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Maximal Marcinkiewicz multipliers

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Abstract. Let $\mathcal{M}=\{m_j\}_{j=1}^{\infty}$ be a family of Marcinkiewicz multipliers of sufficient uniform smoothness in $\mathbb{R}^n$. We show that the L^p norm, $1<p<\infty$, of the related maximal operator

$$M_N f(x)=\sup_{1\le j\le N} |\mathcal{F}^{-1}(m_j \mathcal{F} f)|(x)$$

is at most $C(\log(N+2))^{n/2}$. We show that this bound is sharp.

1. Introduction

A *Marcinkiewicz multiplier* on $\mathbb{R}^n$ is a Fourier multiplier with a symbol which satisfies a set of conditions

$$|\partial_{i_1}...\partial_{i_k} m_j|(\xi)\le A|\xi_{i_1}|^{-1}...|\xi_{i_k}|^{-1} \tag{1}$$

for all $\{i_1,...,i_k\}\subset\{1,...,n\}$. We consider a family of N symbols $m_1,...,m_N$ which satisfy these conditions uniformly. We form a maximal operator

$$M_N f(x)=\sup_{1\le i\le N} |\mathcal{F}^{-1}(m_i \hat{f})|(x).$$

We show that the norm of this operator grows as $C(\log(N+2))^{n/2}$. Previously, a similar theorem was proved for Hörmander–Mikhlin symbols in [3] and [6]. The main difference is that in the case of Hörmander–Mikhlin symbols the bound is $C(\log(N+2))^{1/2}$ independently of dimension.

The smoothness of the symbol is not optimal here, as the Marcinkiewicz multiplier theorem may be formulated with a set of BV (bounded variation) type conditions

$$\int_{I_1}...\int_{I_k} |\partial_{i_1}...\partial_{i_k} m_j|(\xi_1,...,\xi_n)\, d\xi_{i_1}...d\xi_{i_k}\le A, \tag{2}$$

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where $I_l=[r_l,2r_l]$ and $\{i_1,...,i_k\}\subset\{1,...,n\}$. Also a different maximal theorem for Marcinkiewicz multipliers appeared recently in [7], with condition even weaker than BV. It is not clear to us if the smoothness of the symbol may be relaxed in our theorem.

Theorem 1.1. *Suppose $\mathcal{M}=\{m_j\}_{j=1}^{\infty}$ is a family of functions in $\mathbb{R}^n$ such that*

$$|\partial_{i_1}...\partial_{i_k} m_j|(\xi)\le A|\xi_{i_1}|^{-1}...|\xi_{i_k}|^{-1} \tag{3}$$

for all $\{i_1,...,i_k\}\subset\{1,...,n\}$. Then for any $N\in\mathbb{N}$ and $1<p<\infty$ we have

$$\Big\|\sup_{1\le j\le N}|\mathcal{F}^{-1}(m_j\hat{f})|(x)\Big\|_{L_p(x)}\le AC_{n,p}(\log^{n/2}(N+2))\|f\|_p.$$

Also, for any given $N\ge1$ and $1<p<\infty$ there is a sequence $\mathcal{M}=\{m_j\}_{j=1}^{\infty}$ which satisfies (3) and a function g such that

$$\Big\|\sup_{1\le j\le N}|\mathcal{F}^{-1}(m_j\hat{g})|(x)\Big\|_{L_p(x)}\ge A\widetilde{C}_{n,p}(\log^{n/2}(N+2))\|g\|_p.$$

One motivation for the study of the maximal Marcinkiewicz multipliers was the open problem of the maximal hyperbolic Bochner–Riesz means. We give a brief discussion of this problem at the end of the paper.

2. Multiple martingales

In this section we introduce the dyadic multiple martingales, which we use as a tool to study the Marcinkiewicz multipliers. First, let us introduce the classical dyadic martingale. Consider an integrable function f on $[0,1]$. Let us denote by $\mathcal{D}_k$ the set of dyadic intervals of length 2^{-k} and define the expectation operator

$$E_kf(x)=2^k\int_I f(y)\,dy,$$

where $x\in I\in\mathcal{D}_k$. We define the martingale differences as $D_k=E_k-E_{k-1}$ and the square function

$$Sf(x)=\bigg(\sum_{k=0}^{\infty}(D_kf(x))^2\bigg)^{1/2}.$$

The maximal martingale function f^* is defined as $f^*(x)=\sup_k|E_kf|(x)$. There is an equivalent representation of this object using Haar functions. For each dyadic interval I, we define a function h_I which is equal to $|I|^{-1/2}$ on the left half and to $-|I|^{-1/2}$ on the right half of the interval. These functions, together with the function $h_0(x)=1$ form an orthonormal basis in $L^2([0,1])$, and also a Schauder basis in $L^p([0,1])$, $1\le p<\infty$. One sees that for $k\ge 1$,

$$D_k f=\sum_{I\in\mathcal{D}_{k-1}}\langle h_I,f\rangle h_I.$$

The dyadic multiple martingales represent a tensored version of the above object. We stress that the multiple martingales are not martingales and in fact they lack many of the key properties of the classical martingales. Let us have a function g on the cube $[0,1]^n$. For dyadic intervals $I_1,...,I_n$ we define the function

$$h_{I_1,...,I_n}(x_1,...,x_n)=h_{I_1}(x_1)...h_{I_n}(x_n).$$

Using these tensored Haar functions, one can define the multiple versions of the operators E, D and S. In particular, we have

$$D_{k_1,...,k_n}g=\sum_{I_i\in D_{k_i}}\langle h_{I_1,...,I_n},g\rangle h_{I_1,...,I_n},$$

$$E_{k_1,...,k_n}=\sum_{-1\le m_1\le k_1,...,-1\le m_n\le k_n}D_{m_1,...,m_n}$$

and

$$Sf(x)=S_{n+1}f(x)=\Biggl(\sum_{k_1\ge -1,...,k_n\ge -1}(D_{k_1,...,k_n}f(x))^2\Biggr)^{1/2}.$$

If we fix $n-1$ indices and $n-1$ variables, then the expectations $E_{k_1,...,k_n}g(x_1,...,x_n)$ form a dyadic martingale, for example

$$\widetilde{E}_j(y)=E_{k_1,...,k_{n-1},j}\,g(x_1,...,x_{n-1},y)$$

is a sequence of expectations of one-dimensional dyadic martingales.

We use some fine results on the relationship of the square functions and maximal operators. In the case of the dyadic martingale, the sharp good lambda inequality

$$|\{x:f^*(x)>2\lambda \text{ and } Sf(x)<\varepsilon\lambda\}|\le Ce^{-c/\varepsilon^2}|\{x:f^*(x)>\lambda\}| \tag{4}$$

was proved by Chang, Wilson and Wolff [2]. In the case of double dyadic martingales, a similar inequality was proved by Pipher [8]. The argument of Pipher extends to higher dimensions as well by induction. Before we demonstrate this fact, we need to define intermediate square functions. We put for $m=2,\dots,n+1$,

$$S_m f=\Bigg(\sum_{k_1,\dots,k_{m-1}}\Bigg(\sum_{k_m,\dots,k_n}D_{k_1,\dots,k_n}f(x)\Bigg)^2\Bigg)^{1/2}.$$

and

$$S_m^* f=\sup_r\Bigg(\sum_{k_1,\dots,k_{m-1}}\Bigg(\sum_{k_m<r,k_{m+1},\dots,k_n}D_{k_1,\dots,k_n}f(x)\Bigg)^2\Bigg)^{1/2}.$$

We also define the following maximal function (replacing S_1^*)

$$M_1 f(x)=\sup_r\Bigg|\sum_{m_1\le r,\dots,m_n}D_{m_1,\dots,m_n}f(x)\Bigg|.$$

Standard arguments show that all the above operators are L^p bounded for $1<p<\infty$.

In the case $m=1$ and $n=2$ the good lambda inequality was proved by Pipher in [8] (fourth formula on p. 76), but as the inequality is not stated exactly in the form we need it and we need a higher-dimensional version, we feel that we need to reproduce the proof here. The starting point of the proof is the following lemma, proved in the paper [8] as Lemma 2.2 (here d_q^j is a difference of a dyadic martingale):

Lemma 2.1. *Suppose $X_N^j=\sum_{q=0}^N d_q^j$, $j=1,\dots,M$, is a sequence of dyadic martingales on the space Y and set*

$$SX_N^j=\Bigg(\sum_{q=0}^N(d_q^j)^2\Bigg)^{1/2},$$

the square function of X_N^j. Then

$$\int_Y\exp\Bigg(\sqrt{1+\sum_{j=1}^M(X_N^j)^2}-\sum_{j=1}^M(SX_N^j)^2\Bigg)\,dx\le e.$$

The following lemma is the higher-dimensional analogue of the Pipher good lambda inequality.

Lemma 2.2. *Let $f\in L^1([0,1]^n)$. Let us fix N and set $g=E_{N,\dots,N}f$. Then we have for $0<\varepsilon<\frac{1}{2}$, $1<m\le n$ and $x_1,\dots,x_{m-1},x_{m+1},\dots,x_n$,*

$$|\{x_m : S_m^* g(x_1,...,x_n) > 2\lambda \text{ and } S_{m+1}g(x_1,...,x_n) < \varepsilon\lambda\}| \\ \le Ce^{-c/\varepsilon^2} |\{x_m : S_m^* g(x_1,...,x_n) > \lambda\}|.$$

Moreover

$$|\{x_1 : M_1 g(x_1,...,x_n) > 2\lambda \text{ and } S_2 g(x_1,...,x_n) < \varepsilon\lambda\}| \\ \le Ce^{-c/\varepsilon^2} |\{x_1 : M_1 g(x_1,...,x_n) > \lambda\}|.$$

The constants C and c are independent of f, N, λ and ε.

Proof. The second inequality follows directly from the inequality of Chang, Wilson and Wolff (4), since for $x_2,...,x_n$ fixed

$$M_1 g(x_1,...,x_n)$$

represents the dyadic martingale maximal function of the function

$$g_1(x_1) = g(x_1,...,x_n)$$

and

$$S_2 g(x_1,...,x_n)$$

represents its martingale square function $Sg_1(x_1)$.

To prove the first inequality, we use the same argument as Pipher [8] in the proof of her Corollary 2.2a. We fix m and $x_1,...,x_{m-1},x_{m+1},...,x_n$ and set

$$d_q^{(j_1,...,j_{m-1})}(x_m) = \sum_{k_{m+1},...,k_n} D_{j_1,...,j_{m-1},q,k_{m+1}...,k_n} g(x_1,...,x_n).$$

Let $J=(j_1,...,j_{m-1})$. Then

$$X_N^J = \sum_{q=0}^{N} d_q^J$$

is a sequence of martingales on the interval $0 \le x_m \le 1$, as in Lemma 2.1.

The set $\{x : S_m^* g(x) > \lambda\}$ is composed of maximal dyadic intervals I such that for $x_m \in I$,

$$\left(\sum_{k_1,...,k_{m-1}} \left(\sum_{k_m < r,...,k_n} D_{k_1,...,k_n} g(x) \right)^2 \right)^{1/2} > \lambda.$$

The r above is then minimal, and we fix a pair I, r. Assume that

$$I \cap \{x : S_m^* g(x) \ge 2\lambda\} \ne \varnothing.$$

Then we may find maximal dyadic intervals $I' \subset I$ such that for $x_m \in I'$,

$$\left(\sum_{k_1,\dots,k_{m-1}} \left(\sum_{k_m<r',\dots,k_n} D_{k_1,\dots,k_n} g(x) \right)^2 \right)^{1/2} \geq 2\lambda.$$

We localize the $d_q^{(j_1,\dots,j_{m-1})}$ and X_N^J to the interval I' and apply a localized version of Lemma 2.1. We then get for any α and t,

$$\int_{I'} \exp\Bigg[\alpha \left(\sum_{k_1,\dots,k_{m-1}} \left(\sum_{r<k_m<t,\dots,k_n} D_{k_1,\dots,k_n} g(x) \right)^2 \right)^{1/2} \\ - \alpha^2 \sum_{k_1,\dots,k_{m-1},r<k_m<t} \left(\sum_{k_{m+1},\dots,k_n} D_{k_1,\dots,k_n} g(x) \right)^2 \Bigg] \leq e|I'|.$$

Consider the set $A = \{x \in I' : S_{m+1} g(x) \leq \varepsilon\lambda\}$. We have for $x \in A$,

$$\left(\sum_{k_1,\dots,k_{m-1}} \left(\sum_{k_m=r,\dots,k_n} D_{k_1,\dots,k_n} g(x) \right)^2 \right)^{1/2} \leq \varepsilon\lambda$$

and therefore

$$\sum_{k_1,\dots,k_{m-1}} \left(\sum_{r<k_m<t,\dots,k_n} D_{k_1,\dots,k_n} g(x) \right)^2 \geq (2\lambda)^2 - \lambda^2 - \varepsilon^2\lambda^2.$$

This gives

$$|A| \exp[\alpha(3-\varepsilon^2)^{1/2}\lambda - \alpha^2\varepsilon^2\lambda^2] \leq e|I'|.$$

We take $\alpha = (3-\varepsilon^2)^{1/2}/2\varepsilon^2\lambda$ and sum the intervals I' to obtain the result. □

3. Proof of the positive result

The main idea of the proof comes from the article [6], we apply the Lemma 2.2 in each variable separately. In some of the estimates we replace the usual maximal function by the strong maximal function, related to averages over rectangular parallelepipeds with sides parallel to the axes.

In order to prove our theorem, we also need to refer to the proof of the Marcinkiewicz theorem. We use the notation and method of the proof from [5]. We may assume that all the multipliers are supported in the positive cone $\{\xi : \xi_1 > 0, \dots, \xi_n > 0\}$. For a set $A \subset \mathbb{R}$ we introduce the coordinate cutoff operators

$$\Delta_A^{(j)} f = \mathcal{F}^{-1}(\chi_A(\xi_j)\hat{f}).$$

Let us fix $j\in\mathbb{Z}^n$ and take

$$\xi\in R_j=[2^{j_1},2^{j_1+1}]\times...\times[2^{j_n},2^{j_n+1}].$$

The following representation formula follows from the fundamental theorem of calculus:

$$m(\xi)=m(2^{j_1},...,2^{j_n})+\sum_{k=1}^{n}\int_{2^{j_k}}^{\xi_{j_k}}\partial^k m(\xi_1,...,t,...,\xi_n)\,dt$$
$$+...+\int_{2^{j_1}}^{\xi_{j_1}}...\int_{2^{j_n}}^{\xi_{j_n}}\partial^1...\partial^n m(t_1,...,t_n)\,dt_1...dt_n.$$

Using this formula, we can write T_mf as a sum of integral terms, with the leading term being

$$\sum_{j\in\mathbb{Z}^n}\int_{2^{j_1}}^{2^{j_1+1}}...\int_{2^{j_n}}^{2^{j_n+1}}\partial^1...\partial^n m(t_1,...,t_n)\Delta^{(1)}_{[t_1,2^{j_1+1}]}...\Delta^{(n)}_{[t_n,2^{j_n+1}]}f(x)\,dt_1...dt_n$$

Therefore in order to establish L^p control of $\sup_{1\le i\le N}|T_{m_i}f(x)|$ we need to estimate

$$\int_{2^{j_1}}^{2^{j_1+1}}...\int_{2^{j_n}}^{2^{j_n+1}}\sup_{1\le i\le N}|\partial^1...\partial^n m(t_1,...,t_n)\Delta^{(1)}_{[t_1,2^{j_1+1}]}...\Delta^{(n)}_{[t_n,2^{j_n+1}]}f(x)|\,dt_1...dt_n$$
$$\le\int_1^2...\int_1^2 A\sup_{1\le i\le N}\left|\sum_{j\in\mathbb{Z}^n}a_{i,j}(t)\Delta^{(1)}_{[2^{j_1}t_1,2^{j_1+1}]}...\Delta^{(n)}_{[2^{j_n}t_n,2^{j_n+1}]}f(x)\right|dt_1...dt_n,$$

where for each $t\in[1,2]$ we have $\|a_{i,j}(t)\|_\infty\le 1$.

In the light of these considerations we see that the proof of the theorem reduces to obtaining the bound

$$\left\|\sup_{1\le i\le N}|\widetilde{T}_if|(x)\right\|_p\le C(\log(N+2))^{n/2}\|f\|_p, \tag{5}$$

where

$$\widetilde{T}_if(x)=\sum_{j\in\mathbb{Z}^n}a_{i,j}\Delta^{(1)}_{[2^{j_1}t_1,2^{j_1+1}]}...\Delta^{(n)}_{[2^{j_n}t_n,2^{j_n+1}]}f(x),$$

each $t_l\in[1,2]$ and $a_{i,j}$ is a sequence in l^∞ with norm 1. We note that the L^p boundedness of the operator $\widetilde{T}_i$ may be deduced from the boundedness of the square function

$$\widetilde{S}f(x)=\left(\sum_{j\in\mathbb{Z}^n}|\Delta^{(1)}_{[2^{j_1}t_1,2^{j_1+1}]}...\Delta^{(n)}_{[2^{j_n}t_n,2^{j_n+1}]}f(x)|^2\right)^{1/2}.$$

The estimates

$$\|\widetilde{S}f\|_p\le C\|f\|_p\quad\text{and}\quad\|\widetilde{T}_if\|_p\le C\|\widetilde{S}f\|_p$$

are proved in detail for example in [5].

We now fix $t=(t_1,...,t_n)$ and for each $j\in\mathbb{Z}^n$ introduce the notation

$$\Delta_j^\flat f=\Delta^{(1)}_{[2^{j_1}t_1,2^{j_1+1}]}...\Delta^{(n)}_{[2^{j_n}t_n,2^{j_n+1}]}f.$$

Let us recall the *sharp maximal function*. Let $\mathcal{R}$ denote the set of all rectangular parallelepipeds with sides parallel to the coordinate axes. We let

$$M_sf(x)=\sup_{x\in R\in\mathcal{R}}\frac{1}{|R|}\int_R|f|(y)\,dy.$$

Let us introduce the operator

$$G_r(f)=\sum_{j\in\mathbb{Z}^n}(M_sM_sM_s|\Delta_j^\flat f|^r)^{2/r}.$$

This operator is L^p bounded for $p>r$. This follows by repeated application of the Fefferman–Stein theorem for vector-valued maximal functions (see [10]) in each coordinate.

If we take a dyadic cube Q with volume 1, we may adapt the dyadic multiple martingale operators to it. We need the following estimates.

Lemma 3.1. *Let us fix $K>0$ and let $a_{i,j}=0$ whenever some $j_k\le K$. For any dyadic cube Q with $|Q|=1$ and $x\in Q$ we have for $1<r<\infty$,*

$$S(\widetilde{T}_if)(x)\le C_rAG_r(f)(x).$$

Moreover, we have

$$\left|\int_Q\widetilde{T}_if\,dy\right|\le C_rA2^{-K/r}G_r(f)(x).$$

Proof. Let us choose a Schwartz function b on $\mathbb{R}$ with the following properties: $b(0)=0$, $b(\xi)\neq0$ for $\xi\in\left[\frac12,4\right]$ and $\hat b$ is supported in $\left\{x:|x|\le\frac14\right\}$. Then, we select a function a in C_c^∞ such that for $\xi\in[1,2]$ we have $b^2(\xi)a(\xi)=1$. For $\xi\in\mathbb{R}^n$ we put

$$\beta(\xi)=b(\xi_1)...b(\xi_n)\quad\text{and}\quad\psi(\xi)=a(\xi_1)...a(\xi_n).$$

For $k\in\mathbb{Z}^n$ we define the operators

$$B_kf(\xi)=\mathcal{F}^{-1}(\beta(2^{-k_1}\xi_1,...,2^{-k_n}\xi_n)\hat f(\xi))$$

and

$$L_kf(\xi)=\mathcal{F}^{-1}(\psi(2^{-k_1}\xi_1,...,2^{-k_n}\xi_n)\hat f(\xi)).$$

We have the following representation of the operator $\widetilde{T}_i$:

$$\widetilde{T}_i f(x) = \sum_{j\in\mathbb{N}^n} a_{i,j} B_j B_j L_j \Delta_j^\flat(f)(x).$$

Then, we get

$$D_k \widetilde{T}_i f(x) = \sum_{j\in\mathbb{N}^n} a_{i,j} (D_k B_j) B_j L_j \Delta_j^\flat(f)(x).$$

Clearly, we have

$$|B_j L_j f|(x) \le M_s f(x).$$

Therefore in order to prove the lemma, we need to obtain the estimates

$$|D_k B_j| f(x) \le C 2^{-\alpha/r'} (M_s M_s |f|^r(x))^{1/r} \tag{6}$$

and

$$|E_k B_j| f(x) \le C 2^{-\tilde{\alpha}/r'} (M_s M_s |f|^r(x))^{1/r}, \tag{7}$$

where $\alpha=\max\{|k_1-j_1|, ..., |k_n-j_n|\}$ and $\widetilde{\alpha}=\max\{j_1-k_1, ..., j_n-k_n\}$.

The proof of (6) is simple in the case $\alpha=\max\{k_1-j_1, ..., k_n-j_n\}=k_s-j_s$ for some s. Since $j_s>0$, we have $k_s\neq-1$. Smoothness estimate in the variable x_s gives

$$|D_k B_j| f(x) \le C 2^{-\alpha} M_s M_s f(x)$$

and the claim follows.

If on the other hand $\alpha=\widetilde{\alpha}=\max\{j_1-k_1, ..., j_1-k_1\}=j_s-k_s$ for some s, the estimate (6) follows from (7), which we prove next. We write $f=g+h$, where

$$g = f\chi_{\{x:|x_s-l2^{-k_s}|\le 2^{-j_s} \text{ for some } l\in\mathbb{Z}\}}.$$

We observe that since $\int_{\mathbb{R}} \hat{b}(x)\,dx=0$ we get

$$|E_k B_j h(x)| = 0.$$

On the other hand,

$$\begin{aligned} |E_k B_j| g(x) &\le \frac{1}{|R_k(x)|} \int_{R_k(x)\cup\{x:|x_s-l2^{-k_s}|\le 2^{-j_s+1} \text{ for some } l\in\mathbb{Z}\}} |M_s g(y)|\,dy \\ &\le C 2^{-\alpha/r'} (M_s(M_s|f|)^r(x))^{1/r}, \end{aligned}$$

where $R_k(x)\in\mathcal{R}$ is the dyadic parallelepiped with dimensions given by k and containing x. The first inequality follows by the support properties of b and the second follows from Hölder inequality. This finishes the proof of (6) and (7) and the lemma is proved. □

In order finish the proof, we need to estimate the measure of the set

$$\Omega_\lambda = \Big\{x : \sup_{1\le i\le N} |\widetilde{T}_i f|(x) > 2^{n+2}\lambda\Big\}.$$

We may assume that $\hat{f}$ is compactly supported inside the positive cone and then, by a simple scaling, that $a_{i,j}=0$ whenever some $j_k \le N$.

We split $\Omega_\lambda = \Omega_\lambda^1 \cup \Omega_\lambda^2 \cup \Omega_\lambda^3$, where

$$\Omega_\lambda^1 = \Big\{x : \sup_{1\le i\le N} |\widetilde{T}_i f - E_0 \widetilde{T}_i f|(x) > 2^{n+1}\lambda \text{ and } G_r(f)(x) \le \varepsilon^n \lambda\Big\},$$

$$\Omega_\lambda^2 = \{x : G_r(f)(x) > \varepsilon^n \lambda\},$$

$$\Omega_\lambda^3 = \Big\{x : \sup_{1\le i\le N} |E_0 \widetilde{T}_i f|(x) > 2^{n+1}\lambda\Big\}.$$

We have

$$\Omega_\lambda^1 \subset \bigcup_{i=1}^N \Omega_{\lambda,i}^1,$$

where

$$\Omega_{\lambda,i}^1 = \{x : |\widetilde{T}_i f - E_0 \widetilde{T}_i f|(x) > 2^{n+1}\lambda \text{ and } S\widetilde{T}_i f(x) \le \varepsilon^n \lambda\}.$$

Finally, we observe that we may choose $j=(j_1, \dots, j_n)$ such that for each i,

$$\|\widetilde{T}_i f - E_j \widetilde{T}_i f\|_p \le \frac{\|f\|_p}{N}. \tag{8}$$

We may therefore write $\Omega_{\lambda,i}^1 \subset \Gamma_{\lambda,i} \cup \Gamma'_{\lambda,i}$, where

$$\Gamma'_{\lambda,i} = \{x : |\widetilde{T}_i f - E_j \widetilde{T}_i f|(x) > 2^n \lambda\}$$

and

$$\begin{aligned}\Gamma_{\lambda,i} \subset & \{x : |E_j \widetilde{T}_i f - E_0 \widetilde{T}_i f|(x) > 2^n \lambda \text{ and } S_2 E_j \widetilde{T}_i f(x) \le 2^{n-1}\varepsilon\lambda\} \\ & \cup \bigcup_{\nu=1}^{n-1} \{x : |S_\nu E_j \widetilde{T}_i f|(x) > 2^{n-\nu}\varepsilon^\nu \lambda \text{ and } S_{\nu+1} E_j \widetilde{T}_i f(x) \le 2^{n-\nu-1}\varepsilon^{\nu+1}\lambda\}.\end{aligned}$$

We apply Lemma 2.2 and obtain

$$|\Gamma_{\lambda,i}| \le Ce^{-c/\varepsilon^2}\Big(|\{x : |M_1(E_j\widetilde{T}_i f - E_0\widetilde{T}_i f)|(x) > 2^{n-1}\lambda\}| + \sum_{\nu=1}^{n-1} |\{x : |S^*_\nu E_j \widetilde{T}_i f|(x) > 2^{n-\nu-1}\varepsilon^\nu\lambda\}|\Big).$$

We set $\varepsilon = C/\log^{1/2}(N+2)$, collect all the previous estimates and integrate them with respect to λ^{p-1}.

This yields

$$\begin{aligned}\Big\|\sup_{1\le i\le N}|\widetilde{T}_i f|\Big\|_p^p &\le C_{n,p}\int \lambda^{p-1}\Big(\Omega^2_\lambda + \Omega^3_\lambda + \sum_{i=1}^N (|\Gamma_{\lambda,i}| + |\Gamma'_{\lambda,i}|)\Big)\,d\lambda \\ &\le C_{n,p}\Big(\log^{n/2}(N+2)\|G_r(f)\|_p + \sum_{i=1}^N \|E_0\widetilde{T}_i f\|_p \\ &\quad + \frac{1}{N}\sum_{i=1}^N \|M_1(E_j\widetilde{T}_i f - E_0\widetilde{T}_i f)\|_p \\ &\quad + \frac{\log^{n/2}(N+2)}{N}\sum_{i=1}^N\sum_{\nu=1}^{N-1}\|S^*_\nu E_j\widetilde{T}_i f\|_p + \sum_{i=1}^N \|\widetilde{T}_i f - E_j\widetilde{T}_i f\|\Big).\end{aligned}$$

Here we observe, that the first, third and fifth term is bounded by the boundedness of the operators in question, the second is bounded by Lemma 3.1 and the fourth is bounded by (8). Together this finishes the proof of the first part of the theorem.

4. Example

Here we construct the sequence in the second part of the theorem. The example is very similar to the one from [3], the key difference is the observation that the tensored products have less freedom of oscillations than one-dimensional sequences.

Let us take a smooth function $\phi_0(x)$ such that $\phi_0(x)=1$ for $x\in\left[\frac{5}{4},\frac{7}{4}\right]$ and $\phi_0(x)=0$ for $x\in\mathbb{R}\setminus(1,2)$ and a smooth nonzero function ψ_0 supported in $\left[\frac{5}{4},\frac{7}{4}\right]$. For $\xi\in\mathbb{R}^n$ we then define

$$\psi(\xi) = \psi_0(\xi_1)\ldots\psi_0(\xi_n).$$

Suppose that $(20n)^{nK} < N \le (20n)^{n(K+1)}$. For $x\in\mathbb{R}^n$ we define

$$g(x) = \sum_{j_1,\ldots,j_n\in\{1,\ldots,K\}} e^{2\pi i(2^{j_1},\ldots,2^{j_n})\cdot x}(\mathcal{F}^{-1}\psi)(x).$$

A a square function argument shows that $\|g\|_p \approx K^{n/2}$.

Let $\{\omega_1, ..., \omega_{20n}\}$ be the set of the complex $20n$th roots of 1. Let us take a sequence $a \in \{1, ..., 20n\}^K$. We define for $\eta \in \mathbb{R}$,

$$\widetilde{m}_a(\eta) = \sum_{i=1}^{K} \omega_{a(i)} \phi_0(\eta - 2^i).$$

For a sequence $b = (a_1, ..., a_n)$ and $\xi \in \mathbb{R}^n$ we define

$$m_b(\xi) = \widetilde{m}_{a_1}(\xi_1) ... \widetilde{m}_{a_n}(\xi_n).$$

Now we define

$$Mf(x) = \sup_b |\mathcal{F}^{-1}(m_b \hat{f})|(x).$$

Next, let us fix $x \in \mathbb{R}^n$ and select a sequence $b = (a_1(j), ..., a_n(j))_{j=1}^K$ such that

$$|\omega_{a_k(j)} e^{2\pi i 2^j x_k} - 1| \leq \frac{2\pi}{10n} \quad \text{for all } j \text{ and } k.$$

It is easy to check that

$$|\mathcal{F}^{-1}(m_b \hat{g})|(x) \geq |\mathrm{Re}\, \mathcal{F}^{-1}(m_b \hat{g})(x)| \geq CK^n |\mathrm{Re}\, \mathcal{F}^{-1}(\psi)(x)|.$$

Therefore,

$$\|Mg\|_p \geq CK^{n/2} \|g\|_p$$

and we are finished.

5. Maximal hyperbolic Bochner–Riesz means

One motivation for the study of the maximal Marcinkiewicz multipliers was the open problem of the boundedness of the *maximal hyperbolic Bochner–Riesz operator*

$$Mf(x) = \sup_{k \in \mathbb{Z}} |(\mathcal{F})^{-1}(m_\lambda(2^k \cdot) \hat{f})|(x),$$

where

$$m_\lambda(\xi_1, \xi_2) = (1 - |\xi_1 \xi_2|)_+^\lambda.$$

While the boundedness of the operator $(\mathcal{F})^{-1}(m_\lambda \hat{f})$ was settled by El-Cohen [4] and Carbery [1], the boundedness of the maximal operator remains open for any λ. Our result gives logarithmic growth with respect to the number of dilations in case $\lambda > 1$, but the full solution remains elusive (see also [9]).

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