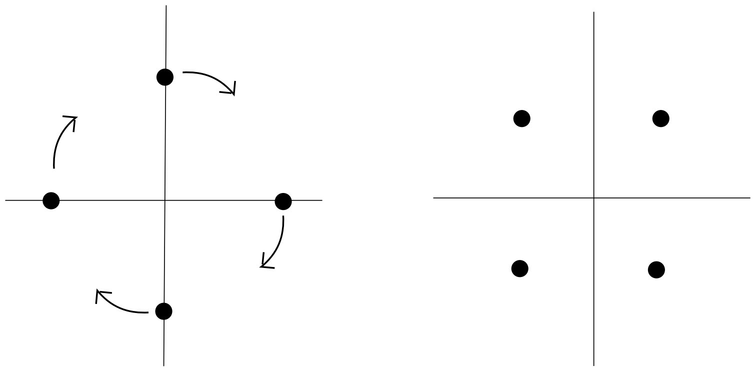

Figure 1. On the left-hand side the pushoff is constructed in the normal plane using Kirby's and Melvin's definition. On the right-hand side the pushoff is constructed using the definitions in this paper. The arrows indicate an isotopy taking the pushoffs to each other.

to construct an isotopy taking their pushoff to $P(C_{S_i})$ as illustrated by Figure 1, depicting the normal plane, by simply rotating, using the fact that $\mathbb{R}P^3$ is orientable.

Remark 3.5. The fourfold pushoff invariant depends on the Riemannian metric chosen. To see this we can change the metric around a solitary self-intersection. The eigenvectors correspond to the axis vectors of the ellipses obtained by putting the quadratic form equal to some small ε. By changing the metric locally we can rotate these ellipses, changing the value of the invariant. As the metric changes, so does γ. The value of the invariant changes just as the chosen parametrization passes through γ.

Theorem 3.6. *The fourfold pushoff invariant changes by* 2 *when passing transversally through* γ.

Proof. This can easily be seen by an examination of the equation $zx^2+y^2+2\varepsilon xy=0$, modeling the crossing, where the sign of ε marks which side of the strata we are located on and $z=1, x=y=0$ is the point on the self-intersection on which the singularity appears. □

The fourfold pushoff invariant changes by 2 exactly when passing through γ. This can obviously only happen when two complex sheets intersect since the eigenvalues have different signs when two real sheets intersect.

3.2. Proof of Theorem 1.1

We restate Theorem 1.1 presented in Section 1.

Theorem 1.1. *The fourfold pushoff invariant is constant on connected components of* $M_S \backslash (\sigma \cup \gamma)$.

Proof. This is obvious, since the pushoff varies continuously and is well defined on each component. □

3.3. Proof of Theorem 1.2

We restate Theorem 1.2 presented in Section 1.

Theorem 1.2. *Counted* mod 8, *the fourfold pushoff invariant is constant on connected components of* $M_S \backslash \gamma$.

Proof. From Theorem 2.4 and Lemma 2.11 we know that σ and γ are of codimension 1 in M_S. Given a path P in a component of $M_S \backslash \gamma$ we can assume that P passes through σ transversally. It is then enough to show that the invariant does not change mod 8 during such a passage. We examine these singular cases and then compute the invariant before and after the singularity. The codimension-1 components of this discriminant are known from Theorem 2.4. Each case can have several subcases depending on if the preimage under the projection is real or not. We recall the different situations. If we have four separate points in the preimage we will locally have a generic intersection of four planes at a single point. This situation is denoted by Q, Q' and Q'' depending on how many of these planes are real. If we have three distinct points, one of them is special and the other two come from two transversal sheets. These situations are denoted by T and T', again depending on if the two transversal sheets are complex conjugate or not. If we have two separate points, we either have a plane traveling through a Whitney umbrella which we denote by C, or two tangentially intersecting planes which we denote by E, E' and H, H' depending on if the tangency is hyperbolic or elliptic. If we have one point, we have two Whitney umbrellas colliding, either along the real self-intersection or the solitary self-intersection. These are denoted by B and K.

For each of these subcomponents of σ we examine the change of the fourfold pushoff.

– Four sheets intersecting at a single point, all of them real. This occurs when a real sheet passes through a triple point. Examining the self-intersections, we see that their positions relative to each other either do not change, or that one passes

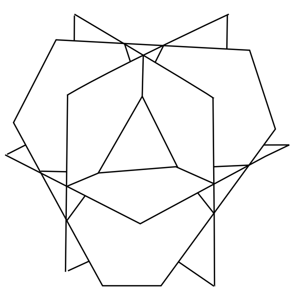

Figure 2. A type Q singularity.

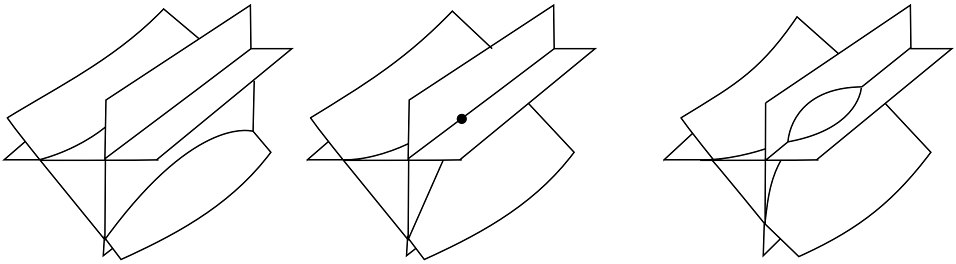

Figure 3. A type T singularity.

through another. If the two curves in the self-intersection pass through each other, the invariant changes by 8 or 0 depending on if the curves are pieces of the same immersed circle or not. See Figure 2.

– Four sheets intersecting, two real and two complex conjugate. This occurs when a solitary piece of the self-intersection passes through a nonsolitary piece. The invariant changes by 8 or 0 depending on if the curves are pieces of the same immersed circle or not.

– Four sheets intersecting, two pairs of complex conjugate sheets. Here two solitary pieces of the self-intersection passes through each other, the invariant changes by 8 or 0 depending on if the curves are pieces of the same immersed circle or not.

– Two triple points meet and annihilate each other along a real self-intersection. See Figure 3. The invariant does not change.

– A solitary self-intersection passes through a real plane, creating two triple points. The invariant does not change.

– Two real sheets intersecting tangentially in a single point. See Figure 4. The value of the invariant on the new circle is 0.

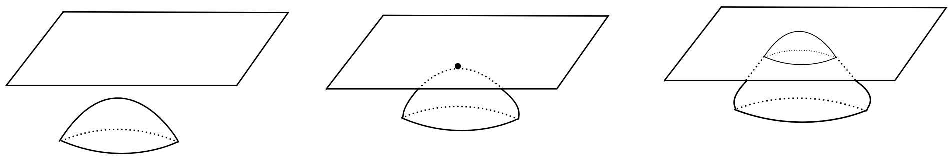

Figure 4. A type E singularity.

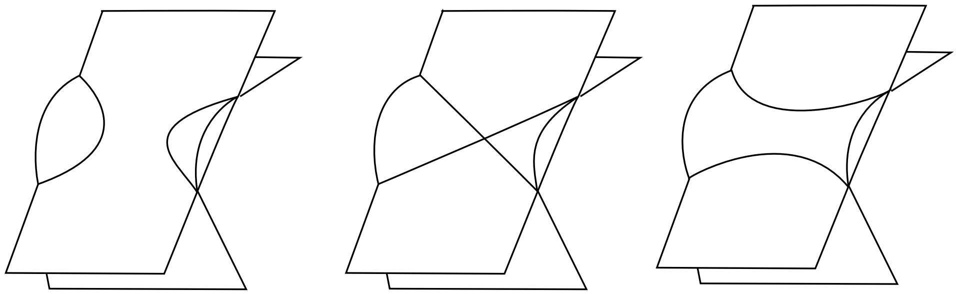

Figure 5. A type H singularity.

– Two complex conjugate tangent sheets intersecting tangentially at a single point. The value of the invariant on the new circle is 0.

– Two sheets intersecting as a plane passing through the surface defined by $z=x^2-y^2$. See Figure 5. If the two components of the self-intersection are different before and after, it may have caused two different immersed circles to join together (or split apart). Globally, this changes the invariant by a multiple of 8.

– Two Whitney umbrellas collide and annihilate each other, either along a solitary self-intersection or a real self-intersection. See Figure 6. The invariant does not change.

– A real plane passes through a Whitney umbrella. See Figure 7. The invariant does not change. □

Remark 3.7. Several of these codimension-1 passages have been studied for the smooth case by Goryunov [1], in our setting additional codimension-1 singularities arose from the additional structure from complex parts of the surface. We have followed his notation for the different kinds of singularities, using $'$ to denote variants arising from solitary/complex conjugate pieces.

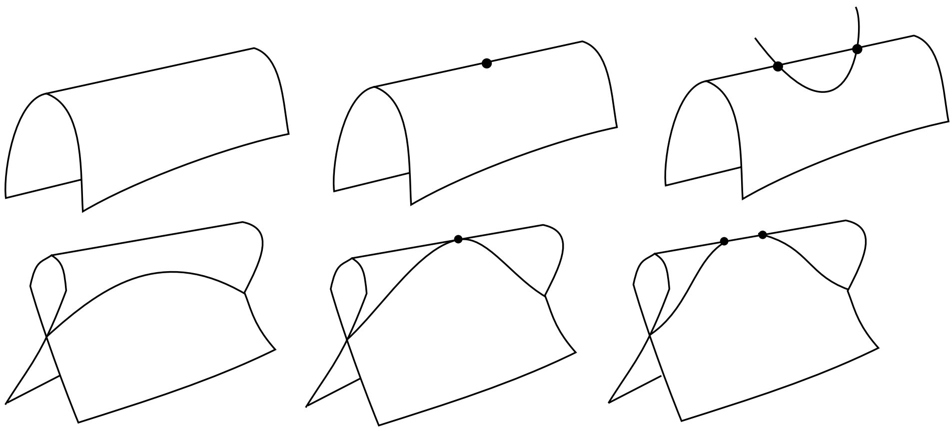

Figure 6. A type B singularity and a type K singularity.

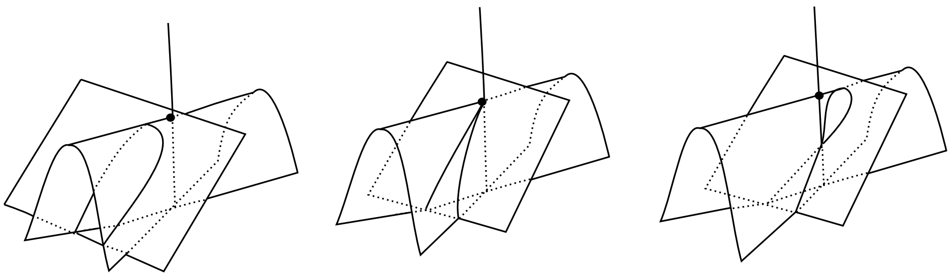

Figure 7. A type C singularity.

4. Examples

In this section we present some examples of surfaces together with the value of their fourfold pushoff invariants.

Example 4.1. The Roman surface is an example of a parametrized projective plane. It has six Whitney umbrellas and the fourfold pushoff takes value 0. It is defined by the equation $x^2y^2+y^2z^2+z^2x^2=xyz$ in affine coordinates.

Example 4.2. The parametrization

$$x(t,s)=\frac{t^3s+ts^3}{t^4+s^4} \quad \text{and} \quad z(t,s)=\frac{2ts(t^2-s^2)}{t^4+s^4}$$

parametrizes a curve looking like the symbol ∞. In affine coordinates we get a similar symbol by the solutions to the equation $y^4+y^2=x^2(1-x^2)$. By rotating it around a line in space we get a real surface parametrized by a torus such that its real self-intersection is diffeomorphic to a circle (this surface will then be parametrized by degree 8). The pushoff of this circle will result in four circles, none of them linked to the self-intersection so the invariant is zero.

Example 4.3. Take our earlier parametrization $x(s,t), z(s,t)$ of the ∞ symbol. We want to rotate this parametrization by applying the following trick. We can parametrize a circle easily by using $p_1(u,v)=2uv/(u^2+v^2)$ and $p_2(u,v)=(u^2-v^2)/(u^2+v^2)$. This allows us to consider p_1 and p_2 as the sin and cos functions. They allow us to first apply a rotation matrix to the parametrization of the ∞ symbol (by just multiplying with the parametrization), then moving it to the side, and then rotating it around the z-axis as earlier. This will result in a degree-16 surface which still has a circle as self-intersection while the pushoff is linked, giving a value of the invariant of ± 4 depending on our choice of direction of rotation, taken modulo 8 the values are of course equal.

Example 4.4. The equation $x^4+y^4+(z^2+t^2)y^2-(z^2+t^2)x^2=0$ defines a surface with two components in the pushoff of the self-intersection. The value of the invariant is 0.

Example 4.5. The equation $x^4+y^4+2(z^2+t^2)y^2+(z^2+t^2)x^2=0$ defines a surface with a real part consisting only of a solitary self-intersection looking like the line defined by $x=y=0$. The value of the invariant is 0.

Example 4.6. The equation $(t^2+z^2)y^2=x^2(x-z)(x-3z-t)$ defines a surface which has a self-intersection located at the line defined by $x=y=0$. The self-intersection consists of both solitary and nonsolitary parts. The value of the invariant is 0.

Example 4.7. The equation $(t^2+z^2)y=\pm 2tzx$ defines a surface which has a self-intersection located at the line defined by $t=z=0$. The value of the invariant is ± 1.

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