Exact Lagrangian caps and non-uniruled Lagrangian submanifolds

Georgios Dimitroglou Rizell

Abstract. We make the elementary observation that the Lagrangian submanifolds of $\mathbf{C}^n$, $n \geq 3$, constructed by Ekholm, Eliashberg, Murphy and Smith are non-uniruled and, moreover, have infinite relative Gromov width. The construction of these submanifolds involve exact Lagrangian caps, which obviously are non-uniruled in themselves. This property is also used to show that if a Legendrian submanifold inside a contactisation admits an exact Lagrangian cap, then its Chekanov–Eliashberg algebra is acyclic.

1. Introduction

1.1. Background

A *contact manifold* is a smooth $(2n+1)$-dimensional manifold Y together with the choice of a maximally non-integrable field of tangent hyperplanes. For us, this field of hyperplanes will be given as $\ker \lambda$ for a fixed choice of one-form λ, and we will think of the contact manifold as the pair (Y, λ). It follows that $\lambda \wedge (d\lambda)^{\wedge n}$ is a volume form on Y. A *Legendrian submanifold* $\Lambda \subset (Y, \lambda)$ is an n-dimensional submanifold, which is tangent to $\ker \lambda$.

A *symplectic manifold* (X, ω) is a $2n$-dimensional manifold X together with a closed non-degenerate two-form ω. We say that (X, ω) is *exact* if $\omega = d\alpha$ is exact. The basic example of an exact symplectic manifold is the standard symplectic n-space $(\mathbf{C}^n, \omega_0 := d\alpha_0)$, where

$$\alpha_0 := -(y_1 \, dx_1 + ... + y_n \, dx_n).$$

A *Lagrangian submanifold* $L \subset (X, \omega)$ is an n-dimensional submanifold satisfying $\omega|_{TL} = 0$. In the case when $\omega = d\alpha$ is exact, we say that an immersion (respectively

embedding) $g\colon L \to (X, d\alpha)$ of an n-dimensional manifold is an *exact Lagrangian immersion* (respectively *embedding*) if $g^*\alpha$ is exact.

By a *Lagrangian cap* of a Legendrian submanifold $\Lambda \subset (Y, \lambda)$ we mean a properly embedded Lagrangian submanifold without boundary

$$L_{\Lambda, \varnothing} \subset (\mathbf{R} \times Y, d(e^t\lambda))$$

of the symplectisation of (Y, λ) satisfying the property that $L_{\Lambda, \varnothing}$ coincides with the cylinder $(-\infty, N) \times \Lambda$ outside of a compact set. Here t denotes the coordinate of the $\mathbf{R}$-factor. We say that the cap is *exact* if the pull-back of the one-form $e^t\lambda$ to the cap is an exact one-form, which, moreover, has a primitive that vanishes outside of a compact set.

Let $(\overline{X}, d\alpha)$ be a compact Liouville domain with contact boundary

$$(Y, \lambda) := (\partial \overline{X}, \alpha|_{TY})$$

and consider its completion $(X, d\alpha)$ obtained by gluing the non-compact cylindrical end

$$([0, +\infty) \times Y, d(e^t\lambda)),$$

i.e. half of the symplectisation, to the boundary of $(\overline{X}, d\alpha)$. We are also interested in *Lagrangian fillings* $L_{\varnothing, \Lambda} \subset (X, d\alpha)$ of a Legendrian submanifold $\Lambda \subset (Y, \lambda)$, by which we mean a properly embedded Lagrangian submanifold without boundary that coincides with the cylinder

$$(N, +\infty) \times \Lambda \subset [0, +\infty) \times Y$$

outside of a compact set.

Given a Legendrian submanifold $\Lambda \subset (Y, \lambda)$ admitting both a filling $L_{\varnothing, \Lambda} \subset X$ and a cap $L_{\Lambda, \varnothing} \subset \mathbf{R} \times Y$, the following construction produces a closed Lagrangian submanifold in $(X, d\alpha)$. After a translation of $L_{\Lambda, \varnothing}$ in the t-direction, which is an isotopy through exact Lagrangian submanifolds, we may suppose that there is some $N \geq 1$ for which

$$L_{\varnothing, \Lambda} \cap \{(t, x)\,;\, t \geq N-1\} = [N-1, +\infty) \times \Lambda$$

holds in the cylindrical end of X, while

$$L_{\Lambda, \varnothing} \cap \{(t, x)\,;\, t \leq N\} = (-\infty, N] \times \Lambda$$

holds inside the symplectisation. We now define the *concatenation of $L_{\varnothing, \Lambda}$ and $L_{\Lambda, \varnothing}$* to be the closed Lagrangian submanifold

$$L := (L_{\varnothing, \Lambda} \cap \{(t, x)\,;\, t \leq N\}) \cup (L_{\Lambda, \varnothing} \cap \{(t, x)\,;\, t \geq N\}) \subset (X, d\alpha),$$

where $L_{\Lambda,\varnothing}\cap\{(t,x); t\geq N\}$ has been canonically identified with an exact Lagrangian submanifold (with boundary) of the cylindrical end of X. The concatenation may thus be seen as the simultaneous resolution of the conical singularities of $\mathbf{R}\times Y$ and $L_{\Lambda,\varnothing}\subset\mathbf{R}\times Y$. Observe that we do not assume the filling to be exact, and hence nor is the obtained Lagrangian submanifold L in general.

We say that a Lagrangian immersion $L\subset(X,\omega)$ is *displaceable* if there exists a time-dependent Hamiltonian on X whose induced Hamiltonian flow $\phi^s\colon X\to X$ satisfies $L\cap\phi^1(L)=\varnothing$.

1.2. Previous results

Consider a Lagrangian submanifold $L\subset(X,\omega)$ of a general symplectic manifold and let $\mathcal{B}(X,L,r)$ be the set of symplectic embeddings of

$$B^{2n}(r):=\{\mathbf{z}\,;\|\mathbf{z}\|<r\}\subset(\mathbf{C}^n,\omega_0)$$

into (X,ω) that map the real-part into L, and which otherwise have image disjoint from L. The following notion was first considered by Barraud and Cornea in [1].

Definition 1.1. The (*relative*) *Gromov width* of a Lagrangian submanifold $L\subset(X,\omega)$ is the number

$$w(L,X):=\sup\{\pi r^2\in[0,+\infty)\,;\mathcal{B}(X,L,r)\neq\varnothing\}.$$

Weinstein's Lagrangian neighbourhood theorem implies that the Gromov width always is positive. Furthermore, it is obviously invariant under symplectomorphisms.

Definition 1.2. A Lagrangian immersion $L\subset(X,\omega)$ whose self-intersections consist of transverse double-points is said to be *uniruled* if there is some $A>0$ for which, given any compatible almost complex structure J on X and a point $x\in L$, there exists a non-constant J-holomorphic disc in X having boundary on L, having a boundary point mapping to x, and whose ω-area is at most A.

Uniruledness is known to imply finiteness of the Gromov width, see for instance [2, Corollary 3.10]. It has been expected that every displaceable closed Lagrangian submanifold is uniruled [1, Conjecture 3.15], given that the symplectic manifold is either compact or given as the completion of a Liouville domain (or, more generally, is convex at infinity). The observations in this paper show that this conjecture is false, since it is not satisfied by the Lagrangian submanifolds of $\mathbf{C}^n$ constructed in [10] by Ekholm, Eliashberg, Murphy and Smith (see Corollary 1.5 below).

For another source of counterexamples we refer to [28], where Murphy constructs exact closed Lagrangian submanifolds inside certain symplectisations (which hence also are displaceable, but not uniruled). However, in this case the non-compact symplectic manifolds are not given as the completion of a compact Liouville domain.

Definition 1.3. Let $L \subset (X, d\alpha)$ be a Lagrangian submanifold of a simply connected exact symplectic manifold having vanishing first Chern class, and fix a primitive α of the symplectic form. Let $\sigma \in H^1(L, \mathbf{R})$ be induced by the pull-back of α to L, i.e. the so-called *symplectic action class*, and let $\mu \in H^1(L, \mathbf{Z})$ denote the Maslov class of L. We say that L is *monotone* if

$$\sigma = K\mu$$

holds for some $K > 0$.

Previous results show that uniruledness indeed holds for many classes of closed displaceable Lagrangian submanifolds. For instance, Biran and Cornea [3] showed this for certain monotone displaceable Lagrangian submanifolds, a result which in particular applies to monotone Lagrangian submanifolds of $(\mathbf{C}^n, d\alpha_0)$. Subsequent work of Charette [5] moreover proves [1, Conjecture 3.15] for monotone Lagrangian submanifolds. See also Cornea and Lalonde [7], and Zehmisch [33] for similar results. Note that Theorem 1.6 below is also in the same spirit.

Finally, Borman and McLean [4] have shown that the Gromov width is finite for displaceable (not necessarily monotone) closed Lagrangian submanifolds of completions of Liouville domains (e.g. $\mathbf{C}^n$), given the topological condition that the Lagrangian submanifold admits a metric of non-positive scalar curvature. More precisely, it is shown that the Gromov width is bounded from above by four times the displacement energy in this case.

1.3. Results

1.3.1. Counterexamples to Conjecture 3.15 in [1]

Again, let $(X, d\alpha)$ be the completion of a Liouville domain having a non-compact cylindrical end

$$([0, +\infty) \times Y, d(e^t \lambda)).$$

Proposition 1.4. *Let* $L \subset (X, d\alpha)$ *be a closed Lagrangian submanifold, which is of the form*

$$L \cap \{(t, x) \, ; \, t \in [N, N+\varepsilon]\} = [N, N+\varepsilon] \times \Lambda$$

for some $N \geq 0$ and $\varepsilon > 0$. If $e^t \lambda$ pulled back to $L_{\mathrm{cap}} := L \cap \{(t,x); t \geq N\}$ is exact and, moreover, has a primitive that vanishes along the boundary, it follows that:

(1) For any $A > 0$, there exists a compatible almost complex structure J_A on $(X, d\alpha)$ for which every non-trivial connected J_A-holomorphic curve whose boundary is contained in L and, moreover, intersects L_{cap}, has $d\alpha$-area at least A. In particular, L is not uniruled;

(2) L has infinite relative Gromov width.

The assumptions of the above proposition are obviously satisfied for L obtained as the concatenation of a Lagrangian filling and an exact Lagrangian cap. However, the construction of displaceable such examples is highly non-trivial. The only examples in $\mathbf{C}^n$, $n \geq 3$, known to the author are the examples in [10], which are constructed using an h-principle. We give an outline in Section 1.4.

In the case of $\mathbf{C}^n$ for $n > 3$ even, the Lagrangian submanifolds produced by the construction in [10] are all non-orientable (see Remark 1.12). To find orientable examples in $\mathbf{C}^n$ for arbitrary $n \geq 3$, we proceed as follows. For each $k > 0$, the construction in [10] provides a non-monotone Lagrangian embedding of $S^1 \times S^{2k}$ into $\mathbf{C}^{1+2k}$ satisfying the assumptions of Proposition 1.4. Given the existence of these embeddings, it is now easy to show the following.

Corollary 1.5. *Let*
$$S := S^{l_1} \times \ldots \times S^{l_m}$$
be an M-dimensional product of spheres of arbitrary dimensions. For each $k > 0$ there exists a Lagrangian embedding of $S^1 \times S^{2k} \times S$ into $\mathbf{C}^{1+2k+M}$ having infinite Gromov width.

Proof. The embedding can be constructed using the following technique. The symplectic manifold
$$(\mathbf{C}^n \times T^* S^l, \omega_0 \oplus d\theta_{S^l}) \simeq (T^* \mathbf{R}^n \times T^* S^l, d\theta_{\mathbf{R}^n} \oplus d\theta_{S^l})$$
$$\simeq (T^*(\mathbf{R}^n \times S^l), d\theta_{\mathbf{R}^n \times S^l}) \subset (T^* \mathbf{R}^{n+l}, d\theta_{\mathbf{R}^{n+l}})$$
symplectically embeds into $\mathbf{C}^{n+l}$, where θ_N denotes the tautological one-form on $T^* N$. Given a Lagrangian submanifold $L \subset \mathbf{C}^n$, its product $L \times 0_{S^l} \subset \mathbf{C}^n \times T^* S^l$ with the zero-section is clearly Lagrangian, and the above inclusion induces a Lagrangian embedding of $L \times S^l$ into $\mathbf{C}^{n+l}$.

Observe that if $L_1 \subset (X_1, \omega_1)$ and $L_2 \subset (X_2, \omega_2)$ both have infinite Gromov width, then the same is true for their product
$$L_1 \times L_2 \subset (X_1 \times X_2, \omega_1 \oplus \omega_2).$$

The result now follows from the fact that the zero-section inside a cotangent bundle has infinite Gromov width, together with the observation that the Lagrangian embedding of $S^1 \times S^{2k}$ into $\mathbf{C}^{1+2k}$ constructed in [10] satisfies Proposition 1.4, and hence has infinite Gromov width as well. $\square$

The construction used in the proof of the above corollary is related to the front-spinning construction due to Ekholm, Etnyre and Sullivan [12, Section 4.4], which later was generalised by Golovko [21] (see also Remark 1.13 below).

It is interesting to note that the situation is very different for Lagrangian embeddings of $S^1 \times S^{2k+1}$ into $\mathbf{C}^{2+2k}$ when $k > 0$. Namely, work by Fukaya and Oh [20, Proposition 2.10] implies that such a Lagrangian submanifold is monotone. See also [7, Corollary 4.6] for a proof of the uniruledness of these Lagrangian submanifolds.

Finally, a sufficiently stabilised Legendrian knot inside the standard contact three-sphere (S^3, λ_0) (see Section 1.4) admits an exact Lagrangian cap in the symplectisation by a result due to Lin [26]. These caps can be used to construct *non-orientable* Lagrangian submanifolds of $\mathbf{C}^2$ satisfying the assumptions of Proposition 1.4. This construction is related to the exact Lagrangian immersions constructed explicitly by Sauvaget in [31].

1.3.2. Implications for the Legendrian contact homology of Λ

We will restrict our attention to the case of a *contactisation* of an exact symplectic manifold $(P, d\theta)$, by which we mean a contact manifold of the form

$$(Y, \lambda) = (P \times \mathbf{R}, dz + \theta),$$

where z denotes a coordinate of the $\mathbf{R}$-factor. For technical reasons we will make the assumption that $(P, d\theta)$ is symplectomorphic to the completion of a Liouville domain. Observe that a Legendrian submanifold of $P \times \mathbf{R}$ projects to an exact Lagrangian immersion in $(P, d\theta)$ and, conversely, a generic exact Lagrangian immersion lifts to a Legendrian submanifold of $P \times \mathbf{R}$.

An important example of a contactisation of the above form is the one-jet space

$$(J^1 M = T^* M \times \mathbf{R}, dz + \theta_M),$$

where θ_M is the Liouville form on $T^* M$, as well as the standard contact $(2n+1)$-space

$$(J^1 \mathbf{R}^n = \mathbf{C}^n \times \mathbf{R}, dz + \theta_{\mathbf{R}^n} = dz - \alpha_0).$$

The displaceability of a Lagrangian submanifold L, together with the non-obstructedness of Floer theories of different kinds associated with L, has in many

cases been shown to imply uniruledness. For instance, see [7, Corollary 1.18]. We will here follow a similar path, by considering the Legendrian contact homology differential graded algebra (DGA for short), also called the *Chekanov–Eliashberg algebra*. This is an invariant that can be associated with the Legendrian lift of an exact Lagrangian immersion. It was introduced in [18] by Eliashberg, Givental, and Hofer, and in [6] by Chekanov. We will be using the version in [15] by Ekholm, Etnyre, and Sullivan, which we outline in Section 3.1 below.

By Weinstein's Lagrangian neighbourhood theorem, the exact Lagrangian immersion of L can be extended to a symplectic immersion of the co-disc bundle $(D^*L, d\theta_L)$ of some radius. Given a Riemannian metric g on L, there is a natural construction of a compatible almost complex structure J_g on T^*L. We use $\mathcal{J}_L$ to denote the compatible almost complex structures on $(X, d\alpha)$, which, in some neighbourhood of L, is obtained by the push-forward of such an almost complex structure J_g. We refer to [11, Remark 6.1] for more details. The proof of the following theorem is obtained by adapting the proof of [11, Theorem 5.5] to the algebraic setup developed in Section 3.2.

Theorem 1.6. *Let R be a unital ring and $L \subset (X, d\alpha)$ be a displaceable exact Lagrangian immersion of a closed manifold. If $1+1 \neq 0$ in R, then we moreover make the assumption that L is a spin manifold, and we fix the choice of a spin structure. Suppose that the Chekanov–Eliashberg algebra with coefficients in R of the Legendrian lift of L is not acyclic (with or without Novikov coefficients). It follows that, for any $J \in \mathcal{J}_L$ and $x \in L$, there exists a J-holomorphic disc in X having boundary on L, one positive boundary puncture, possibly several negative boundary punctures, and a boundary-point passing through x.*

Remark 1.7. An *augmentation* is a unital DGA morphism taking values in the coefficient ring (considered as a trivial DGA). The existence of an augmentation should be seen as a certain non-obstructedness of the Legendrian contact homology, and in particular it implies that the DGA is not acyclic. Under the additional assumption that the Chekanov–Eliashberg algebra of L admits an augmentation, the consequence of Theorem 1.6 can be seen to follow from [11, Theorem 5.5]. However, because of the non-commutativity of the DGA under consideration, this condition is strictly stronger than being non-acyclic. We refer to [17] for examples.

Corollary 1.8. *If $L \subset (X, d\alpha)$ is a displaceable exact Lagrangian immersion of a closed manifold obtained as the concatenation of an immersed exact Lagrangian filling and an embedded exact Lagrangian cap, then the Chekanov–Eliashberg algebra of L is acyclic (even when using Novikov coefficients).*

Proof. L can be seen to satisfy the assumptions of Proposition 1.4, with the only difference that the complement of L_{cap} is immersed. Part (1) of Proposition 1.4 still applies, producing a compatible almost complex structure J_A satisfying the property that there exists no J_A-holomorphic discs in X with boundary on L, one positive puncture, possibly several negative boundary punctures, and boundary passing through L_{cap}. Here we have used the fact that the pseudo-holomorphic discs of the latter kind have a uniform bound on their symplectic area. Furthermore, it is readily seen that the almost complex structure J_A produced by Proposition 1.4 can be taken to satisfy $J_A \in \mathcal{J}_L$.

The Chekanov–Eliashberg algebra of L can now be seen to be acyclic, since we otherwise could apply Theorem 1.6, whose conclusion clearly contradicts the non-existence of the J_A-holomorphic discs established above. $\square$

Assume that $\Lambda \subset (P \times \mathbf{R}, dz + \theta)$ admits an exact Lagrangian cap in the symplectisation. In Section 3.3 below we construct a displaceable exact Lagrangian immersion

$$L_\Lambda \subset (P \times \mathbf{C}, d(\theta \oplus x\, dy)),$$

which can be obtained as the concatenation of an exact immersed Lagrangian filling with the exact Lagrangian cap. Moreover, the Chekanov–Eliashberg algebra of L_Λ without Novikov coefficients is homotopy equivalent to that of Λ (see Lemma 3.5). Using the above corollary, we can now make the following conclusion.

Corollary 1.9. *Suppose that the closed Legendrian submanifold*

$$\Lambda \subset (P \times \mathbf{R}, dz + \theta)$$

admits an exact Lagrangian cap

$$L_{\Lambda, \varnothing} \subset (\mathbf{R} \times (P \times \mathbf{R}), d(e^t(dz + \theta))).$$

Then the Chekanov–Eliashberg algebra of Λ with $\mathbf{Z}_2$-coefficients is acyclic when not using Novikov coefficients.

If $L_{\Lambda, \varnothing}$ is a spin cobordism, it follows that the same is true with $\mathbf{Z}$-coefficients, given that the Chekanov–Eliashberg algebra of Λ is induced by the choice of a spin structure on the cap.

Remark 1.10. All the exact Lagrangian caps constructed in [19] are caps of so-called *loose Legendrian submanifolds*, as defined in [27]. By an explicit computation (see [12, Section 4.3] and [27, Section 8]) it follows that the Chekanov–Eliashberg algebra of a loose Legendrian submanifold, with or without Novikov coefficients, is acyclic.

It is important to note that a Legendrian submanifold satisfying the assumptions of Corollary 1.9 still may have a non-acyclic Chekanov–Eliashberg algebra *when using Novikov coefficients*, as the following example shows. In particular, a Legendrian submanifold admitting an exact Lagrangian cap is not necessarily loose.

Example 1.11. In [9] the Legendrian torus $L_{1,1} \subset J^1 \mathbf{R}^2$ was constructed, and it was shown to have a Chekanov–Eliashberg algebra having non-trivial homology when Novikov coefficients are used, but which becomes acyclic without Novikov coefficients. We will now argue that $L_{1,1}$ admits an exact Lagrangian cap inside the symplectisation

$$(\mathbf{R} \times J^1 \mathbf{R}^2, d(e^t(dz + \theta_{\mathbf{R}^2}))).$$

Let $\Lambda_0 \subset J^1 \mathbf{R}^2$ denote the *loose Legendrian two-sphere* (see [27]). Observe that the loose two-sphere admits an exact Lagrangian cap by [19]. The existence of an exact Lagrangian cap of $L_{1,1}$ will be established by showing the existence of a proper embedded exact Lagrangian cobordism

$$V \subset (\mathbf{R} \times J^1 \mathbf{R}^2, d(e^t(dz + \theta_{\mathbf{R}^2})))$$

inside the symplectisation, which, outside of a compact set, is of the form

$$((-\infty, -N) \times L_{1,1}) \cup ((N, +\infty) \times \Lambda_0)$$

for some $N > 0$. We will moreover require that the pull-back of $e^t \lambda$ to V has a primitive that vanishes outside of a compact set.
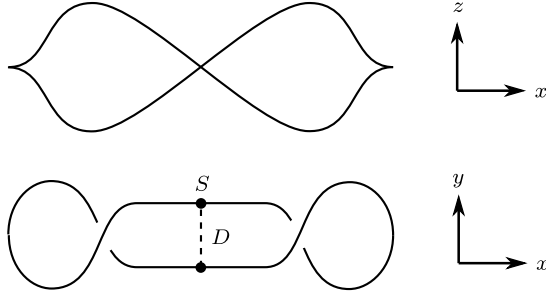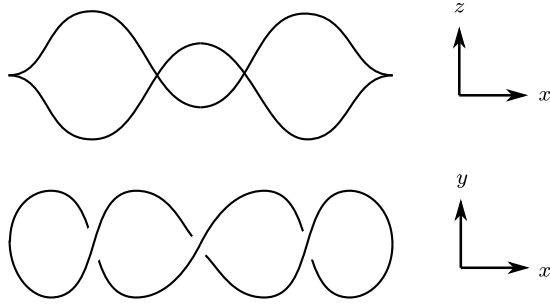
We start by describing the Legendrian torus $L_{1,1} \subset J^1 \mathbf{R}^2$. Let

$$\mathbf{R}/2\pi \mathbf{Z} \ni \theta \longmapsto (x_K(\theta), y_K(\theta), z_K(\theta))$$

be a parametrisation of the Legendrian knot $K \subset J^1 \mathbf{R} = \mathbf{R}^2 \times \mathbf{R}$ shown in Figure 1. This parametrisation may be supposed to satisfy the relation

$$(x_K(\theta + \pi), y_K(\theta + \pi), z_K(\theta + \pi)) = (-x_K(\theta), -y_K(\theta), z_K(\theta)),$$

from which it follows that one can rotate this knot to form an embedded Legendrian torus

Figure 1. The Legendrian knot K.

Figure 2. The Legendrian knot K' obtained by a Legendrian ambient surgery on $S \subset K$.

$$(\mathbf{R}/\pi\mathbf{Z}) \times (\mathbf{R}/2\pi\mathbf{Z}) \longrightarrow J^1\mathbf{R}^2 = \mathbf{R}^2 \times \mathbf{R}^2 \times \mathbf{R},$$

$$(\theta_1, \theta_2) \longmapsto \begin{pmatrix} (x_K(\theta_2+\theta_1)\cos\theta_1, \, x_K(\theta_2+\theta_1)\sin\theta_1) \\ (y_K(\theta_2+\theta_1)\cos\theta_1, \, y_K(\theta_2+\theta_1)\sin\theta_1) \\ z_K(\theta_2+\theta_1) \end{pmatrix}.$$

This torus coincides with $L_{1,1}$ as constructed in [9].

We choose the following representative of the loose two-sphere $\Lambda_0 \subset J^1\mathbf{R}^2$. Let

$$\mathbf{R}/2\pi\mathbf{Z} \ni \theta \longmapsto (x_{K'}(\theta), y_{K'}(\theta), z_{K'}(\theta))$$

be a parametrisation of the Legendrian knot $K' \subset J^1\mathbf{R} = \mathbf{R}^2 \times \mathbf{R}$ shown in Figure 2. This parametrisation may be supposed to satisfy the relation

$$(x_{K'}(-\theta), y_{K'}(-\theta), z_{K'}(-\theta)) = (-x_{K'}(\theta), -y_{K'}(\theta), z_{K'}(\theta)),$$

from which it follows that the image of

$$(\mathbf{R}/2\pi\mathbf{Z})\times(\mathbf{R}/2\pi\mathbf{Z})\longrightarrow J^1\mathbf{R}^2=\mathbf{R}^2\times\mathbf{R}^2\times\mathbf{R},$$

$$(\theta_1,\theta_2)\longmapsto\begin{pmatrix}(x_{K'}(\theta_2)\cos\theta_1,x_{K'}(\theta_2)\sin\theta_1)\\(y_{K'}(\theta_2)\cos\theta_1,y_{K'}(\theta_2)\sin\theta_1)\\z_{K'}(\theta_2)\end{pmatrix},$$

can be identified with an embedded Legendrian sphere. In this case, the obtained sphere is the loose Legendrian two-sphere $\Lambda_0\subset J^1\mathbf{R}^2$.

Legendrian ambient surgery is a construction, which was introduced in [8, Section 4] and which roughly can be described as follows. Let $\Lambda\subset(Y,\lambda)$ be a Legendrian embedding. Suppose that we are given a so-called *isotropic surgery disc* in Y with boundary $S\subset\Lambda$, together with the choice of a Lagrangian frame of its symplectic normal bundle. This data determines a Legendrian embedding $\Lambda_S\subset(Y,\lambda)$ of the manifold obtained from Λ by surgery on the sphere $S\subset\Lambda$. Moreover, the construction provides an exact Lagrangian cobordism from Λ to Λ_S, a so-called *elementary exact Lagrangian cobordism*.

We claim that one can construct the sought cobordism V as the elementary exact Lagrangian cobordism associated with a Legendrian ambient surgery on $L_{1,1}$. More precisely, the Legendrian plane $\{(0,0)\}\times\mathbf{R}^2\times\{0\}\subset J^1\mathbf{R}^2$ intersects $L_{1,1}$ in a circle S, and we use D to denote the Legendrian disc bounded by S in this plane. The disc D is an isotropic surgery disc with boundary on $S\subset L_{1,1}$, and it uniquely determines a Legendrian ambient surgery on $S\subset L_{1,1}$. The produced Legendrian embedding $(L_{1,1})_S\subset J^1\mathbf{R}^2$ is thus a sphere, which, moreover, can be seen to be the loose two-sphere Λ_0. The sought exact Lagrangian cobordism V can now be taken to be the elementary exact Lagrangian cobordism from $L_{1,1}$ to Λ_0 provided by the construction.

There is an analogous construction for the Legendrian knot K. Namely, choose the isotropic surgery disc to be given by $D\subset J^1\mathbf{R}$ shown in Figure 1, whose boundary is the 0-sphere $S\subset K$. The corresponding Legendrian ambient surgery on S produces an elementary exact Lagrangian cobordism from K to the Legendrian knot $K'\subset J^1\mathbf{R}$ shown in Figure 2. In this dimension, the cobordism and the surgery construction was also described in [16].

1.4. Existence of exact Lagrangian caps and constructions

In [27] Murphy introduced the concept of a *loose* Legendrian n-dimensional submanifold for $n\geq2$. Moreover, loose Legendrian submanifolds where shown to satisfy an h-principle.

Assume that we are given a loose Legendrian submanifold $\Lambda\subset(Y,\lambda)$. Using the above h-principle it was shown in [19] by Eliashberg and Murphy that, under

some homotopy-theoretic assumptions, an exact *immersed* Lagrangian cap of Λ is Lagrangian regular homotopic (through exact Lagrangian immersions) to an embedded exact Lagrangian cap. Recall that exact Lagrangian immersions satisfy an h-principle by Gromov [23] and Lees [25], but that there is no such h-principle for general Lagrangian embeddings.

Ekholm, Eliashberg, Murphy and Smith [10] used the h-principle for exact Lagrangian caps of loose Legendrian submanifolds to construct many interesting examples of Lagrangian embeddings inside $\mathbf{C}^n$ when $n \geq 3$. These examples are all constructed as follows. First, consider an embedded exact Lagrangian cap inside

$$(\mathbf{R} \times S^{2n-1}, d(e^t \lambda_0)), \quad \lambda_0 := \frac{1}{2} \sum_{j=1}^{n} (x_j \, dy^j - y_j \, dx^j),$$

where λ_0 thus restricts to the standard contact one-form on $S^{2n-1} \subset \mathbf{C}^n$. This cap is then concatenated with a (non-exact) Lagrangian filling inside

$$(\mathbf{C}^n, \omega_0 = d\lambda_0),$$

which moreover can be constructed explicitly. Observe that the result is a closed Lagrangian submanifold of the standard symplectic manifold $(\mathbf{C}^n, \omega_0)$, which hence is displaceable. These Lagrangian submanifolds have been shown to satisfy many surprising properties.

We now restrict our attention to the construction inside $\mathbf{C}^3$. Starting with the loose Legendrian two-sphere $\Lambda_0 \subset (S^5, \lambda_0)$ and given any three-manifold M, it follows from the theory in [19] that there exists an exact Lagrangian cap

$$L_{\Lambda_0, \varnothing} \subset (\mathbf{R} \times S^5, d(e^t \lambda_0))$$

inside the symplectisation, which is diffeomorphic to $M \backslash \{\mathrm{pt}\}$ and whose Maslov class vanishes.

A standard construction (see [10]) produces an exact *immersed* filling $\tilde{L}_{\varnothing, \Lambda_0} \subset \mathbf{C}^3$ of Λ_0, which is diffeomorphic to a three-ball and which has a single transverse double-point.

Observe that all fillings of Λ_0 indeed must have a double-point, since the concatenation of the cap and the filling otherwise would be a closed exact Lagrangian submanifold of $\mathbf{C}^3$, thus contradicting a theorem of Gromov [22]. Finally, it can be checked that the grading of the double-point of $\tilde{L}_{\varnothing, \Lambda_0}$, thought of as a generator in Legendrian contact homology, here must be equal to one (see also Remark 1.12 below). We refer to [13] for the definition of this grading.

The exact Lagrangian immersion of M obtained by concatenating the exact Lagrangian cap $L_{\Lambda_0, \varnothing}$ with the exact Lagrangian immersion $\tilde{L}_{\varnothing, \Lambda_0}$ is interesting

in itself. For instance, its number of double-points is in general strictly less than $\frac{1}{2} \dim H(M; \mathbf{Z}_2)$. By [11, Theorem 1.2], this is a lower bound on the number of transverse double-points for an exact Lagrangian immersion in $\mathbf{C}^n$ whose Legendrian lift has a Chekanov–Eliashberg algebra admitting an augmentation.

The double-point of $\tilde{L}_{\varnothing,\Lambda}$ can be removed by a Lagrange surgery, as defined in [24] and [30]. A suitable choice of such a surgery produces a (non-exact) embedded Lagrangian filling $L_{\varnothing,\Lambda_0}$ of Λ_0 diffeomorphic to $(S^2 \times S^1) \backslash \{\mathrm{pt}\}$. The fact that the grading of the double-point is equal to one implies that the Maslov class of $L_{\varnothing,\Lambda_0}$ vanishes for this choice of Lagrange surgery.

In conclusion, for every three-manifold M, the construction produces a Lagrangian embedding of $M \# (S^2 \times S^1)$ into $\mathbf{C}^3$ with vanishing Maslov class. These were the first known examples of closed Lagrangian submanifolds of $\mathbf{C}^n$ satisfying this property. Moreover, by construction, these Lagrangian submanifolds satisfy the assumptions of Proposition 1.4.

In higher dimensions, the same technique gives Lagrangian embeddings of $S^1 \times S^{2k}$ into $\mathbf{C}^{1+2k}$ for each $k > 0$, see [10, Corollary 1.6]. Furthermore, the Maslov class of these embeddings can be seen to evaluate to $2-2k$ on the unique generator $\gamma \in H_1(S^1 \times S^{2k}; \mathbf{Z})$ of positive symplectic action.

Remark 1.12. A Lagrange surgery on the double-points of an orientable exact Lagrangian immersion $L \subset \mathbf{C}^n$ obtained by concatenating an exact immersed Lagrangian filling and an exact embedded Lagrangian cap always produces a non-orientable Lagrangian submanifold in the case $n=2k$. To see this, first observe that Corollary 1.8 implies that the Chekanov–Eliashberg algebra of L is acyclic. Second, there must exist a double-point of odd degree, since the differential decreases the degree by one, and since the unit lives in degree zero. Finally, a Lagrange surgery on a double-point of odd degree always creates a non-orientable submanifold in these dimensions.

Remark 1.13. The examples in $\mathbf{C}^n$ for $n=2k$ produced by Corollary 1.5 are constructed by a version of the front-spinning construction. It can be seen that these examples still are obtained by the concatenation of a Lagrangian filling and an exact Lagrangian cap. Namely, suppose that we are given an exact Lagrangian cap

$$L_{\Lambda,\varnothing} \subset \mathbf{R} \times J^1 \mathbf{R}^n$$

and a non-exact Lagrangian filling

$$L_{\varnothing,\Lambda} \subset \mathbf{R} \times J^1 \mathbf{R}^n.$$

The front-spinning construction produces a Legendrian embedding of $\Lambda \times S^l$ inside $J^1 \mathbf{R}^{n+l}$. This spinning extends to the symplectisation by [21], producing an exact Lagrangian cap

$$L_{\Lambda \times S^l, \varnothing} \subset \mathbf{R} \times J^1 \mathbf{R}^{n+l}$$

and a non-exact Lagrangian filling

$$L_{\varnothing, \Lambda \times S^l} \subset \mathbf{R} \times J^1 \mathbf{R}^{n+l}$$

of $\Lambda \times S^l$, which are diffeomorphic to $L_{\Lambda, \varnothing} \times S^l$ and $L_{\varnothing, \Lambda} \times S^l$, respectively.

2. Proof of Proposition 1.4

Consider a smooth cut-off function $\rho \colon [0, +\infty) \to \mathbf{R}$ satisfing $\rho'(t) \geq 0$ for all t, $\rho(t) = 0$ for $t \leq N$, and $\rho(t) = 1$ for $t \geq N + \varepsilon$. The flow

$$\phi^s \colon [0, +\infty) \times Y \longrightarrow [0, +\infty) \times Y,$$
$$(t, y) \longmapsto (t + \rho(t)s, y),$$

defined on the cylindrical end of X satisfies

$$(\phi^s)^*(e^t \lambda) = e^{\rho(t)s} e^t \lambda.$$

In particular, ϕ^s coincides with the so-called *Liouville flow* on $[N + \varepsilon, +\infty) \times Y$. We extend ϕ^s to an isotopy defined on all of X by prescribing it to be the identity outside of the cylindrical end.

The condition that

$$L \cap \{(t, x) \,; t \in [N, N + \varepsilon]\} = [N, N + \varepsilon] \times \Lambda \subset (\mathbf{R} \times Y, d(e^t \lambda))$$

is a Lagrangian submanifold is equivalent to the pull-back of $e^t \lambda$ being closed. Moreover, the equality

$$d(e^t \lambda) = e^t \, dt \wedge \lambda + e^t \, d\lambda$$

shows that the pull-back of λ to this Lagrangian cylinder in fact vanishes. Let $f \colon L_{\mathrm{cap}} \to \mathbf{R}$ be a primitive of $e^t \lambda$ pulled back to L_{cap} that vanishes along the boundary $\partial L_{\mathrm{cap}}$ and, by the above, on $L \cap \{(t, x) ; t \in [N, N + \varepsilon]\}$ as well. The function $e^s f$ can now be seen to be a primitive of $(\phi^s)^*(e^t \lambda)$ pulled back to L_{cap} (which thus also vanishes along the boundary). Since the isotopy ϕ^s fixes $L \setminus L_{\mathrm{cap}}$, we see that

$$(\phi^s|_L)^* \alpha = (\phi^0|_L)^* \alpha + d((e^s - 1)f)$$

is a path of cohomologous one-forms on L. From this it follows that

$$L_s := \phi^s(L) \subset (X, d\alpha), \quad s \geq 0,$$

is a path of Lagrangian submanifolds and, moreover, that the isotopy L_s may be realised by a time-dependent Hamiltonian isotopy.

1. *Non-uniruledness.* Weinstein's neighbourhood theorem produces a contact-form preserving identification of a neighbourhood of $\Lambda \subset Y$ with a neighbourhood of the zero-section in $(J^1\Lambda, dz + \theta_\Lambda)$. Using this map, one constructs a symplectic identification of a neighbourhood of $[N, +\infty) \times \Lambda \subset \mathbf{R} \times Y$ and

$$D^*\Lambda \times [\log N, +\infty) \times [-\delta, \delta] \subset (D^*\Lambda \times \mathbf{C}, d\theta_\Lambda \oplus d(x\,dy))$$

for some fixed radius of the co-disc bundle and $\delta > 0$, such that moreover

$$0_\Lambda \times [\log N, +\infty) \times \{0\} \subset D^*\Lambda \times \mathbf{C}$$

is identified with $[N, +\infty) \times \Lambda$ (see Section 3.3 for a similar identification). We choose a compatible almost complex structure $\tilde{J}$ on $(X, d\alpha)$, which, in the above neighbourhood, is taken to coincide with an almost complex structure on $D^*\Lambda \times \mathbf{C}$ that is invariant under translations of the x-coordinate.

Since $e^t\lambda$ pulled back to the set $L_s \cap \{(t, x); t \geq N\}$ is exact, any connected $\tilde{J}$-holomorphic curve whose boundary is located on L_s and, moreover, intersects $L_s \cap \{(t, x); t \geq N+s\}$, has the property that its boundary passes through each slice

$$L_s \cap \{(t, x); t = t_0\} = \{t_0\} \times \Lambda \subset \mathbf{R} \times Y, \quad t_0 \in [N, N+s],$$

unless the curve is constant.

The monotonicity property for the symplectic area of pseudo-holomorphic curves with boundary on a Lagrangian submanifold (see [32, Proposition 4.7.2(ii)]) gives a constant $C > 0$ for which the following holds. Any non-constant $\tilde{J}$-holomorphic curve in $D^*\Lambda \times [-\delta, \delta]^2$ with boundary on

$$(0_\Lambda \times [-\delta, \delta] \times \{0\}) \cup \partial(D^*\Lambda \times [-\delta, \delta]^2),$$

whose boundary moreover passes through $0_\Lambda \times \{(0,0)\}$, has symplectic area at least $C > 0$.

Fix $A > 0$. Using the above monotonicity property, together with the translation invariance of $\tilde{J}$ and the above behaviour of the $\tilde{J}$-holomorphic curves with boundary on L_s, we finally make the following conclusion. For $s_A > 0$ sufficiently large, any non-constant connected $\tilde{J}$-holomorphic curve whose boundary is located on L_{s_A} and, moreover, intersects $L_{s_A} \cap \{(t, x); t \geq N+s_A\}$, has symplectic area at least A.

Finally, we can take J_A to be the pull-back of $\tilde{J}$ under a symplectomorphism that takes L to L_{s_A} and $L \cap \{(t,x); t \geq N\}$ to $L_{s_A} \cap \{(t,x); t \geq N + s_A\}$. Here we have used the fact that L_{s_A} is Hamiltonian isotopic to L.

2. *The Gromov width.* Observe that Weinstein's Lagrangian neighbourhood theorem can be used to symplectically embed the standard symplectic ball of some positive radius $r > 0$ inside $\{(t,x); t \geq N + \varepsilon\} \subset X$, so that the ball moreover intersects L precisely in the real part. By composing this embedding with ϕ^s, we see that L_s admits such a symplectic ball of radius $e^s r$. Finally, since L_s is Hamiltonian isotopic to L, it follows that L admits a symplectic ball of radius $e^s r$ as well. In conclusion, L has infinite Gromov width.

3. The Chekanov–Eliashberg algebra of a Legendrian submanifold admitting an exact Lagrangian cap

A smooth one-parameter family of Legendrian embeddings is called a *Legendrian isotopy*. Legendrian contact homology is an algebraic Legendrian isotopy invariant introduced in [18] by Eliashberg, Givental and Hofer and in [6] by Chekanov. Here we will use the version constructed in [14] and [15] by Ekholm, Etnyre and Sullivan, which is defined in the setting of a contactisation $(P \times \mathbf{R}, dz + \theta)$ equipped with the standard contact form, given some technical assumptions on $(P, d\theta)$. In particular, the theory is well-defined when $(P, d\theta)$ is symplectomorphic to the completion of a Liouville domain.

3.1. Legendrian contact homology

In the following we let $\Lambda \subset P \times \mathbf{R}$ be a closed Legendrian submanifold. We will use $\mathcal{Q}(\Lambda)$ to denote the set of *Reeb chords* on Λ, which are the non-trivial integral curves of ∂_z in $P \times \mathbf{R}$ having both end-points on Λ. Observe that the Reeb chords correspond bijectively to the double-points of the image of Λ under the canonical projection

$$\pi_{\mathrm{Lag}} \colon P \times \mathbf{R} \longrightarrow P.$$

We will now sketch the definition of Legendrian contact homology in this setting, we refer to [15] for more details.

The Chekanov–Eliashberg algebra $(\mathcal{A}(\Lambda; R), \partial)$ of Λ is a unital DGA induced by the choice of a compatible almost complex structure J on P. The underlying algebra is the unital, strictly non-commutative, graded algebra that is freely generated by $\mathcal{Q}(\Lambda)$ over the ring R. Here we assume that $\pi_{\mathrm{Lag}}(\Lambda)$ is a generic immersion having transverse double-points, which means that $\mathcal{Q}(\Lambda)$ is a finite set. The grading of

each generator is induced by the so-called *Conley–Zehnder index* associated with the Reeb chord. Furthermore, to each Reeb chord there is an associated action

$$\ell(c) := \int_c dz > 0.$$

Fix a generic compatible almost complex structure J on $(P, d\theta)$. The differential ∂ is an R-linear map defined on each generator by counting the rigid J-holomorphic discs in P having boundary on $\pi_{\mathrm{Lag}}(\Lambda)$ and boundary-punctures mapping to double-points, of which exactly one is positive. The differential is then extended to all of $\mathcal{A}(\Lambda; R)$ by the Leibniz rule

$$\partial(\mathbf{ab}) = \partial(\mathbf{a})\mathbf{b} + (-1)^{|\mathbf{a}|}\mathbf{a}\partial(\mathbf{b}),$$

where $\mathbf{a}$ and $\mathbf{b}$ denote words of generators. It can be checked that, if the coefficient in front of the word $b_1...b_m$ in the expression $\partial(a)$ is non-zero, then necessarily

$$\ell(a) > \ell(b_1) + ... + \ell(b_m),$$

as follows from the positivity of the symplectic area of J-holomorphic discs.

We may always use coefficients $R = \mathbf{Z}_2$. In the case when Λ is a spin manifold, we may also use $R = \mathbf{Z}$ after fixing a spin structure (see [14]). Hence, in the latter case, it is possible to define the theory with coefficients in an arbitrary ring R. It is also possible to define a richer invariant using so-called *Novikov coefficients*, where coefficients are chosen in the group ring $R[H_1(L)]$.

We have $\partial^2 = 0$ by [15, Lemma 2.5]. By [15, Theorem 1.1] it follows that the stable-tame isomorphism type of the Chekanov–Eliashberg algebra is independent of the choice of a generic J and invariant under Legendrian isotopy of Λ. In particular, its homology $\mathrm{HC}_\bullet(\Lambda; R)$ is a Legendrian isotopy invariant.

3.2. A linear complex over the characteristic algebra

Given a unital DGA morphism to a commutative algebra (viewed as a DGA with trivial differential), i.e. an *augmentation*, one obtains an induced complex with coefficients in the latter algebra by a so-called *linearisation* of the DGA [6]. The construction in this section can be seen as a straightforward generalisation of this construction to the case when the algebra is non-commutative (in particular, we will be considering the so-called *characteristic algebra* of the DGA). The non-commutativity implies that the obtained complex is a *bimodule* over the algebra.

Given a decomposition $\Lambda = \Lambda_1 \sqcup \Lambda_2$ of a disconnected Legendrian submanifold, where Λ_j is closed but not necessarily connected, we will use $\mathcal{Q}(\Lambda_1, \Lambda_2)$ to denote the set of Reeb chords starting on Λ_1 and ending on Λ_2.

Consider the R-submodule

$$\mathcal{A}(\Lambda_1, \Lambda_2; R) := \bigoplus_{c \in \mathcal{Q}(\Lambda_1, \Lambda_2)} \mathcal{A}(\Lambda_1; R) c \mathcal{A}(\Lambda_2; R) \subset \mathcal{A}(\Lambda_1 \cup \Lambda_2; R),$$

which naturally carries the structure of a free left $\mathcal{A}(\Lambda_2; R) \otimes \mathcal{A}(\Lambda_1; R)^{\mathrm{op}}$-module (note the order!). The module multiplication is here induced by

$$(a_2 \otimes a_1) \cdot \mathbf{a}_1 c \mathbf{a}_2 = \mathbf{a}_1 a_1 c a_2 \mathbf{a}_2, \quad c \in \mathcal{Q}(\Lambda_1, \Lambda_2),$$

where $a_j \in \mathcal{Q}(\Lambda_j)$ are generators, and $\mathbf{a}_j \in \mathcal{A}(\Lambda_j; R)$ are words of generators. Observe that $\mathcal{A}(\Lambda_1, \Lambda_2; R)$ can be identified with the canonical free $\mathcal{A}(\Lambda_2; R) \otimes \mathcal{A}(\Lambda_1; R)^{\mathrm{op}}$-module generated by $\mathcal{Q}(\Lambda_1, \Lambda_2)$ via

$$\Psi \colon \mathcal{A}(\Lambda_1, \Lambda_2; R) \longrightarrow \bigoplus_{c \in \mathcal{Q}(\Lambda_1, \Lambda_2)} (\mathcal{A}(\Lambda_2; R) \otimes \mathcal{A}(\Lambda_1; R)^{\mathrm{op}}) c,$$

$$\mathbf{a}_1 c \mathbf{a}_2 \longmapsto (\mathbf{a}_2 \otimes \mathbf{a}_1) c.$$

There is a filtration

$$\mathcal{A}(\Lambda_1 \cup \Lambda_2; R) =: \mathcal{A}^0 \supset \mathcal{A}^1 \supset \mathcal{A}^2 \supset ...,$$

where $\mathcal{A}^k$ is spanned by words in which at least k generators come from $\mathcal{Q}(\Lambda_1, \Lambda_2)$. By the topological properties of the boundary condition of the pseudo-holomorphic discs used in the definition of the differential, ∂ preserves this filtration. In other words, ∂ induces a differential ∂^1 on $\mathcal{A}^0 / \mathcal{A}^2$. Using the natural inclusion

$$\mathcal{A}(\Lambda_1, \Lambda_2; R) \subset \mathcal{A}^0 / \mathcal{A}^2,$$

∂^1 can moreover be seen to induce an R-linear differential on $\mathcal{A}(\Lambda_1, \Lambda_2; R)$. Obviously, ∂^1 is not $\mathcal{A}(\Lambda_2; R) \otimes \mathcal{A}(\Lambda_1; R)^{\mathrm{op}}$-linear in general. We will amend this problem by taking a suitable quotient of this algebra.

In the following we will work under the assumption that there is an isomorphism $\mathcal{A}(\Lambda_1; R) \simeq \mathcal{A}(\Lambda_2; R)$ of unital DGAs. This is for example true if Λ_2 is a translation of Λ_1 in the z-coordinate, followed by a sufficiently small perturbation. We use

$$\mathcal{C}_{\Lambda_j; R} := \mathcal{A}(\Lambda_j; R) / \mathcal{A}(\Lambda_j; R) \operatorname{Im}(\partial) \mathcal{A}(\Lambda_j; R), \quad j = 1, 2,$$

to denote the so-called *characteristic algebra* of $\mathcal{A}(\Lambda_j; R)$, which is the quotient by the two-sided ideal generated by the boundaries. This invariant was introduced by Ng and studied in [29]. Observe that, since we are working with unital algebras, we have the following characterisation.

Lemma 3.1. *The DGA* $\mathcal{A}(\Lambda_j; R)$ *is acyclic if and only if* $\mathcal{C}_{\Lambda_j; R} = 0$.

Since $\mathcal{C}_{\Lambda_1;R} \simeq \mathcal{C}_{\Lambda_2;R}$ holds by assumption, we will use $\mathcal{C}_R$ to denote either of them. The free left $\mathcal{C}_R$-module

$$\mathcal{C}(\Lambda_1, \Lambda_2; R) := \bigoplus_{c \in \mathcal{Q}(\Lambda_1, \Lambda_2)} \mathcal{C}_R c$$

has a natural structure as a non-free left $\mathcal{C}_R \otimes_R \mathcal{C}_R^{\mathrm{op}}$-module (i.e. its natural bimodule structure). Furthermore, we have the canonical projection

$$\pi \colon \bigoplus_{c \in \mathcal{Q}(\Lambda_1, \Lambda_2)} (\mathcal{C}_R \otimes_R \mathcal{C}_R^{\mathrm{op}}) c \longrightarrow \mathcal{C}(\Lambda_1, \Lambda_2; R),$$

$$([\mathbf{a}] \otimes [\mathbf{b}]) c \longmapsto [\mathbf{ab}] c,$$

of $\mathcal{C}_R \otimes_R \mathcal{C}_R^{\mathrm{op}}$-modules, where the domain is free.

Also, consider the surjective unital R-algebra morphism

$$\phi \colon \mathcal{A}(\Lambda_2; R) \otimes_R \mathcal{A}(\Lambda_1; R)^{\mathrm{op}} \longrightarrow \mathcal{C}_R \otimes_R \mathcal{C}_R^{\mathrm{op}}$$

induced by the quotient projection, which for generators $a_j \in \mathcal{Q}(\Lambda_j)$, $j = 1, 2$, takes the form

$$\phi(a_2 \otimes 1) = [a_2] \otimes [1] \quad \text{and} \quad \phi(1 \otimes a_1) = (-1)^{|a_1|} [1] \otimes [a_1].$$

There is an induced R-module morphism

$$\Phi \colon \bigoplus_{c \in \mathcal{Q}(\Lambda_1, \Lambda_2)} (\mathcal{A}(\Lambda_2; R) \otimes_R \mathcal{A}(\Lambda_1; R)^{\mathrm{op}}) c \longrightarrow \bigoplus_{c \in \mathcal{Q}(\Lambda_1, \Lambda_2)} (\mathcal{C}_R \otimes_R \mathcal{C}_R^{\mathrm{op}}) c.$$

Observe that, since ∂ is of degree -1, the kernel of ϕ is the same as the kernel of the canonical projection $\mathcal{A}(\Lambda_2; R) \otimes_R \mathcal{A}(\Lambda_1; R)^{\mathrm{op}} \to \mathcal{C}_R \otimes_R \mathcal{C}_R^{\mathrm{op}}$. The Leibniz rule now implies that ∂^1 preserves the kernel of the R-module morphism

$$\pi \circ \Phi \circ \Psi \colon \mathcal{A}(\Lambda_1, \Lambda_2; R) \longrightarrow \mathcal{C}(\Lambda_1, \Lambda_2; R),$$

and hence descends to an R-linear differential

$$\partial_{\mathcal{C}} \colon \mathcal{C}(\Lambda_1, \Lambda_2; R) \longrightarrow \mathcal{C}(\Lambda_1, \Lambda_2; R).$$

Finally, the Leibniz rule can also be seen to imply the following result.

Lemma 3.2. *The R-linear differential $\partial_{\mathcal{C}}$ is $\mathcal{C}_R \otimes_R \mathcal{C}_R^{\mathrm{op}}$-linear.*

In other words, we have produced a complex $(\mathcal{C}(\Lambda_1, \Lambda_2; R), \partial_{\mathcal{C}})$, which is finitely generated by the Reeb chords from Λ_1 to Λ_2, and which has coefficients in $\mathcal{C}_R \otimes_R \mathcal{C}_R^{\mathrm{op}}$.

We will now see that the invariance proof of the Chekanov–Eliashberg algebra implies an invariance result for this complex as well. However, there is one caveat— the coefficients $\mathcal{C}_R \otimes_R \mathcal{C}_R^{\mathrm{op}}$ are constructed using the characteristic algebra $\mathcal{C}_R$ of Λ_j, and it is not true that the characteristic algebra is invariant under Legendrian isotopy. Namely, it changes by a stabilisation under the birth/death of a pair of Reeb chords (see [29, Theorem 3.4] for more details). However, for us it will be sufficient to restrict our attention to Legendrian isotopies of a special form.

Theorem 3.3. ([15, Theorem 1.1]) *The homotopy type of $(\mathcal{C}(\Lambda_1, \Lambda_2; R), \partial_{\mathcal{C}})$ is independent of the choice of a regular compatible almost complex structure and invariant under Legendrian isotopies of $\Lambda_1 \cup \Lambda_2$, given that the Legendrian isotopy restricted to either of Λ_j, $j=1,2$, induces no births or deaths of Reeb chords.*

Proof. This result is an algebraic consequence of the invariance result [15, Theorem 1.1], which, in turn, depends on [14, Section 4.3]. We let Λ_j^s, $s \in [0,1]$, be the isotopy restricted to Λ_j, where $\Lambda_j^0 = \Lambda_j$.

Observe that the isomorphism class of the Chekanov–Eliashberg algebra does not depend on the choice of a generic almost complex structure by [14, Lemma 4.13]. The same result also shows that, since there are no births or deaths of Reeb chords in the one-parameter family Λ_j^s by assumption, the Chekanov–Eliashberg algebras of Λ_j^s are all isomorphic (for generic $s \in [0,1]$). In particular, we may identify the coefficient ring of the complex $(\mathcal{C}(\Lambda_1^s, \Lambda_2^s; R), \partial_{\mathcal{C},s})$ with $\mathcal{C}_R \otimes_R \mathcal{C}_R^{\mathrm{op}}$ for generic $s \in [0,1]$.

The proof of invariance for the Chekanov–Eliashberg algebra provides a so-called *stable tame isomorphism*

$$\Phi \colon (\mathcal{A}(\Lambda_1^0 \cup \Lambda_2^0; R), \partial_0) \longrightarrow (\mathcal{A}(\Lambda_1^1 \cup \Lambda_2^1; R), \partial_1)$$

of DGAs. Considering the topology of the boundary condition used for the pseudo-holomorphic discs that appear in the construction of Φ, we see that Φ preserves the filtration considered above, and hence that it descends to an R-module chain map

$$\Phi^1 \colon (\mathcal{A}(\Lambda_1^0, \Lambda_2^0; R), \partial_0^1) \longrightarrow (\mathcal{A}(\Lambda_1^1, \Lambda_2^1; R), \partial_1^1).$$

To establish the sought homotopy equivalence, there are two basic cases that one now needs to consider: First, the case when Φ^1 is an isomorphism is immediate. Second, the case when Φ^1 is an inclusion into a stabilisation follows by an explicit construction of a chain homotopy. $\square$

3.3. Construction of the exact Lagrangian immersion L_Λ

In the following we assume that the Legendrian submanifold $\Lambda \subset (P \times \mathbf{R}, dz + \theta)$ admits an exact Lagrangian cap $L_{\Lambda, \varnothing}$ inside the symplectisation

$$(\mathbf{R} \times (P \times \mathbf{R}), d(e^t(dz + \theta))).$$

Let $\phi^t \colon P \to P$ denote the time-t flow of the Liouville vector field ζ associated with θ, which is uniquely defined by the requirement that $\iota_\zeta \, d\theta = \theta$. Observe that $(\phi^t)^* \theta = e^t \theta$. Endow $\mathbf{R}_{>0} \times \mathbf{R} \subset \mathbf{R}^2 = \mathbf{C}$ with the standard symplectic form $\omega_0 = dx \wedge dy$, and consider the exact symplectomorphism

$$F \colon (\mathbf{R} \times (P \times \mathbf{R}), d(e^t(dz + \theta))) \longrightarrow (P \times (\mathbf{R}_{>0} \times \mathbf{R}), d(\theta \oplus x \, dy)),$$

$$(t, (p, z)) \longmapsto (\phi^t(p), (e^t, z)).$$

Let $(g, h) \colon \Lambda \to (P \times \mathbf{R}, dz + \theta)$ be the Legendrian embedding under consideration, which thus satisfies $g^* \theta + dh = 0$. We construct the immersion

$$G \colon \mathbf{R} \times \Lambda \longrightarrow (P \times \mathbf{R}^2, d(\theta \oplus x \, dy)),$$

$$(t, q) \longmapsto (\phi^{\log \rho(t)} \circ g, (t, \rho'(t) h(q))),$$

which is an exact Lagrangian immersion, as follows by the identity

$$G^*(\theta \oplus x \, dy) = d((t\rho'(t) - \rho)h).$$

We will choose $\rho \colon \mathbf{R} \to \mathbf{R}_{>0}$ to be a Morse function satisfying

$$\begin{cases} \rho(t) = \rho(-t), & t \in \mathbf{R}, \\ \rho(t) = |t|, & |t| \geq 2, \\ \rho(t) = 1 + t^2/2, & |t| \leq \varepsilon, \\ \rho'(t) \neq 0, & t \neq 0, \end{cases}$$

for some $0 < \varepsilon < 1$. In particular, it follows that $G|_{\{t=0\}} = (g, (0,0))$, while

$$G(\mathbf{R} \times \Lambda) \cap \{(t, x) \, ; 2 \leq x \leq e^N\} = F(L_{\Lambda, \varnothing}) \cap \{(t, x) \, ; 2 \leq x \leq e^N\}$$

is an embedding. Here, we have translated the cap so that

$$L_{\Lambda, \varnothing} \cap \{(t, x) \, ; t \leq N\} = (-\infty, N] \times \Lambda$$

is satisfied for some $N > 1$. See [13, Section 10.2] and [14, Section 4.3.2] for a similar construction of an exact Lagrangian immersion of a cylinder $\mathbf{R} \times \Lambda$ using a Legendrian embedding Λ.

Consider the exact symplectomorphism

$$\tau \colon (P \times \mathbf{R}^2, d(\theta \oplus x \, dy)) \longrightarrow (P \times \mathbf{R}^2, d(\theta \oplus x \, dy)),$$

$$(p, (x, y)) \longmapsto (p, (-x, -y)),$$

which maps the Lagrangian immersion $G(\mathbf{R} \times \Lambda)$ to itself. Defining $L_\Lambda \subset P \times \mathbf{R}^2$ to be the union

$$(\tau(F(L_{\Lambda, \varnothing})) \cap \{(t, x) \, ; \, x \le -e^N\}) \cup (G(\mathbf{R} \times \Lambda) \cap \{(t, x) \, ; \, -e^N \le x \le e^N\})$$

$$\cup (F(L_{\Lambda, \varnothing}) \cap \{(t, x) \, ; \, x \ge e^N\}),$$

it follows that this is an exact Lagrangian immersion of the closed manifold formed by taking two copies of the cap $L_{\Lambda, \varnothing} \cap \{(t, x) ; t \ge 0\}$ and gluing them along the boundary using the canonical identification. Observe that all double-points of L_Λ are contained inside

$$L_\Lambda \cap \{(t, x) \, ; \, x = 0\} = \pi_{\mathrm{Lag}}(\Lambda) \times \{(0, 0)\}.$$

Lemma 3.4. ([14, Lemmata 4.14 and 4.15]) *For a compatible almost complex structure J on $(P, d\theta)$, every non-constant $J \oplus i$-holomorphic curve in $P \times \mathbf{C}$ with boundary on L_Λ which has compact image is contained inside $P \times \{(0, 0)\}$.*

Proof. Use $u \colon C \to P \times \mathbf{C}$ to denote the pseudo-holomorphic curve. Observe that the projection $\pi_{\mathbf{C}} \circ u$ to the $\mathbf{C}$-factor is holomorphic, and that the boundary is contained inside the projection $\pi_{\mathbf{C}}(L_\Lambda)$.

Using the exactness of the boundary condition, it follows that the pseudo-holomorphic curve must have a positive boundary puncture that is mapped to a double point of L_Λ. In particular, $\pi_{\mathbf{C}} \circ u$ has the origin in its image. By the argument in [14, Lemmata 4.14 and 4.15], which we now outline, it follows that $\pi_{\mathbf{C}} \circ u$ in fact must vanish constantly.

By construction, the projection $\pi_{\mathbf{C}}(L_\Lambda)$ satisfies

$$\pi_{\mathbf{C}}(L_\Lambda) \cap \{(t, x) \, ; \, -\varepsilon \le x \le \varepsilon\} = \pi_{\mathbf{C}} \circ G(\mathbf{R} \times \Lambda) \cap \{(t, x) \, ; \, -\varepsilon \le x \le \varepsilon\}$$

$$\subset V := \left\{ x + iy \, ; \, y \in \left[x \min_\Lambda h, x \max_\Lambda h \right] \text{ and } x \in [-\varepsilon, \varepsilon] \right\}$$

for some sufficiently small $\varepsilon > 0$. The open mapping principle can be seen to imply that

$$\pi_{\mathbf{C}} \circ u(C) \cap \{(t, x) \, ; \, -\varepsilon \le x \le \varepsilon\} \subset V.$$

However, unless $\pi_{\mathbf{C}} \circ u$ vanishes constantly, this contradicts the behaviour of the asymptotics of the positive boundary-puncture of u. $\square$

Lemma 3.5. *The Chekanov–Eliashberg algebras of L_Λ and Λ with $\mathbf{Z}_2$-coefficients are homotopy equivalent when not using Novikov coefficients. If $L_{\Lambda,\varnothing}$ is a spin cobordism and if the DGA of Λ is induced by a spin structure on the cap, then the same is true with $\mathbf{Z}$-coefficients.*

Proof. First, observe that there is a canonical identification of the generators, which preserves the grading. What remains is to show that the differentials agree.

By the above lemma we know that every $J\oplus i$-holomorphic disc with boundary on L_Λ lies in $P\times\{(0,0)\}$ and, hence, has boundary on $\pi_{\mathrm{Lag}}(\Lambda)\times\{(0,0)\}$.

In particular, we have established a bijective correspondence between the $J\oplus i$-holomorphic discs in $P\times\mathbf{C}$ with boundary on L_Λ that contribute to the differential of the Chekanov–Eliashberg algebra of L_Λ, and the J-holomorphic discs in P with boundary on $\pi_{\mathrm{Lag}}(\Lambda)$ that contribute to the differential of the Chekanov–Eliashberg algebra of Λ. For these choices of almost complex structure, this shows that the Chekanov–Eliashberg algebras of Λ and L_Λ are equal. $\square$

3.4. Proof of Theorem 1.6

The proof is an adaptation of the proof of [11, Theorem 5.5] to the current algebraic setup.

We start by describing the two-copy lift as constructed in [11, Section 3.1]. In a Weinstein neighbourhood of $L_1:=L\subset(X,d\alpha)$ given as the image of a co-disc bundle D^*L under a symplectic immersion, we let L_2 be given as the section df for a sufficiently C^1-small Morse function $f\colon L\to\mathbf{R}$.

We choose a Legendrian lift of $L_1\cup L_2$ to the contactisation $(X\times\mathbf{R},dz+\alpha)$ for which the z-coordinate satisfies

$$\min_{L_2}(z) > \max_{L_1}(z),$$
$$\max_{L_2}(z) = \max_{L_1}(z)+N,$$

for some large number $N>0$. In particular, this implies that a Reeb chord on $L_1\cup L_2$ either has both endpoints of one of L_j, $j=1,2$, or starts on L_1 and ends on L_2. In other words, there is a decomposition

$$\mathcal{Q}(L_1\cup L_2) = \mathcal{Q}(L_1)\sqcup\mathcal{Q}(L_2)\sqcup\mathcal{Q}(L_1,L_2)$$

of the set of Reeb chords.

Recall that Reeb chords correspond to double-points of $L_1 \cup L_2 \subset X$. First, we choose f so that L_1 and L_2 are uniformly C^1-close, after which we can assume that there is a natural bijective correspondence between $\mathcal{Q}(L_1)$ and $\mathcal{Q}(L_2)$. We may also assume that a Reeb chord in $\mathcal{Q}(L_1, L_2)$ either corresponds to a critical point $\mathrm{Crit}(f)$ of f, or corresponds to a double-point, which is contained in an arbitrarily small neighbourhood of a double-point in $\mathcal{Q}(L)$. Furthermore, for each Reeb chord $c \in \mathcal{Q}(L)$, there are precisely two Reeb chords p_c and q_c of the latter kind, where the action of p_c is strictly smaller than the action of q_c. Using these identifications, we write

$$\mathcal{P} := \{p_c \,;\, c \in \mathcal{Q}(L)\} \quad \text{and} \quad \mathcal{Q} := \{q_c \,;\, c \in \mathcal{Q}(L)\}.$$

We thus have a decomposition

$$\mathcal{Q}(L_1, L_2) = \mathcal{P} \sqcup \mathrm{Crit}(f) \sqcup \mathcal{Q},$$

where the actions of the different Reeb chords satisfy

$$\ell(\mathcal{P}) < \ell(\mathrm{Crit}(f)) < \ell(\mathcal{Q}).$$

Recall that $\mathcal{C}_R := \mathcal{C}_{L;R}$ is the characteristic algebra of the Chekanov–Eliashberg algebra of $L = L_1$, which, in the current situation, may be assumed to be naturally isomorphic to that of L_2. We will consider the complex

$$(\mathcal{C}(L_1, L_2; R), \partial := \partial_{\mathcal{C}}),$$

defined in Section 3.2 above, which is the free $\mathcal{C}_R$-module generated by $\mathcal{Q}(L_1, L_2)$, but considered as a non-free $\mathcal{C}_R \otimes_R \mathcal{C}_R^{\mathrm{op}}$-module. Since L is displaceable by assumption, Theorem 3.3 implies that this complex is acyclic. Also, since $\mathrm{HC}_\bullet(L; R) \neq 0$ holds by assumption, which is equivalent to the fact that $\mathcal{C}_R \neq 0$ (see Lemma 3.5), it moreover follows that the complex $(\mathcal{C}(L_1, L_2; R), \partial)$ is non-trivial.

By simply considering the actions of the different kinds of generators, and using the fact that the differential is action-decreasing, ∂ is seen to be of the form

$$\mathcal{C}(L_1, L_2; R) = Q \oplus C \oplus P, \quad \partial = \begin{pmatrix} \partial_Q & 0 & 0 \\ \rho & -\partial_C & 0 \\ \eta & \sigma & \partial_P \end{pmatrix},$$

where Q, C and P are the submodules generated by the Reeb chords in $\mathcal{Q}$, $\mathrm{Crit}(f)$ and $\mathcal{P}$, respectively.

Fix a point $x \in L$ and an almost complex structure $J \in \mathcal{J}_L$ induced by a metric on L as in [11, Remark 6.1]. By the Gromov-type compactness for the pseudo-

holomorphic discs under consideration, which was established in [13], it suffices to show the claim of the theorem after choosing both x and $J \in \mathcal{J}_L$ to be generic.

By the disc count in [11, Proposition 3.7(2)] it follows that, for a generic $J \in \mathcal{J}_L$, we have an isomorphism

$$(C_\bullet, \partial_C) \simeq (C^{\mathrm{Morse}}_{\bullet+1}(f) \otimes_R \mathcal{C}_R, \partial_f),$$

where the latter is the Morse complex with coefficients in $\mathcal{C}_R$ (with shifted grading).

Let $c_L \in C$ denote a $\mathcal{C}_R$-fundamental class of L, which is a linear combination of the maxima of f. Observe that, in the case when $1+1 \neq 0$ holds in $\mathcal{C}_R$ (and hence in R as well) the closed manifold L is a spin manifold by assumption. In particular, L is orientable in the latter case, and there always exists such a fundamental class.

We begin by arguing that $\partial(c_L) = 0$. First, $\partial_C(c_L) = 0$ holds by the definition of being an R-fundamental class. It remains to show that $\sigma(c_L) = 0$. This follows by the count of discs analogous to the count conducted in the proof of [11, Theorem 5.5]. For a Reeb chord $c \in \mathcal{Q}(L)$ we let $c_j \in \mathcal{Q}(L_j)$ denote the corresponding Reeb chord on L_j (i.e. $c_1 = c$), and p_c the corresponding Reeb chord in $\mathcal{P}$. Let $c_M \in \mathrm{Crit}(f)$ be any local maximum of f. By [11, Theorem 3.6(3)], the J-holomorphic discs in the definition of $\sigma(c_M)$ that contribute to the coefficient in front of p_c correspond bijectively to rigid generalised pseudo-holomorphic discs consisting of

• a J-holomorphic disc $u \colon (D^2, \partial D^2) \to (X, L)$ with boundary on L, one positive puncture, and negative punctures of which at least one is mapped to c; together with

• an oriented flow-line of $-\nabla f$ on L connecting c_M to the boundary of u.

Formula (3.11) in [11] implies that u must live in a moduli space of expected dimension -1, and must hence have constant image in c. In particular, all boundary punctures of u must map to c.

In addition, we now claim that the above generalised pseudo-holomorphic discs that contribute to σ must satisfy the property that u has exactly one negative puncture. This holds by the correspondence given by [11, Theorem 3.6(3)] together with the fact that a J-holomorphic disc with boundary on $L_1 \cup L_2$ having one positive boundary puncture at c_M, and negative boundary punctures at either $\{p_c, c_1\}$ or $\{p_c, c_2\}$, cannot have any additional negative boundary punctures at c_j. Namely, such a disc would have negative symplectic area, as follows from the action considerations

$$\ell(c_M) \approx N, \quad \ell(p_c) \approx N - \ell(c) \quad \text{and} \quad \ell(c_1) \approx \ell(c_2).$$

For each $c \in \mathcal{Q}(L)$, and for a generic f, the above analysis shows that there are exactly two J-holomorphic discs with boundary on $L_1 \cup L_2$ that contribute to $\sigma(c_M)$; one having negative punctures at p_c and c_1, and one having negative punctures at c_2 and p_c, where the punctures moreover come in this order with respect to the boundary orientation. We thus compute

$$\sigma(c_M) = \sum_{c \in \mathcal{Q}(L)} \pi \circ \Phi((c_2 \otimes 1 + (-1)^{|c_1|+1} 1 \otimes c_1) p_c) = \sum_{c \in \mathcal{Q}(L)} (c - c) p_c = 0.$$

In order to obtain the sign $(-1)^{|c_1|+1}$ in the above formula, one must employ the convention that the orientations of L_1 and L_2 agree (respectively, differ) in the case when $\dim L_j$ is even (respectively, odd). Namely, in this case, the capping operators associated with the negative punctures both have index one (respectively, zero) in the case when $|c_1|$ is even (respectively, $|c_1|$ odd), as can be seen by the results in [14, Section 3.3.6]. To that end, one can use [14, Lemma 4.3] together with the formula $|c_j| = -|p_c| + \dim L_j - 2$ in [11, (3.3)]. See [11, Remark 5.6] for the analogous cancellation with signs in the proof of [11, Theorem 5.5].

We have thus shown that any fundamental class $c_L \in \mathcal{C}(L_1, L_2)$ has the property that $\partial(c_L) = 0$. Since the complex is acyclic, c_L must be a boundary. As it is not in the image of the Morse differential ∂_C, the only possibility is for c_L to be in the image of ρ. We now consider the case when c_L is represented by a single maximum of f, which is taken to be situated exactly at $x \in L$.

Let $q_c \in Q$ be a generator. By [11, Theorem 3.6(3)], the J-holomorphic discs in the definition of $\rho(q_c)$ that contribute to the coefficient in front of c_L correspond to rigid generalised pseudo-holomorphic discs consisting of

- a J-holomorphic disc $u \colon (D^2, \partial D^2) \to (X, L)$ with boundary on L, one positive puncture at c, and possibly several negative punctures; together with
- an oriented flow-line of $-\nabla f$ on L connecting the boundary of u to $c_L = x$.

Rigidity of the configuration means that the flow-line must be trivial, and hence that the boundary of u actually must pass through x. This gives the existence of the sought disc.

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Georgios Dimitroglou Rizell
Département de Mathématiques
Université Paris-Sud
Bat. 425
FR-91405 Orsay
France
georgios.dimitroglou@math.u-psud.fr