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Monomial ideals whose depth function has any given number of strict local maxima

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In recent years there have been several publications concerning the stable set of prime ideals of a monomial ideal, see for example [4], [7], [10] and [9]. It is known by Brodmann [2] that for any graded ideal I in the polynomial ring S (or any proper ideal I in a local ring) there exists an integer k_0 such that $\mathrm{Ass}(I^k)=\mathrm{Ass}(I^{k+1})$ for $k\geq k_0$. The smallest integer k_0 with this property is called the *index of stability* of I and $\mathrm{Ass}(I^{k_0})$ is the set of *stable prime ideals* of I. A prime ideal $P\in\bigcup_{k=1}^{\infty}\mathrm{Ass}(I^k)$ is said to be *persistent* with respect to I if whenever $P\in\mathrm{Ass}(I^k)$ then $P\in\mathrm{Ass}(I^{k+1})$, and the ideal I is said to satisfy the *persistence property* if all prime ideals $P\in\bigcup_{k=1}^{\infty}\mathrm{Ass}(I^k)$ are persistent. It is an open question (see [6] and [13, Question 3.28]) whether any square-free monomial ideal satisfies the persistence property.

We call the numerical function $f(k)=\mathrm{depth}(S/I^k)$ the *depth function* of I. It is easy to see that a monomial ideal I satisfies the persistence property if all monomial localizations of I have non-increasing depth functions. In view of the above mentioned open question it is natural to ask whether all square-free monomial ideals have non-increasing depth functions. The situation for non-square-free monomial ideals is completely different. Indeed, in [8, Theorem 4.1] it is shown that for any non-decreasing numerical function f, which is eventually constant, there exists a monomial ideal I such that $f(k)=\mathrm{depth}(S/I^k)$ for all k. Note that a similar result for non-increasing depth functions is not known, even though it is expected that all square-free monomial ideals have non-increasing depth functions. In general the depth function of a monomial ideal does not need to be monotone. Examples of monomial ideals with non-monotone depth functions are given in [12, Example 4.18] and [8]. The question arises which numerical functions are depth functions of monomial ideals. Since $\mathrm{depth}(S/I^k)$ is constant for all $k\gg 0$ (see [1]), any depth function is eventually constant. So the most wild conjecture one could make is that any numerical function which is eventually constant is indeed the depth function of a monomial ideal. In support of this conjecture we show in our theorem that for any given number n there exists a monomial ideal whose depth function has precisely

n strict local maxima. Sathaye in [11, Example, p. 2] gives an example of an ideal I in a graded ring R and a prime ideal P of R such that $P\in\mathrm{Ass}(I^k)$ for k even and $P\notin\mathrm{Ass}(I^k)$ for k odd, for all k up to any given bound. Our example has this property, too, but in contrast to Sathaye's example it is defined in a regular ring, indeed in the polynomial ring. The price that we have to pay is that the number of variables needed to define our ideal with n strict local maxima is relatively large, namely $2n+4$.

For the class of examples considered here the depth function is constant beyond the number of variables. In all other examples known to us, in particular those discussed in [8], this is also the case. Thus we are tempted to conjecture that for any monomial ideal I in a polynomial ring in n variables $\mathrm{depth}(I^k)$ is constant for $k\geq n$.

In the following theorem we present the monomial ideals admitting a depth function as announced in the title of the paper. For the calculation of some preliminary examples we used the computer algebra system CoCoA [5].

Theorem 1. *Let $n\geq 0$ be an integer and $I\subset S=K[a,b,c,d,x_1,y_1,\dots,x_n,y_n]$ be the monomial ideal in the polynomial ring S with generators*

$$a^6,\ a^5b,\ ab^5,\ b^6,\ a^4b^4c,\ a^4b^4d,\ a^4x_1y_1^2,\ b^4x_1^2y_1,\ \dots,\ a^4x_ny_n^2,\ b^4x_n^2y_n.$$

Then

$$\mathrm{depth}(S/I^k)=\begin{cases}0, & \text{if } k \text{ is odd and } k\leq 2n+1,\\ 1, & \text{if } k \text{ is even and } k\leq 2n,\\ 2, & \text{if } k>2n+1.\end{cases}$$

In particular, the depth function of this ideal has precisely n strict local maxima.

Proof. First of all, for each odd integer $k=2t-1$ with $t\leq n+1$, we show that $\mathrm{depth}(S/I^k)=0$. For this purpose we find a monomial belonging to $(I^k:\mathfrak{m})\setminus I^k$, where $\mathfrak{m}=(a,b,c,d,x_1,y_1,\dots,x_n,y_n)$. We claim that the monomial

$$u=a^4b^4(a^4x_1y_1^2)(b^4x_1^2y_1)\dots(a^4x_{t-1}y_{t-1}^2)(b^4x_{t-1}^2y_{t-1})x_ty_t\dots x_ny_n$$

satisfies $u\in(I^k:\mathfrak{m})\setminus I^k$. Let

$$v_1=a^5b\cdot b^6(a^4x_{t-1}y_{t-1}^2)\prod_{i=1}^{t-2}(a^4x_iy_i^2)(b^4x_i^2y_i),$$

$$v_2=ab^5\cdot a^6(b^4x_{t-1}^2y_{t-1})\prod_{i=1}^{t-2}(a^4x_iy_i^2)(b^4x_i^2y_i),$$

$$v_3 = a^4b^4c\prod_{i=1}^{t-1}(a^4x_iy_i^2)(b^4x_i^2y_i),$$

$$v_4 = a^4b^4d\prod_{i=1}^{t-1}(a^4x_iy_i^2)(b^4x_i^2y_i),$$

$$v_{2l+3} = a^6(b^4x_l^2y_l)^2\prod_{i=1}^{l-1}(a^4x_iy_i^2)(b^4x_i^2y_i)\prod_{i=l+1}^{t-1}(a^4x_iy_i^2)(b^4x_i^2y_i), \quad 1\le l\le t-1,$$

$$v_{2l+4} = b^6(a^4x_ly_l^2)^2\prod_{i=1}^{l-1}(a^4x_iy_i^2)(b^4x_i^2y_i)\prod_{i=l+1}^{t-1}(a^4x_iy_i^2)(b^4x_i^2y_i), \quad 1\le l\le t-1,$$

$$v_{2l+3} = (b^4x_l^2y_l)\prod_{i=1}^{t-1}(a^4x_iy_i^2)(b^4x_i^2y_i), \quad t\le l\le n,$$

$$v_{2l+4} = (a^4x_ly_l^2)\prod_{i=1}^{t-1}(a^4x_iy_i^2)(b^4x_i^2y_i), \quad t\le l\le n.$$

Clearly $v_i\in I^k$ for $1\le i\le 2n+4$. One easily sees that

$$v_1|au, \quad v_2|bu, \quad v_3|cu \quad \text{and} \quad v_4|du.$$

Since

$$a^4b^4(a^4x_ly_l^2)x_l = a^2(a^6(b^4x_l^2y_l))y_l \quad \text{and} \quad a^4b^4(b^4x_l^2y_l)y_l = b^2(b^6(a^4x_ly_l^2))x_l,$$

it follows that

$$v_{2l+3}|x_lu \text{ and } v_{2l+4}|y_lu, \quad 1\le l\le t-1.$$

Moreover,

$$v_{2l+3}|x_lu \text{ and } v_{2l+4}|y_lu, \quad t\le l\le n.$$

Hence $u\mathfrak{m}\subseteq I^k$. In other words, $u\in I^k:\mathfrak{m}$.

Now, we wish to prove that $u\notin I^k$. Since neither c nor d divides u, it is enough to show that $u\notin \bar{I}^k$, where

$$\bar{I} = (a^6, a^5b, ab^5, b^6, a^4x_1y_1^2, b^4x_1^2y_1, \ldots, a^4x_ny_n^2, b^4x_n^2y_n).$$

Suppose that there exists a monomial $w=u_1\ldots u_k\in\bar{I}^k$ with each $u_i\in G(\bar{I})$ such that w divides u. Since $\deg_{x_i}(u)=\deg_{y_i}(u)=1$ for $i=t,\ldots,n$, each u_i belongs to

$$\mathcal{M} = \{a^6, a^5b, ab^5, b^6, a^4x_1y_1^2, b^4x_1^2y_1, \ldots, a^4x_{t-1}y_{t-1}^2, b^4x_{t-1}^2y_{t-1}\}.$$

Since $\deg_{x_i}(u)=\deg_{y_i}(u)=3$ for $i=1,\dots,t-1$, it follows that, for u_i and u_j belonging to

$$\mathcal{N}=\{a^4x_1y_1^2, b^4x_1^2y_1, \dots, a^4x_{t-1}y_{t-1}^2, b^4x_{t-1}^2y_{t-1}\}$$

with $i\neq j$, one has $u_i\neq u_j$. As $|\mathcal{N}|=2t-2=k-1$, there exists $1\leq j\leq k$ with $u_j\notin\mathcal{N}$. Let ρ denote the number of integers $1\leq j\leq k$ with $u_j\notin\mathcal{N}$. Since w divides u, one has $\deg_a(w)\leq 4t$ and $\deg_b(w)\leq 4t$.

Case 1. $\rho=1$. Since $|\mathcal{N}|=k-1$, each monomial belonging to $\mathcal{N}$ divides w. Thus $\deg_a(w)=4(t-1)+c$ and $\deg_b(w)=4(t-1)+d$, where (c,d) belongs to

$$\{(0,6),(1,5),(5,1),(6,0)\}.$$

Hence one has either $\deg_a(w)>4t$ or $\deg_b(w)>4t$, a contradiction.

Case 2. $\rho=2$. Then we may assume that $\deg_a(w)=4(t-1)+c_1+c_2$ and $\deg_b(w)=4(t-2)+d_1+d_2$, where each (c_i,d_i) belongs to $\{(0,6),(1,5),(5,1),(6,0)\}$. Again, one has either $\deg_a(w)>4t$ or $\deg_b(w)>4t$, a contradiction.

Case 3. $\rho=h$ with $h>2$. Suppose that a^4 divides each of the monomials $u_1,\dots,u_s$, where $s\leq k-h$. Let $\deg_a(w)=4s+c_1+\dots+c_h$ and $\deg_b(w)=4(k-h-s)+d_1+\dots+d_h$, where $c_i+d_i=6$ for each $1\leq i\leq h$. Since

$$\deg_b(w)=4(k-h-s)+(6-c_1)+\dots+(6-c_h)\leq 4t=2(k+1),$$

it follows that

$$\deg_a(w)=4s+c_1+\dots+c_h\geq 4(k-h)+6h-2(k+1)=2k+2h-2.$$

However, since $h>2$, one has

$$2k+2h-2>2k+4-2=2k+2=2(k+1)=4t.$$

Thus $\deg_a(w)>4t$, a contradiction.

The above three cases complete the proof of $u\notin I^k$. Hence u belongs to $(I^k:\mathfrak{m})\setminus I^k$ and $\operatorname{depth}(S/I^k)=0$, as desired.

Now we are going to prove that $\operatorname{depth}(S/I^k)\geq 1$ for any even number $0<k\leq 2n$. For the proof we introduce the ideals $J=(a^6,a^5b,ab^5,b^6,a^4b^4c,a^4b^4d)$ and $L=(a^4x_1y_1^2, b^4x_1^2y_1,\dots,a^4x_ny_n^2,b^4x_n^2y_n)$. Then $I=J+L$, and hence

$$I^k=J^k+J^{k-1}L+\dots+J^2L^{k-2}+JL^{k-1}+L^k.$$

We first show that for $k\geq 2$, the factor module $(I^k:(c,d))/I^k$ is generated by the residue classes of the elements of the set

$$\mathcal{S}_k=\{a^4b^4v_1\dots v_{k-1}: v_i\in G(L) \text{ and } v_i\neq v_j \text{ for } i\neq j\}. \tag{1}$$

Observe that the minimal set of generators of J^2 only consists of monomials in a and b. Therefore, the only monomials in I^k which are divisible by c or d are the generators of JL^{k-1}. It follows that the generators of $I^k:(c,d)$ which do not belong to I^k are the monomials of the form $a^4b^4v_1...v_{k-1}$ with $v_i\in G(L)$.

Suppose that $v_i=v_j$ for some $i\neq j$, say $v_i=v_j=a^4x_ly_l^2$. We may assume that $i=1$ and $j=2$. Then

$$u=a^4b^4v_1...v_{k-1}=a^{12}b^4x_l^2y_l^4v_3...v_{k-1}=(a^{12})(b^4x_l^2y_l)v_3...v_{k-1}y_l^3.$$

Since $a^{12}\in J^2$ and $b^4x_l^2y_l\in L$, we see that $u\in J^2L^{k-2}\subset I^k$. This proves (1).

For a monomial $u=a^4b^4v_1...v_{k-1}\in\mathcal{S}_k$, we set

$$Z_u=\{x_l:\deg_{x_l}(v_i)=1 \text{ for some } i\}\cup\{y_l:\deg_{y_l}(v_i)=1 \text{ for some } i\}$$

and

$$W_u=\bigcup_{x_l\notin \operatorname{supp}(u)}\{x_l^2y_l,x_ly_l^2\}.$$

Note that $u+I^k$ is annihilated by a, b, c and d, all variables in Z_u and all monomials in W_u. Indeed, it is obvious that a, b, c and d and all monomials in W_u annihilate $u+I^k$. Now let $x_l\in Z_u$; we shall show that $ux_l\in I^k$. We can assume that $v_1=a^4x_ly_l^2$. Hence $a^6(b^4x_l^2y_l)v_2...v_{k-1}\in I^k$ and $a^6(b^4x_l^2y_l)v_2...v_{k-1}|ux_l$, so $ux_l\in I^k$. Similarly for $y_s\in Z_u$, we show that $uy_s\in I^k$.

It follows from this observation that $(I^k:(c,d))/I^k$ is generated as a K-module by the residue classes of monomials uvw where $u\in\mathcal{S}_k$, v is a monomial in the variables x_i and y_j belonging to $V_u=\operatorname{supp}(u)\setminus Z_u$, and w is a monomial in the variables x_i and y_j not belonging to the support of u and not divisible by a monomial in W_u.

Fix $u=a^4b^4v_1...v_{k-1}\in\mathcal{S}_k$ and let the residue class of $m=uvw$ be a generator of $(I^k:(c,d))/I^k$ as described in the preceding paragraph. Then v is a monomial with $\deg_{x_i}(u)=\deg_{y_j}(u)=2$ for each $x_i,y_j\in\operatorname{supp}(v)$. After relabeling of the variables we may assume that

$$\operatorname{supp}(u)=\{a,b,x_1,y_1,...,x_t,y_t\}.$$

Then

$$uv=a^4b^4\prod_{i=1}^{r}(a^4x_iy_i^2)(b^4x_i^2y_i)\prod_{j=r+1}^{s}a^4x_jy_j^{h_j}\prod_{l=s+1}^{t}b^4x_l^{g_l}y_l \tag{2}$$

with $h_j\geq 2$ and $g_l\geq 2$, and $k-1=r+t$.

Claim 2. *None of the monomials $m=uvw$ belong to I^k.*

For the proof of Claim 2 we first observe the following claim.

Claim 3. *If* $w_1 \ldots w_s$ *divides* m *with* $w_1, \ldots, w_s \in G(L)$, *then* $s \leq k-1$ *and after renumbering of the* v_i *we have* $w_i = v_i$ *for* $i=1, \ldots, s$.

Proof. Indeed we may assume that $w_1 = a^4 x_j y_j^2$. It follows from (2) that $x_j y_j^2$ appears in one of the v_i. Hence after renumbering we may assume that $w_1 = v_1$. Then $w_2 \ldots w_s$ divides m/v_1. Induction on k completes the proof. □

Proof of Claim 2. We assume on the contrary that $m \in I^k$. Then there exist $w_i \in G(I)$ such that $w_1 \ldots w_k$ divides m. We may assume that $w_1, \ldots, w_s \in G(L)$ and $w_{s+1}, \ldots, w_k \in G(J)$. By Claim 3 we may assume that $w_i = v_i$ for $i=1, \ldots, s$. We next need the following claim.

Claim 4. $s = k-1$.

Proof. For the proof we consider the following two cases.

Case 1. $s = k-2$. Then, $w_{k-1} w_k$ divides $a^4 b^4 v_{k-1} v w$. However, since $w_{k-1}, w_k \in G(J)$, we have

$$\deg_a(w_{k-1} w_k) > \deg_a(a^4 b^4 v_{k-1} v w) \quad \text{or} \quad \deg_b(w_{k-1} w_k) > \deg_b(a^4 b^4 v_{k-1} v w),$$

a contradiction.

Case 2. $s = k-h$ with $h > 2$. Then $w_{k-h+1} \ldots w_k$ divides $a^4 b^4 v_{k-h+1} \ldots v_{k-1} v w$. Since $w_{k-h+1}, \ldots, w_k \in G(J)$, it follows that

$$\deg_a(w_{k-h+1} \ldots w_k) + \deg_b(w_{k-h+1} \ldots w_k) \geq 6h.$$

On the other hand,

$$\deg_a(a^4 b^4 v_{k-h+1} \ldots v_{k-1} v w) + \deg_b(a^4 b^4 v_{k-h+1} \ldots v_{k-1} v w) = 4h + 4.$$

Now since $h > 2$, it follows that $4h + 4 < 6h$. This means that

$$\deg_a(a^4 b^4 v_{k-h+1} \ldots v_{k-1} v) + \deg_b(a^4 b^4 v_{k-h+1} \ldots v_{k-1} v) < \deg_a(w_{k-h+1} \ldots w_k) + \deg_b(w_{k-h+1} \ldots w_k),$$

a contradiction. This concludes the proof. □

We now continue with the proof of Claim 2. As we know that $s = k-1$, it follows that w_k divides $a^4 b^4 v w$. This is a contradiction, since $w_k \in G(J)$. Thus the proof of Claim 2 is complete. □

From Claim 2 it follows that $\operatorname{depth}(S/I^k)>0$ for even $0<k\leq 2n$. Indeed suppose that $\operatorname{depth}(S/I^k)=0$. Then $I^k:\mathfrak{m}\neq I^k$. Since $I^k:\mathfrak{m}\subset I^k:(c,d)$, it follows that there exists a monomial $m=uvw\in I^k:\mathfrak{m}$ of the form as described above. Now since k is even and $0<k\leq 2n$ and $v_i\neq v_j$ for $i\neq j$, the set $V_u\neq\varnothing$. It follows that $mv'\notin I^k$ for any $v'\in V_u$, a contradiction.

In the next step we show that $\operatorname{depth}(S/I^k)\leq 1$ (and hence $\operatorname{depth}(S/I^k)=1$) for even k with $0<k\leq 2n$. Indeed, we claim that

$$P=(a,b,c,d,x_1,y_1,\ldots,x_{n-1},y_{n-1},x_n)$$

belongs to $\operatorname{Ass}(I^k)$ for even k with $0<k\leq 2n$. Then, since

$$\operatorname{depth}(S/I^k)\leq \min\{\dim(S/Q):Q\in\operatorname{Ass}(I^k)\}$$

(see [3, Proposition 1.2.13]), the required inequality follows.

To show this we note that $P\in\operatorname{Ass}(I^k)$ if and only if $\operatorname{depth}(S(P)/I(P)^k)=0$, see for example [9, Lemma 2.3]. Here $S(P)$ is the polynomial ring in the variables which generate P, and $I(P)$ is obtained from I by the substitution $y_n\mapsto 1$.

In our case $I(P)$ is generated by

$$a^6,a^5b,ab^5,b^6,a^4b^4c,a^4b^4d,a^4x_1y_1^2,b^4x_1^2y_1,\ldots,a^4x_{n-1}y_{n-1}^2,b^4x_{n-1}^2y_{n-1},a^4x_n,b^4x_n^2.$$

We claim that for $k=2t$ with $t\leq n$ the monomial

$$u'=a^8b^4(a^4x_1y_1^2)(b^4x_1^2y_1)\ldots(a^4x_{t-1}y_{t-1}^2)(b^4x_{t-1}^2y_{t-1})x_ty_t\ldots x_{n-1}y_{n-1}x_n$$

satisfies $u'\in(I(P)^k:\mathfrak{m}(P))\setminus I(P)^k$. This shows that $\operatorname{depth}(S(P)/I(P)^k)=0$. Let

$$v_i'=(a^4x_n)v_i \text{ for } i=1,\ldots,2n+2 \quad\text{and}\quad v_{2n+3}'=a^6(b^4x_n^2)\prod_{i=1}^{t-1}(a^4x_iy_i^2)(b^4x_i^2y_i),$$

where v_i is defined as in the first part of the proof. Clearly $v_i'\in I(P)^k$ for $1\leq i\leq 2n+3$. One easily sees that

$$v_1'|au',\quad v_2'|bu',\quad v_3'|cu'\quad\text{and}\quad v_4'|du'.$$

Moreover

$$v_{2l+3}'|x_lu',\ v_{2l+4}'|y_lu',\ 1\leq l\leq n-1,\quad\text{and}\quad v_{2n+3}'|x_nu'.$$

Hence $u'\in(I(P)^k:\mathfrak{m}(P))$.

With the same argument as in the first part of the proof one can easily see that $u'\notin I(P)^k$. Therefore $u'\in(I(P)^k:\mathfrak{m}(P))\setminus I(P)^k$, so $\operatorname{depth}(S(P)/I(P)^k)=0$, as desired.

Finally we show that $\operatorname{depth}(S/I^k)=2$ for $k>2n+1$. Since the only generators of I^k which are divisible by c are among the generators of JL^{k-1}, we see that $I^k:(c)/I^k$ is generated by the residue classes of the set of monomials $\bigcup_{u\in\mathcal{S}_k}\{u, ud\}$. Since $k>2n+1$, it follows that $\mathcal{S}_k=\varnothing$. Hence $I^k:(c)=I^k$ for $k>2n+1$. Similarly, $I^k:(d)=I^k$ for $k>2n+1$. It follows that c, d is a regular sequence on S/I^k for $k>2n+1$. This implies that $\operatorname{depth}(S/I^k)\geq 2$ for all $k>2n+1$.

Let $\bar{S}=K[a, b, x_1, y_1, ..., x_n, y_n]$ and

$$\bar{I}=(a^6, a^5b, ab^5, b^6, a^4x_1y_1^2, b^4x_1^2y_1, ..., a^4x_ny_n^2, b^4x_n^2y_n)\subset\bar{S}.$$

Then $(S/I^k)/(c,d)(S/I^k)=\bar{S}/\bar{I}^k$.

We claim that $w=a^5b^{6k-6}x_1y_1x_2y_2...x_ny_n\in(\bar{I}^k:\mathfrak{n})\setminus\bar{I}^k$ for $k\geq 2$, where $\mathfrak{n}$ is the graded maximal ideal of $\bar{S}$. The claim implies that $\operatorname{depth}((S/I^k)/(c,d)(S/I^k))=0$ for all $k\geq 2$. In particular it follows that $\operatorname{depth}(S/I^k)=2$ for all $k>2n+1$, as desired.

To prove the claim we notice that aw is divisible by $(a^6)(b^6)^{k-1}\in\bar{I}^k$, and bw is divisible by $(a^5b)(b^6)^{k-1}\in\bar{I}^k$. Hence $aw, bw\in\bar{I}^k$.

Next observe that x_iw is divisible by $(a^5b)(b^6)^{k-2}(b^4x_i^2y_i)\in\bar{I}^k$ and y_iw is divisible by $(b^6)^{k-1}(a^4x_iy_i^2)\in\bar{I}^k$. This implies that $x_iw, y_iw\in\bar{I}^k$ for all i. Thus we have shown that $w\in\bar{I}^k:\mathfrak{n}$.

It remains to be shown that $w\notin\bar{I}^k$. Indeed, none of the generators of L divides w, because each of these generators has x_i-degree or y_i-degree 2. Therefore, if $w\in\bar{I}^k$, it follows that w is divisible by a monomial in a and b of degree $6k$. However, a^5b^{6k-6} has only degree $6k-1$, a contradiction. □

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