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Mappings onto multiplicative subsets of function algebras and spectral properties of their products

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Dedicated to Junzo Wada.

Abstract. We characterize mappings S_i and T_i, not necessarily linear, from sets $\mathcal{J}_i$, $i=1,2$, onto multiplicative subsets of function algebras, subject to the following conditions on the peripheral spectra of their products: $\sigma_\pi(S_1(a)S_2(b))\subset\sigma_\pi(T_1(a)T_2(b))$ and $\sigma_\pi(S_1(a)S_2(b))\cap\sigma_\pi(T_1(a)T_2(b))\neq\varnothing$, $a\in\mathcal{J}_1$, $b\in\mathcal{J}_2$. As a direct consequence we obtain a large number of previous results about mappings subject to various spectral conditions.

1. Introduction and motivation

Let A and B be commutative Banach algebras. A mapping $S\colon A\to B$ is *spectrum-preserving* if $\sigma(S(a))\subset\sigma(a)$ for all $a\in A$, where $\sigma(a)$ is the spectrum of a. The study of spectrum-preserving mappings has a long history. For semisimple commutative Banach algebras A, Gleason [1] and Kahane and Żelazko [15] have proven independently that every surjective *linear* operator $S\colon A\to\mathbb{C}$ with $S(a)\in\sigma(a)$ for all $a\in A$ is multiplicative, i.e. $S(ab)=S(a)S(b)$ for all $a,b\in A$. This result is known as the Gleason–Kahane–Żelazko theorem. It yields that every spectrum-preserving linear mapping $S\colon A\to B$ between semisimple commutative Banach algebras is multiplicative. It is imperative for the operator S in the Gleason–Kahane–Żelazko theorem to be linear. Kowalski and Słodkowski [16] have found spectral conditions for *a priori* nonlinear mappings to be automatically linear. Namely, a surjective mapping S from a commutative Banach algebra A onto a semisimple commuta-

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tive Banach algebra with $S(0)=0$ and $\sigma(S(a)-S(b))\subset\sigma(a-b)$ for all $a,b\in A$ is an algebra isomorphism (see [7, Theorem 3.1]).

Let $C(X)$ be the algebra of all complex-valued continuous functions on a compact Hausdorff space X. Molnár [22] proved a multiplicative version of the theorem of Kowalski and Słodkowski, namely, if X is a first countable space, then a surjective mapping $S\colon C(X)\to C(X)$ with $S(1)=1$ and either $\sigma(S(f)S(g))=\sigma(fg)$, or, $\sigma(S(f)\overline{S(g)})=\sigma(f\bar{g})$ for all $f,g\in C(X)$ is an isometric algebra isomorphism, where $\overline{(\,\cdot\,)}$ stands for the complex conjugate. The study of *a priori* nonlinear mappings S with the property $\sigma(S(f)S(g))=\sigma(fg)$, or, $\sigma(S(f)\overline{S(g)})=\sigma(f\bar{g})$ between uniform or function algebras was initiated independently in several papers (e.g. [6], [8] and [23]). The involvement of the peripheral spectrum, $\sigma_\pi(f)$, of algebra elements $f\in A$ in spectral preserver problems was initiated in [2] (see also [20]), where a new spectral condition, $\sigma_\pi(S(f)S(g))=\sigma_\pi(fg)$ was introduced and mappings satisfying it were completely characterized. These results were unified further in [5] and [10], where pairs of mappings between subsets of algebras satisfying corresponding spectral conditions were studied and characterized.

In this paper, we characterize mappings, not necessarily linear, onto multiplicative subsets of function algebras that are subject to certain conditions on the peripheral spectra of their products. As a direct consequence we obtain a large number of previous results about mappings that satisfy various spectral conditions.

2. Preliminaries and main results

Spectra and peripheral spectra of algebra elements are essential notions in spectral preserver problems for mappings between commutative Banach algebras (e.g. [2], [4], [6], [11], [14], [17], [20], [22], [23], [24] and [26]) and their pairs (e.g. [5], [10] and [18]). In this paper we characterize mappings into multiplicative subsets of function algebras, not necessarily with units, the products of which satisfy general spectral conditions. Most of the previous results, mentioned above, are direct consequences of the results obtained here.

We assume that a *function algebra*, A, on a locally compact Hausdorff space X is a uniformly closed subalgebra of $C_0(X)$, the commutative Banach algebra of all complex-valued continuous functions on X that vanish at infinity, with respect to the pointwise operations and the *sup*-norm $\|\cdot\|$, which strongly separates the points of X in the sense that for each $x,y\in X$ with $x\neq y$ there exists an $f\in A$ such that $f(x)\neq f(y)$ and for each $x\in X$ there exists a $g\in A$ with $g(x)\neq 0$. *Uniform algebras* are function algebras on compact Hausdorff spaces that contain the constant function 1. The underlying space X of a function algebra A can be identified with a subset of the maximal ideal space of A, not necessarily coinciding with it, that contains the

Choquet boundary δA of A, namely, the set of all $x\in X$ such that the evaluation map at x, $f\rightsquigarrow f(x)$, $f\in A$, is an extreme point of the unit ball of the dual space of A. The *range* of $f\in A$ is the set $\mathrm{Ran}(f)=f(X)=\{f(x):x\in X\}$, which is such that either $\mathrm{Ran}(f)$ or $\mathrm{Ran}(f)\cup\{0\}$ are compact sets in $\mathbb{C}$. The *peripheral spectrum* $\sigma_\pi(f)$ of $f\in A$ is the compact set $\sigma_\pi(f)$ of $\mathbb{C}$ defined as

$$\sigma_\pi(f)=\{\lambda\in\sigma(f):|\lambda|=\|f\|\}=\{\lambda\in\mathrm{Ran}(f):|\lambda|=\|f\|\}$$

(see e.g. [2] and [20]). Moreover, $\sigma_\pi(f)=\{\lambda\in f(\delta A):|\lambda|=\|f\|\}$ (cf. Lemma 4.1 in Section 4 below).

Let $\mathcal{S}$ be an arbitrary subset of a function algebra A. A set $E\subset X$ is a *boundary* of $\mathcal{S}$ if every $f\in\mathcal{S}$ attains its maximum modulus on E. An $h\in\mathcal{S}$ is a *peaking function* of $\mathcal{S}$ if $\sigma_\pi(h)=\{1\}$. The set of all peaking functions of $\mathcal{S}$ that peak at $x\in\delta A$ will be denoted by $P_{\mathcal{S}}(x)$. Clearly, $P_{\mathcal{S}}(x)\subset P_A(x)$ and $P_{\mathcal{S}}(x)=P_A(x)\cap\mathcal{S}$. A compact subset E of X is a *peak set* of $\mathcal{S}$ if $E=h^{-1}(1)=\{x\in X:h(x)=1\}$ for some peaking function $h\in\mathcal{S}$. A point $x\in X$ is a *peak point* of $\mathcal{S}$ if $\{x\}$ is a peak set of $\mathcal{S}$, or, equivalently, if there is a function $h\in P_{\mathcal{S}}(x)$ such that $|h(y)|<1$ for any $y\neq x$. Functions h with this property are called *peak functions* of $\mathcal{S}$ at x. We denote by $p^\circ(\mathcal{S})$ the set of all peak points of $\mathcal{S}$. The set of all peak functions of $\mathcal{S}$ at $x\in X$ will be denoted by $P^\circ_{\mathcal{S}}(x)$. Clearly, $P^\circ_{\mathcal{S}}(x)\subset P_{\mathcal{S}}(x)$, $P^\circ_{\mathcal{S}}(x)\subset P^\circ_A(x)$ and $P^\circ_{\mathcal{S}}(x)=P^\circ_A(x)\cap\mathcal{S}$.

A set $E\subset X$ is a *weak peak set* of $\mathcal{S}$ if E is the intersection of a family of peak sets of $\mathcal{S}$. A point $x\in X$ is a *weak peak point* (or, a *p-point*) of $\mathcal{S}$ if $\{x\}$ is a weak peak set of $\mathcal{S}$. Equivalently, x is a weak peak point of $\mathcal{S}$ if for every open set $U\subset X$ containing x there is a peak set E of $\mathcal{S}$ such that $x\in E\subset U$. The set of all weak peak points of $\mathcal{S}$ will be denoted by $p(\mathcal{S})$. Clearly, $p(\mathcal{S})$ is a subset of $p(A)=\delta A$, though it may not coincide with it. Actually, $p(\mathcal{S})$ can be empty. However, if $\mathcal{S}\neq\{0\}$ is a uniformly closed subalgebra of A strongly separating the points of X (and thus $\mathcal{S}$ is a function algebra on X) then $p(\mathcal{S})$ is nonempty and actually coincides with the Choquet boundary $\delta\mathcal{S}$ of $\mathcal{S}$.

Let $\mathcal{J}_i$, $i=1,2$, be arbitrary sets without any particular structure. Throughout this paper we will use the following products of mappings of $\mathcal{J}_i$ into function algebras.

Definition 2.1. If $S_1\colon\mathcal{J}_1\to A$ and $S_2\colon\mathcal{J}_2\to A$ are two mappings we denote by $S_1\otimes S_2\colon\mathcal{J}_1\times\mathcal{J}_2\to A$ the mapping $(S_1\otimes S_2)(a,b)=S_1(a)S_2(b)$, where $a\in\mathcal{J}_1$ and $b\in\mathcal{J}_2$.

If S_i maps $\mathcal{J}_i$ onto a subset $\mathcal{S}_i$ of A, for $i=1,2$, then, clearly, the mapping $S_1\otimes S_2\colon\mathcal{J}_1\times\mathcal{J}_2\to A$ maps $\mathcal{J}_1\times\mathcal{J}_2$ onto the set $\mathcal{S}_1\cdot\mathcal{S}_2\subset A$ and, in fact, provides a $\mathcal{J}_1\times\mathcal{J}_2$-parametrization of $\mathcal{S}_1\cdot\mathcal{S}_2$.

In Section 3 we show that if the norms of two $\otimes$-products of mappings onto multiplicative subsets of function algebras are equal, then their moduli coincide up to the composition with a homeomorphism. Recall that a subset $\mathcal{S}\subset A$ is a *multiplicative set in* A if $fg\in\mathcal{S}$ for all $f,g\in\mathcal{S}$. In Proposition 3.1 we prove that if $\mathcal{S}_i\subset A$ and $\mathcal{T}_i\subset B$ are multiplicative sets in function algebras A and B on locally compact Hausdorff spaces X and Y with $p(\mathcal{S}_i)=\delta A$ and $p(\mathcal{T}_i)=\delta B$, $\mathcal{J}_i$ are arbitrary sets without any particular structure and $S_i\colon \mathcal{J}_i\to\mathcal{S}_i$ and $T_i\colon \mathcal{J}_i\to\mathcal{T}_i$ are surjections for $i=1,2$ such that $\|(T_1\otimes T_2)(a,b)\|=\|(S_1\otimes S_2)(a,b)\|$ for all $a\in\mathcal{J}_1$ and $b\in\mathcal{J}_2$, then there is a homeomorphism $\phi\colon \delta B\to\delta A$ such that

$$|((T_1\otimes T_2)(a,b))(y)|=|((S_1\otimes S_2)(a,b))(\phi(y))| \tag{2.1}$$

for all $a\in\mathcal{J}_1$, $b\in\mathcal{J}_2$ and $y\in\delta B$.

In Section 4 we introduce several spectral conditions for $\otimes$-products of two pairs of mappings into multiplicative subsets of function algebras so that the mappings in the first pair equal the mappings in the second one up to certain weighted composition operators on the corresponding Choquet boundaries.

Theorem 2.2. *Let A and B be function algebras on locally compact Hausdorff spaces X and Y, let $\mathcal{S}_1$ and $\mathcal{S}_2$ be multiplicative subsets of A, and $\mathcal{T}_1$ and $\mathcal{T}_2$ be subalgebras of B such that $p(\mathcal{S}_i)=\delta A$ and $p(\mathcal{T}_i)=\delta B$ for $i=1,2$. If $\mathcal{J}_i$ are arbitrary sets of parameters and the pairs of surjective maps $S_i\colon \mathcal{J}_i\to\mathcal{S}_i$ and $T_i\colon \mathcal{J}_i\to\mathcal{T}_i$, $i=1,2$, satisfy the conditions*:

(i) $\|(T_1\otimes T_2)(a,b)\|=\|(S_1\otimes S_2)(a,b)\|$;

(ii) $((S_1\otimes S_2)(a,b))(\delta A)\subset\mathrm{Ran}((T_1\otimes T_2)(a,b))$;

for all $a\in\mathcal{J}_1$ and $b\in\mathcal{J}_2$, then there is a homeomorphism $\phi\colon \delta B\to\delta A$ and a continuous function $\alpha\colon \delta B\to\mathbb{C}\setminus\{0\}$ so that

$$T_1(a)(y)=\alpha(y)S_1(a)(\phi(y)) \quad \textit{and} \quad T_2(b)(y)=\frac{1}{\alpha(y)}S_2(b)(\phi(y))$$

for all $a\in\mathcal{J}_1$, $b\in\mathcal{J}_2$ and $y\in\delta B$.

A direct consequence of Theorem 2.2 is the main result in [9], where $\mathcal{S}_1=\mathcal{S}_2$ and $\mathcal{T}_1=\mathcal{T}_2$ are Banach function algebras. For subsets $\mathcal{T}_i$ with more specific properties Theorem 2.2 holds under more general spectral conditions on $\otimes$-products.

Definition 2.3. Let $\mathcal{S}$ and $\mathcal{S}'$ be subsets of a function algebra A with $p(\mathcal{S})\neq\varnothing$ and $p(\mathcal{S}')\neq\varnothing$. The pair $(\mathcal{S},\mathcal{S}')$ is said to be a *Bishop pair* if for every $x\in p(\mathcal{S})$ and each $f'\in\mathcal{S}'$ with $f'(x)\neq 0$ there is a peaking function $h\in P_{\mathcal{S}}(x)$ such that $\sigma_\pi(f'h)=\{f'(x)\}$, and for every $x'\in p(\mathcal{S}')$ and each $f\in\mathcal{S}$ with $f(x')\neq 0$ there is a peaking function $h' \in P_{\mathcal{S}'}(x')$ such that $\sigma_\pi(fh') = \{f(x')\}$. A set $\mathcal{S}$ is a *Bishop set* if $(\mathcal{S}, \mathcal{S})$ is a Bishop pair.

In his celebrated lemma (see e.g. [19]), E. Bishop has shown that every uniform algebra is in fact a Bishop set. Examples of Bishop sets $\mathcal{S}$ are also function algebras (see [6, Lemma 2.3], and also [26, Proposition 3.1]), unit balls of function algebras, and the sets of exponents of uniform algebras. Lipschitz algebras on metric spaces are Bishop sets (cf. [12, Lemma 2.2]). Note that all these sets are also multiplicative subsets of function algebras such that $p(\mathcal{S}) = \delta\mathcal{S}$.

Theorem 2.4. *Let $\mathcal{S}_1, \mathcal{S}_2 \subset A$ and $\mathcal{T}_1, \mathcal{T}_2 \subset B$ be multiplicative sets in function algebras A and B on locally compact Hausdorff spaces X and Y such that $p(\mathcal{S}_i) = \delta A$ and $p(\mathcal{T}_i) = \delta B$, and $(\mathcal{T}_1, \mathcal{T}_2)$ is a Bishop pair. If $\mathcal{J}_i$ are arbitrary sets of parameters and the pairs of surjective maps $S_i\colon \mathcal{J}_i \to \mathcal{S}_i$ and $T_i\colon \mathcal{J}_i \to \mathcal{T}_i$, $i=1,2$, satisfy the condition*

$$\sigma_\pi((S_1 \otimes S_2)(a,b)) \subset \sigma_\pi((T_1 \otimes T_2)(a,b)) \tag{2.2}$$

for all $a \in \mathcal{J}_1$ and $b \in \mathcal{J}_2$, then there is a homeomorphism $\phi\colon \delta B \to \delta A$ and a continuous function $\alpha\colon \delta B \to \mathbb{C} \setminus \{0\}$ so that

$$T_1(a)(y) = \alpha(y) S_1(a)(\phi(y)) \quad \textit{and} \quad T_2(b)(y) = \frac{1}{\alpha(y)} S_2(b)(\phi(y))$$

for all $a \in \mathcal{J}_1$, $b \in \mathcal{J}_2$ and $y \in \delta B$.

Earlier versions of Theorem 2.4 have been proven previously under the following particular conditions: $\mathcal{S}_i$ and $\mathcal{T}_i$ are uniform algebras, $S_1(a) = T(a)^m$, $S_2(b) = T(b)^n$, $T_1(a) = a^m$ and $T_2(b) = b^n$, where T is a surjective map and m and n are natural numbers—in [3]; $\mathcal{S}_i$ and $\mathcal{T}_i$ are uniform algebras or algebras of type $C_0(X)$—in [5]; $S_2(b) = \bar{b}$ and $T_2(b) = \overline{T_1(b)}$ —in [8]. Theorem 2.4 holds also if the spectral condition (2.2) is replaced by the conditions $\sigma_\pi((S_1 \otimes S_2)(a,b)) \cap \sigma_\pi((T_1 \otimes T_2)(a,b)) \neq \varnothing$ and $\sigma_\pi(S_i(a)) \subset \sigma_\pi(T_i(a))$, $i=1,2$, (Proposition 4.2), which implies one of the main results in [14].

Theorem 2.5. *Let $\mathcal{S}_1, \mathcal{S}_2 \subset A$ and $\mathcal{T}_1, \mathcal{T}_2 \subset B$ be multiplicative sets in function algebras A and B on locally compact Hausdorff spaces X and Y such that $p(\mathcal{S}_i) = \delta A$, $p^\circ(\mathcal{T}_i) = \delta B$ and $(\mathcal{T}_1, \mathcal{T}_2)$ is a Bishop pair. If $\mathcal{J}_i$ are arbitrary sets of parameters and the pairs of surjective maps $S_i\colon \mathcal{J}_i \to \mathcal{S}_i$ and $T_i\colon \mathcal{J}_i \to \mathcal{T}_i$, $i=1,2$, satisfy the condition*

$$\sigma_\pi((S_1 \otimes S_2)(a,b)) \cap \sigma_\pi((T_1 \otimes T_2)(a,b)) \neq \varnothing \tag{2.3}$$

for all $a\in\mathcal{J}_1$ and $b\in\mathcal{J}_2$, then there is a homeomorphism $\phi\colon \delta B\to\delta A$ and a continuous function $\alpha\colon \delta B\to\mathbb{C}\setminus\{0\}$ so that

$$T_1(a)(y)=\alpha(y)S_1(a)(\phi(y)) \quad \textit{and} \quad T_2(b)(y)=\frac{1}{\alpha(y)}S_2(b)(\phi(y))$$

for all $a\in\mathcal{J}_1$, $b\in\mathcal{J}_2$ and $y\in\delta B$.

The hypotheses on the sets $\mathcal{S}_1,\mathcal{S}_2$ and $\mathcal{T}_1,\mathcal{T}_2$ here are interchangeable. Theorem 2.5 is proven previously in the following cases: $\mathcal{S}_i$ and $\mathcal{T}_i$ are pointed Lipschitz algebras—in [10]; $\mathcal{S}_i$ and $\mathcal{T}_i$ are uniform algebras—in [25].

As a bi-product of the main theorems we obtain that under their conditions the moduli in the equality (2.1) can be removed so that the equality $((T_1\otimes T_2)(a,b))(y)=((S_1\otimes S_2)(a,b))(\phi(y))$ holds for all $a\in\mathcal{J}_1$, $b\in\mathcal{J}_2$ and $y\in\delta B$.

If in the main theorems we choose $\mathcal{J}_i=\mathcal{S}_i$ and $S_i=\mathrm{id}(\mathcal{S}_i)$ for $i=1,2$, we obtain the following result.

Corollary 2.6. *Let $\mathcal{S}_1,\mathcal{S}_2\subset A$ and $\mathcal{T}_1,\mathcal{T}_2\subset B$ be multiplicative sets in function algebras A and B on locally compact Hausdorff spaces X and Y such that $p(\mathcal{S}_i)=\delta A$, $p(\mathcal{T}_i)=\delta B$ for $i=1,2$. If the two surjective maps $T_i\colon \mathcal{S}_i\to\mathcal{T}_i$, $i=1,2$, satisfy one of the conditions*:

(1) $\|T_1(f)T_2(g)\|=\|fg\|$, $(fg)(\delta A)\subset\mathrm{Ran}(T_1(f)T_2(g))$ *for all $f\in\mathcal{S}_1$ and $g\in\mathcal{S}_2$, and $\mathcal{T}_1,\mathcal{T}_2$ are subalgebras of B*;

(2) $\sigma_\pi(fg)\subset\sigma_\pi(T_1(f)T_2(g))$ *for all $f\in\mathcal{S}_1$, $g\in\mathcal{S}_2$ and $(\mathcal{T}_1,\mathcal{T}_2)$ is a Bishop pair*;

(3) $\sigma_\pi(fg)\cap\sigma_\pi(T_1(f)T_2(g))\neq\varnothing$ *for all $f\in\mathcal{S}_1$ and $g\in\mathcal{S}_2$, $p^\circ(\mathcal{T}_i)=\delta B$, $i=1,2$, and $(\mathcal{T}_1,\mathcal{T}_2)$ is a Bishop pair*;

then there is a homeomorphism $\phi\colon \delta B\to\delta A$ and a continuous function $\alpha\colon \delta B\to\mathbb{C}\setminus\{0\}$ so that

$$T_1(f)(y)=\alpha(y)f(\phi(y)) \quad \textit{and} \quad T_2(g)(y)=\frac{1}{\alpha(y)}g(\phi(y)) \tag{2.4}$$

for all $f\in\mathcal{S}_1$, $g\in\mathcal{S}_2$ and $y\in\delta B$. Therefore, T_1 and T_2 are weighted composition operators on δB.

In Section 5 we show that the main theorems follow from Corollary 2.6 (see Remark 5.1).

Earlier versions of part (1) of Corollary 2.6 have been proven previously under the following particular conditions: $\mathcal{S}_i$ and $\mathcal{T}_i$ are uniform algebras—in [6]; $\mathcal{S}_i$ and $\mathcal{T}_i$ are unital semisimple commutative Banach algebras (without assuming $p(\mathcal{S}_i)=\delta A$ and $p(\mathcal{T}_i)=\delta B$)—in [7]; $\mathcal{S}_i$ and $\mathcal{T}_i$ are Banach function algebras—in [9].

If, in addition to the hypotheses of Corollary 2.6, we choose $T_1 = T_2 = T$, then equality (2.4) becomes $T(f)(y) = \alpha(y) f(\phi(y))$, where $\alpha^2(y) = 1$. In particular, we obtain the following result.

Corollary 2.7. *Let $\mathcal{S} \subset A$ and $\mathcal{T} \subset B$ be multiplicative sets in function algebras A and B on locally compact Hausdorff spaces X and Y such that $p(\mathcal{S}) = \delta A$ and $p(\mathcal{T}) = \delta B$. If a surjective map $T\colon \mathcal{S} \to \mathcal{T}$ satisfies one of the conditions:*

(1) $\|T(f)T(g)\| = \|fg\|$, $(fg)(\delta A) \subset \mathrm{Ran}(T(f)T(g))$ *for all $f, g \in \mathcal{S}$, and $\mathcal{T}$ is a subalgebra of B;*

(2) $\sigma_\pi(fg) \subset \sigma_\pi(T(f)T(g))$ *for all $f, g \in \mathcal{S}$ and $\mathcal{T}$ is a Bishop set;*

(3) $\sigma_\pi(fg) \cap \sigma_\pi(T(f)T(g)) \neq \varnothing$ *for all $f, g \in \mathcal{S}$, $p^\circ(\mathcal{T}) = \delta B$ and $\mathcal{T}$ is a Bishop set;*

then there is a homeomorphism $\phi\colon \delta B \to \delta A$ and a continuous function $\alpha\colon \delta B \to \{\pm 1\}$ so that

$$T(f)(y) = \alpha(y) f(\phi(y))$$

for all $f \in \mathcal{S}$ and $y \in \delta B$. Therefore, T is a weighted composition operator on δB.

Earlier versions of part (2) of Corollary 2.7 have been proven previously under the following particular conditions: $\mathcal{S}$ and $\mathcal{T}$ are uniform algebras—in [17], [20], [22], [23], [24]; $\mathcal{S}$ and $\mathcal{T}$ are function algebras —in [13], [14], [24]; $\mathcal{S}$ and $\mathcal{T}$ are Lipschitz algebras—in [12]. Earlier versions of parts (2) and (3) have been proven previously in the case when $\mathcal{T}$ is a function algebra on Y and $\mathcal{S}$ is a dense subalgebra of a function algebra on X ([13], [14]); since any uniform algebra A on a first-countable compact Hausdorff space is a multiplicative set, its weak peak points are peak points and A is a Bishop set, part (3) holds also in the case when $\mathcal{S}$, $\mathcal{T}$ are uniform algebras on first-countable compact Hausdorff spaces. This yields some of the main results in [18], [22] and [25]. In the case when $\mathcal{S}$ and $\mathcal{T}$ are uniform algebras part (3) is proven in [17], and when $\mathcal{S}$ and $\mathcal{T}$ are function algebras—in [26]. Part (1) of Corollary 2.7 implies the following consequence.

Corollary 2.8. *Let A and B be dense subalgebras of function algebras on locally compact Hausdorff spaces X and Y such that $p(A) = \delta(\overline{A})$ and $p(B) = \delta(\overline{B})$, where $\overline{A}$ and $\overline{B}$ are the uniform closures of A and B, respectively. If a surjection $T\colon A \to B$ satisfies the conditions $\|T(f)T(g)\| = \|fg\|$ and $(fg)(\delta A) \subset \mathrm{Ran}(T(f)T(g))$ for all $f, g \in A$, then there is a homeomorphism $\phi\colon \delta B \to \delta A$ and a continuous function $\alpha\colon \delta B \to \{\pm 1\}$ such that*

$$T(f)(y) = \alpha(y) f(\phi(y))$$

for all $f \in A$ and $y \in \delta B$, i.e. T is a weighted composition operator on δB.

3. The homeomorphism ϕ

In this section we prove the following result.

Proposition 3.1. *Let $\mathcal{S}_1, \mathcal{S}_2 \subset A$ and $\mathcal{T}_1, \mathcal{T}_2 \subset B$ be multiplicative sets in function algebras A and B on locally compact Hausdorff spaces X and Y, such that $p(\mathcal{S}_i) = \delta A$ and $p(\mathcal{T}_i) = \delta B$ for $i = 1, 2$. If $\mathcal{J}_i$ are arbitrary sets of parameters and $S_i \colon \mathcal{J}_i \to \mathcal{S}_i$ and $T_i \colon \mathcal{J}_i \to \mathcal{T}_i$, $i = 1, 2$, are pairs of surjective maps satisfying the condition*

$$\|(T_1 \otimes T_2)(a, b)\| = \|(S_1 \otimes S_2)(a, b)\| \tag{3.1}$$

for all $a \in \mathcal{J}_1$ and $b \in \mathcal{J}_2$, then there exists a homeomorphism $\phi \colon \delta B \to \delta A$ such that

$$|((T_1 \otimes T_2)(a, b))(y)| = |((S_1 \otimes S_2)(a, b))(\phi(y))| \tag{3.2}$$

for all $a \in \mathcal{J}_1$, $b \in \mathcal{J}_2$ and $y \in \delta B$.

Lemma 3.2. *Let $\mathcal{S}$ be a multiplicative set in a function algebra A on a locally compact Hausdorff space X with $p(\mathcal{S}) = \delta A$. For every $x_0 \in \delta A$, $\varepsilon > 0$ and any closed set $K \subset X$ with $x_0 \notin K$ there is an $h \in P_{\mathcal{S}}(x_0)$ such that $|h| < \varepsilon$ on K.*

Proof. If $x_0 \in \delta A = p(\mathcal{S})$ choose a $k \in P_{\mathcal{S}}(x_0) = P_A(x_0) \cap \mathcal{S}$ with $k^{-1}(1) \subset X \setminus K$. Then $\max_{x \in K} |k(x)| < 1$ and, therefore, $|k^n| < \varepsilon$ on K for some sufficiently large power of k. Clearly, $h = k^n \in P_{\mathcal{S}}(x_0)$ since $k \in P_{\mathcal{S}}(x_0)$ and $\mathcal{S}$ is a multiplicative subset of A. $\square$

The following lemma is an $\mathcal{S}$-version of [26, Lemma 2.2].

Lemma 3.3. *Let $\mathcal{S}$ be a multiplicative set in a function algebra A on a locally compact Hausdorff space X with $p(\mathcal{S}) = \delta A$. If $f \in A$ and $x_0 \in \delta A$ then*

$$|f(x_0)| = \inf_{h \in P_{\mathcal{S}}(x_0)} \|fh\|.$$

Proof. Let $x_0 \in \delta A$ and $f \in A$. Since $|f(x_0)| = |f(x_0)h(x_0)| \le \|fh\|$ for $h \in P_{\mathcal{S}}(x_0)$, we have $|f(x_0)| \le \inf_{h \in P_{\mathcal{S}}(x_0)} \|fh\|$. Let $\varepsilon > 0$, and consider the set

$$U = \{x \in X : |f(x)| < |f(x_0)| + \varepsilon\}.$$

Clearly, $x_0 \in U$. Choose a $k \in P_{\mathcal{S}}(x_0)$ with $k^{-1}(1) \subset U$. Then $\max_{X \setminus U} |k(x)| < 1$ and for some power of k, $h_0 = k^n \in P_{\mathcal{S}}(x_0)$, with n big enough, we have $|f(x)h_0(x)| < |f(x_0)| + \varepsilon$ for all $x \in X \setminus U$. In addition, for any $x \in U$ we have $|f(x)h_0(x)| \le |f(x)| <$

$|f(x_0)|+\varepsilon$. Hence, $\|fh_0\|<|f(x_0)|+\varepsilon$, and, consequently, $\inf_{h\in P_{\mathcal{S}}(x_0)}\|fh\|\le|f(x_0)|$. Therefore, $|f(x_0)|=\inf_{h\in P_{\mathcal{S}}(x_0)}\|fh\|$, as claimed. □

Throughout this section we assume, without mention, that A and B are function algebras, not necessarily with units, on locally compact Hausdorff spaces X and Y, respectively, $\mathcal{S}_1,\mathcal{S}_2\subset A$ and $\mathcal{T}_1,\mathcal{T}_2\subset B$ are multiplicative sets such that $p(\mathcal{S}_i)=\delta A$ and $p(\mathcal{T}_i)=\delta B$, $\mathcal{J}_i$ are arbitrary sets without any particular structure, and $S_i\colon \mathcal{J}_i\to\mathcal{S}_i$ and $T_i\colon \mathcal{J}_i\to\mathcal{T}_i$ are surjective mappings for $i=1,2$, not necessarily linear, that satisfy the equality (3.1), i.e. $\|(T_1\otimes T_2)(a,b)\|=\|(S_1\otimes S_2)(a,b)\|$ for all $a\in\mathcal{J}_1$ and $b\in\mathcal{J}_2$. Note that the spectral properties (2.2) and (2.3) in Theorems 2.4 and 2.5 imply automatically the equality (3.1). First we construct the homeomorphism $\phi\colon \delta B\to\delta A$ and then we show that it satisfies the equality (3.2).

Lemma 3.4. *If $|T_i(a)(y)|\le|T_i(b)(y)|$ for some $a,b\in\mathcal{J}_i$ and all $y\in\delta B$, then $|S_i(a)(x)|\le|S_i(b)(x)|$, $i=1,2$, for all $x\in\delta A$.*

Proof. Consider first the case of S_1 and T_1 and suppose that $|T_1(a)(y)|\le|T_1(b)(y)|$ for all $y\in\delta B$. For each $x\in\delta A$ and $h\in P_{\mathcal{S}_2}(x)$ let $c_h\in\mathcal{J}_2$ be such that $S_2(c_h)=h$. By hypothesis

$$\begin{aligned}|((T_1\otimes T_2)(a,c_h))(y)|&=|(T_1(a)T_2(c_h))(y)|\le|(T_1(b)T_2(c_h))(y)|\\&=|((T_1\otimes T_2)(b,c_h))(y)|\le\|(T_1\otimes T_2)(b,c_h)\|\end{aligned}$$

for all $y\in\delta B$. Therefore, $\|(T_1\otimes T_2)(a,c_h)\|\le\|(T_1\otimes T_2)(b,c_h)\|$. By using twice the equality (3.1) we obtain

$$\begin{aligned}\|S_1(a)h\|&=\|(S_1\otimes S_2)(a,c_h)\|=\|(T_1\otimes T_2)(a,c_h)\|\\&\le\|(T_1\otimes T_2)(b,c_h)\|=\|(S_1\otimes S_2)(b,c_h)\|=\|S_1(b)h\|,\end{aligned}$$

i.e. $\|S_1(a)h\|\le\|S_1(b)h\|$. Since $h\in P_{\mathcal{S}_2}(x)$ was chosen arbitrarily, Lemma 3.3 yields

$$|S_1(a)(x)|=\inf_{h\in P_{\mathcal{S}_2}(x)}\|S_1(a)h\|\le\inf_{h\in P_{\mathcal{S}_2}(x)}\|S_1(b)h\|=|S_1(b)(x)|.$$

Consequently, $|S_1(a)(x)|\le|S_1(b)(x)|$ for all $x\in\delta A$, as claimed. A similar argument applies to S_2 and T_2. □

For any $x\in\delta A$ we denote by $V_{\mathcal{S}_i}(x)$ the set $V_{\mathcal{S}_i}(x)\overset{\text{def}}{=}\{f\in\mathcal{S}_i:|f(x)|=1=\|f\|\}$, $i=1,2$. Clearly, $V_{\mathcal{S}_i}(x)$ is a multiplicative set in A and $P_{\mathcal{S}_i}(x)\subset V_{\mathcal{S}_i}(x)$. The set $V_{\mathcal{T}_i}(y)$ is defined in a similar way for any $y\in\delta B$. For any $y\in\delta B$ we denote by $\mathcal{F}_y$ the set $\mathcal{F}_y\overset{\text{def}}{=}\{(a,b)\in\mathcal{J}_1\times\mathcal{J}_2:T_1(a)\in V_{\mathcal{T}_1}(y)\text{ and }T_2(b)\in V_{\mathcal{T}_2}(y)\}$.

Lemma 3.5. *For any* $y \in \delta B$,

$$\bigcap_{(a,b)\in\mathcal{F}_y} |(S_1 \otimes S_2)(a,b)|^{-1}(1) \neq \varnothing,$$

where $|(S_1 \otimes S_2)(a,b)|^{-1}(1) = \{x \in X : |((S_1 \otimes S_2)(a,b))(x)| = 1\}$.

Proof. Let $y_0 \in \delta B$. We claim that the family

$$\{|(S_1 \otimes S_2)(a,b)|^{-1}(1) : (a,b) \in \mathcal{F}_{y_0}\}$$

has the finite intersection property. Let $m \in \mathbb{N}$ and let $(a_i, b_i) \in \mathcal{F}_{y_0}$ for $1 \le i \le m$. The definition of $\mathcal{F}_{y_0}$ implies that

$$|T_1(a_i)(y_0)| = 1 = \|T_1(a_i)\| \quad \text{and} \quad |T_2(b_i)(y_0)| = 1 = \|T_2(b_i)\|.$$

Then, by (3.1), $\|(S_1 \otimes S_2)(a_i, b_i)\| = \|(T_1 \otimes T_2)(a_i, b_i)\| = 1$ for all $1 \le i \le m$, as

$$1 = |T_1(a_i)(y_0) T_2(b_i)(y_0)| \le \|(T_1 \otimes T_2)(a_i, b_i)\| \le \|T_1(a_i)\| \, \|T_2(b_i)\| = 1.$$

Since T_1 and T_2 are surjections from $\mathcal{J}_1$ and $\mathcal{J}_2$ onto the multiplicative subsets $\mathcal{T}_1$ and $\mathcal{T}_2$ of B, respectively, there exist $a \in \mathcal{J}_1$ and $b \in \mathcal{J}_2$ such that $\prod_{i=1}^m T_1(a_i) = T_1(a)$ and $\prod_{i=1}^m T_2(b_i) = T_2(b)$. We assert that $\|(S_1 \otimes S_2)(a,b)\| = 1$. In fact, as

$$1 = \prod_{i=1}^m |T_1(a_i)(y_0)| \le \left\| \prod_{i=1}^m T_1(a_i) \right\| \le \prod_{i=1}^m \|T_1(a_i)\| = 1,$$

we have $|T_1(a)(y_0)| = \prod_{i=1}^m |T_1(a_i)(y_0)| = 1 = \|T_1(a)\|$. By the same arguments, we see that $|T_2(b)(y_0)| = 1 = \|T_2(b)\|$, and thus $\|(T_1 \otimes T_2)(a,b)\| = \|T_1(a) T_2(b)\| = 1$. Hence by (3.1), $\|(S_1 \otimes S_2)(a,b)\| = \|(T_1 \otimes T_2)(a,b)\| = 1$ as claimed. Therefore, there exists an $x_0 \in \delta A$ such that $|((S_1 \otimes S_2)(a,b))(x_0)| = 1$.

Next we show that $\prod_{i=1}^m |((S_1 \otimes S_2)(a_i, b_i))(x_0)| = 1$. Since $\|T_1(a_i)\| = 1$, we have $|T_1(a_i)(y)| \le 1$ for any $y \in \delta B$, and thus $|T_1(a)(y)| = |\prod_{i=1}^m T_1(a_i)(y)| \le |T_1(a_i)(y)|$ for all $y \in \delta B$ and every $1 \le i \le m$. By Lemma 3.4, $|S_1(a)(x)| \le |S_1(a_i)(x)|$ for all $x \in \delta A$ and $1 \le i \le m$. A similar argument shows that $|S_2(b)(x)| \le |S_2(b_i)(x)|$ for all $x \in \delta A$ and $1 \le i \le m$. Hence,

$$|((S_1 \otimes S_2)(a,b))(x)| = |S_1(a)(x) S_2(b)(x)| \le |((S_1 \otimes S_2)(a_i, b_i))(x)|$$

for all $x\in\delta A$ and $1\leq i\leq m$. Therefore,

$$1=|((S_1\otimes S_2)(a,b))(x_0)|^m\leq\prod_{i=1}^m|((S_1\otimes S_2)(a_i,b_i))(x_0)|$$

$$\leq\prod_{i=1}^m\|(S_1\otimes S_2)(a_i,b_i)\|=\prod_{i=1}^m\|(T_1\otimes T_2)(a_i,b_i)\|=1,$$

where we made use of (3.1). Consequently $\prod_{i=1}^m|((S_1\otimes S_2)(a_i,b_i))(x_0)|=1$ as claimed. Hence $|((S_1\otimes S_2)(a_i,b_i))(x_0)|=1$ since $\|(S_1\otimes S_2)(a_i,b_i)\|=1$ for $1\leq i\leq m$. Hence $x_0\in\bigcap_{i=1}^m|(S_1\otimes S_2)(a_i,b_i)|^{-1}(1)$. Thus the family $\{|(S_1\otimes S_2)(a,b)|^{-1}(1):(a,b)\in\mathcal{F}_{y_0}\}$ has the finite intersection property as claimed. Since $|(S_1\otimes S_2)(a,b)|^{-1}(1)$ is a compact set for every $(a,b)\in\mathcal{F}_{y_0}$, we deduce that $\bigcap_{(a,b)\in\mathcal{F}_{y_0}}|(S_1\otimes S_2)(a,b)|^{-1}(1)\neq\varnothing$ for every $y_0\in\delta B$. $\square$

For any $y\in\delta B$ we consider the set $\mathfrak{S}_y=\bigcap_{(a,b)\in\mathcal{F}_y}|(S_1\otimes S_2)(a,b)|^{-1}(1)$.

Lemma 3.6. *$\delta A\cap\mathfrak{S}_y\neq\varnothing$ for every $y\in\delta B$.*

Proof. If $y\in\delta B$ then $\mathfrak{S}_y\neq\varnothing$ by Lemma 3.5. Let $x_0\in\mathfrak{S}_y$. By definition, $|((S_1\otimes S_2)(a,b))(x_0)|=1$ for every $(a,b)\in\mathcal{F}_y$. Therefore, $\|(S_1\otimes S_2)(a,b)\|=1$ for every $(a,b)\in\mathcal{F}_y$, since, by (3.1),

$$1=|((S_1\otimes S_2)(a,b))(x_0)|\leq\|(S_1\otimes S_2)(a,b)\|$$

$$=\|(T_1\otimes T_2)(a,b)\|\leq\|T_1(a)\|\,\|T_2(b)\|=1.$$

Let $f_{(a,b)}=(S_1\otimes S_2)(a,b)$ and $\lambda=\overline{f_{(a,b)}(x_0)}$. For each $(a,b)\in\mathcal{F}_y$ define the function $p_{(a,b)}=\frac{1}{2}((\lambda f_{(a,b)})^2+\lambda f_{(a,b)})\in A$. Since $\|f_{(a,b)}\|=\|(S_1\otimes S_2)(a,b)\|=1$, we see that $p_{(a,b)}$ is a peaking function of A such that $x_0\in p_{(a,b)}^{-1}(1)\subset|f_{(a,b)}|^{-1}(1)$ for all $(a,b)\in\mathcal{F}_y$. Hence the set $L_y=\bigcap_{(a,b)\in\mathcal{F}_y}p_{(a,b)}^{-1}(1)$ is a nonempty weak peak set of A. According to Zorn's lemma L_y contains a minimal weak peak set relative to the inclusion. As minimal weak peak sets are singletons (see e.g. [21, proof of Proposition 2.1]), there exists an $x\in\delta A\cap L_y$. Hence $x\in p_{(a,b)}^{-1}(1)\subset|f_{(a,b)}|^{-1}(1)=|(S_1\otimes S_2)(a,b)|^{-1}(1)$ for all $(a,b)\in\mathcal{F}_y$, and thus, $x\in\delta A\cap(\bigcap_{(a,b)\in\mathcal{F}_y}|(S_1\otimes S_2)(a,b)|^{-1}(1))=\delta A\cap\mathfrak{S}_y$. $\square$

Lemma 3.7. *For each $a\in\mathcal{J}_1$, $b\in\mathcal{J}_2$, all $y\in\delta B$ and all $x\in\delta A\cap\mathfrak{S}_y$ we have*

$$|((S_1\otimes S_2)(a,b))(x)|\leq|((T_1\otimes T_2)(a,b))(y)|.$$

Proof. Let $u\in P_{\mathcal{T}_1}(y)$, $v\in P_{\mathcal{T}_2}(y)$, $a_u\in\mathcal{J}_1$ and $b_v\in\mathcal{J}_2$ be such that $T_1(a_u)=u$ and $T_2(b_v)=v$. Thus $T_1(a_u)\in P_{\mathcal{T}_1}(y)\subset V_{\mathcal{T}_1}(y)$ and $T_2(b_v)\in V_{\mathcal{T}_2}(y)$, and therefore, by definition, $(a_u,b_v)\in\mathcal{F}_y$. Since $x\in\mathfrak{S}_y=\bigcap_{(a,b)\in\mathcal{F}_y}|(S_1\otimes S_2)(a,b)|^{-1}(1)$, we see that $|((S_1\otimes S_2)(a_u,b_v))(x)|=1$. Therefore, for every $a\in\mathcal{J}_1$ and $b\in\mathcal{J}_2$, we have

$$\begin{aligned}|((S_1\otimes S_2)(a,b))(x)| &= |((S_1\otimes S_2)(a,b))(x)|\,|((S_1\otimes S_2)(a_u,b_v))(x)|\\ &= |S_1(a)(x)S_2(b)(x)|\,|S_1(a_u)(x)S_2(b_v)(x)|\\ &\le \|(S_1\otimes S_2)(a,b_v)\|\,\|(S_1\otimes S_2)(a_u,b)\|\\ &= \|(T_1\otimes T_2)(a,b_v)\|\,\|(T_1\otimes T_2)(a_u,b)\|\\ &= \|T_1(a)v\|\,\|T_2(b)u\|,\end{aligned}$$

where we made use of (3.1). Since $u\in P_{\mathcal{T}_1}(y)$ and $v\in P_{\mathcal{T}_2}(y)$ were arbitrarily chosen, Lemma 3.3 yields

$$\begin{aligned}|((S_1\otimes S_2)(a,b))(x)| &\le \inf_{v\in P_{\mathcal{T}_2}(y)}\|T_1(a)v\|\inf_{u\in P_{\mathcal{T}_1}(y)}\|T_2(b)u\|\\ &= |T_1(a)(y)|\,|T_2(b)(y)| = |((T_1\otimes T_2)(a,b))(y)|\end{aligned}$$

for all $a\in\mathcal{J}_1$ and $b\in\mathcal{J}_2$. $\square$

Lemma 3.8. *For every $y\in\delta B$ there exists an $x\in\delta A\cap\mathfrak{S}_y$ such that*

$$(S_1\otimes S_2)^{-1}(V_A(x))\subset(T_1\otimes T_2)^{-1}(V_B(y)).$$

Recall that $V_A(x)=\{f\in A{:}|f(x)|=1=\|f\|\}$ by definition. $V_B(y)$ is defined in the same way. Note that $(T_1\otimes T_2)^{-1}(V_B(y))\neq\varnothing$ for every $y\in\delta B=p(\mathcal{T}_i)$, since

$$(T_1\otimes T_2)(a,b)=T_1(a)T_2(b)\in P_B(y)\subset V_B(y)$$

for any $a\in T_1^{-1}(P_{\mathcal{T}_1}(y))$ and $b\in T_2^{-1}(P_{\mathcal{T}_2}(y))$. Similarly, $(S_1\otimes S_2)^{-1}(V_A(x))\neq\varnothing$.

Proof. If $y\in\delta B$, then by Lemma 3.6 there exists some $x\in\delta A\cap\mathfrak{S}_y$. For any $(a,b)\in(S_1\otimes S_2)^{-1}(V_A(x))$ we have $|((S_1\otimes S_2)(a,b))(x)|=1=\|(S_1\otimes S_2)(a,b)\|$. We prove that $(a,b)\in(T_1\otimes T_2)^{-1}(V_B(y))$. By Lemma 3.7 and (3.1),

$$\begin{aligned}1 &= |((S_1\otimes S_2)(a,b))(x)|\le|((T_1\otimes T_2)(a,b))(y)|\\ &\le\|(T_1\otimes T_2)(a,b)\|=\|(S_1\otimes S_2)(a,b)\|=1,\end{aligned}$$

and therefore, $|((T_1\otimes T_2)(a,b))(y)|=1=\|(T_1\otimes T_2)(a,b)\|$. This implies that $(a,b)\in(T_1\otimes T_2)^{-1}(V_B(y))$, and consequently, $(S_1\otimes S_2)^{-1}(V_A(x))\subset(T_1\otimes T_2)^{-1}(V_B(y))$, as desired. $\square$

Lemma 3.9. *If $y, y' \in \delta B$ are so that $(T_1 \otimes T_2)^{-1}(V_B(y')) \subset (T_1 \otimes T_2)^{-1}(V_B(y))$, then $y = y'$.*

Proof. Suppose, on the contrary, that $y \neq y'$. By Lemma 3.2 there exist $u_i \in P_{\mathcal{T}_i}(y')$ such that $|u_i(y)| < 1$ for $i = 1, 2$. Let $a \in \mathcal{J}_1$ and $b \in \mathcal{J}_2$ be such that $T_1(a) = u_1$ and $T_2(b) = u_2$. Then $(T_1 \otimes T_2)(a, b) = u_1 u_2 \in P_B(y') \subset V_B(y')$. By the assumption, $(a, b) \in (T_1 \otimes T_2)^{-1}(V_B(y')) \subset (T_1 \otimes T_2)^{-1}(V_B(y))$. Hence $u_1 u_2 = (T_1 \otimes T_2)(a, b) \in V_B(y)$, and therefore $|(u_1 u_2)(y)| = 1$. This contradicts the inequality $|u_i(y)| < 1$, $i = 1, 2$. Consequently, $y = y'$, as desired. □

Since the conditions for the mappings $S_1 \otimes S_2$ and $T_1 \otimes T_2$ are symmetric, Lemmas 3.8 and 3.9 yield the following lemma.

Lemma 3.10. *For every $x \in \delta A$ there is a $y \in \delta B$ such that $(T_1 \otimes T_2)^{-1}(V_B(y)) \subset (S_1 \otimes S_2)^{-1}(V_A(x))$. If $(S_1 \otimes S_2)^{-1}(V_A(x')) \subset (S_1 \otimes S_2)^{-1}(V_A(x))$ for some $x, x' \in \delta A$, then $x = x'$.*

Lemma 3.11. *For each $y \in \delta B$ there exists a unique $x \in \delta A \cap \mathfrak{S}_y$ such that*

$$(T_1 \otimes T_2)^{-1}(V_B(y)) = (S_1 \otimes S_2)^{-1}(V_A(x)).$$

Proof. Let $y \in \delta B$. According to Lemma 3.8 there exists an $x \in \delta A \cap \mathfrak{S}_y$ such that $(S_1 \otimes S_2)^{-1}(V_A(x)) \subset (T_1 \otimes T_2)^{-1}(V_B(y))$. By Lemma 3.10 there exists a $y' \in \delta B$ with $(T_1 \otimes T_2)^{-1}(V_B(y')) \subset (S_1 \otimes S_2)^{-1}(V_A(x))$, and therefore

$$(T_1 \otimes T_2)^{-1}(V_B(y')) \subset (S_1 \otimes S_2)^{-1}(V_A(x)) \subset (T_1 \otimes T_2)^{-1}(V_B(y)).$$

Lemma 3.9 yields that $y = y'$, and thus $(T_1 \otimes T_2)^{-1}(V_B(y)) = (S_1 \otimes S_2)^{-1}(V_A(x))$.

We now prove the uniqueness of the element $x \in \delta A \cap \mathfrak{S}_y$. If $x' \in \delta A \cap \mathfrak{S}_y$ is such that

$$(S_1 \otimes S_2)^{-1}(V_A(x')) = (T_1 \otimes T_2)^{-1}(V_B(y)),$$

then

$$(S_1 \otimes S_2)^{-1}(V_A(x)) = (S_1 \otimes S_2)^{-1}(V_A(x')),$$

and hence $x = x'$ by Lemma 3.10, as desired. □

Lemma 3.12. *There exists a bijection $\phi\colon \delta B \to \delta A$ such that for every $y \in \delta B$, $\phi(y) \in \mathfrak{S}_y$ and $(T_1 \otimes T_2)^{-1}(V_B(y)) = (S_1 \otimes S_2)^{-1}(V_A(\phi(y)))$.*

Proof. According to Lemma 3.11, for each $y \in \delta B$ there exists a unique $x \in \delta A \cap \mathfrak{S}_y$ such that $(T_1 \otimes T_2)^{-1}(V_B(y)) = (S_1 \otimes S_2)^{-1}(V_A(x))$. Hence the correspondence $\phi(y) = x$ is a well-defined mapping from δB to δA such that $\phi(y) \in \mathfrak{S}_y$ and $(T_1 \otimes T_2)^{-1}(V_B(y)) = (S_1 \otimes S_2)^{-1}(V_A(\phi(y)))$ for every $y \in \delta B$. We claim that the mapping ϕ is surjective. Let $x \in \delta A$. Lemma 3.10 shows that there exists a $y_0 \in \delta B$ such that $(S_1 \otimes S_2)^{-1}(V_A(\phi(y_0))) = (T_1 \otimes T_2)^{-1}(V_B(y_0)) \subset (S_1 \otimes S_2)^{-1}(V_A(x))$. Hence $(S_1 \otimes S_2)^{-1}(V_A(\phi(y_0))) \subset (S_1 \otimes S_2)^{-1}(V_A(x))$. By Lemma 3.10 we see that $\phi(y_0) = x$, and therefore ϕ is surjective, as claimed. Suppose that $\phi(y_1) = \phi(y_2)$ for $y_1, y_2 \in \delta B$. Then

$$\begin{aligned}(T_1 \otimes T_2)^{-1}(V_B(y_1)) &= (S_1 \otimes S_2)^{-1}(V_A(\phi(y_1))) \\ &= (S_1 \otimes S_2)^{-1}(V_A(\phi(y_2))) = (T_1 \otimes T_2)^{-1}(V_B(y_2)),\end{aligned}$$

and thus $y_1 = y_2$ by Lemma 3.9. Therefore, the mapping ϕ is also injective. □

Lemma 3.13. *If $\phi\colon \delta B \to \delta A$ is the bijection from Lemma 3.12, then*

$$|((T_1 \otimes T_2)(a,b))(y)| = |((S_1 \otimes S_2)(a,b))(\phi(y))|$$

for all $a \in \mathcal{J}_1$, $b \in \mathcal{J}_2$ and $y \in \delta B$.

Proof. If $\phi\colon \delta B \to \delta A$ is the bijection from Lemma 3.12, then $\phi(y) \in \mathfrak{S}_y$ and $(T_1 \otimes T_2)^{-1}(V_B(y)) = (S_1 \otimes S_2)^{-1}(V_A(\phi(y)))$ for every $y \in \delta B$. The fact that $\phi(y) \in \mathfrak{S}_y$ allows us to apply Lemma 3.7 to $\phi(y)$ instead of to x. Let $a \in \mathcal{J}_1$, $b \in \mathcal{J}_2$ and $y \in \delta B$. We show that $|((T_1 \otimes T_2)(a,b))(y)| = |((S_1 \otimes S_2)(a,b))(\phi(y))|$. Applied to $\phi(y)$, Lemma 3.7 yields $|((S_1 \otimes S_2)(a,b))(\phi(y))| \le |((T_1 \otimes T_2)(a,b))(y)|$.

Next we prove the converse inequality. If $u \in P_{\mathcal{S}_1}(\phi(y))$, $v \in P_{\mathcal{S}_2}(\phi(y))$, $a_u \in \mathcal{J}_1$ and $b_v \in \mathcal{J}_2$ are such that $S_1(a_u) = u$ and $S_2(b_v) = v$, then

$$(S_1 \otimes S_2)(a_u, b_v) = S_1(a_u) S_2(b_v) = uv \in V_A(\phi(y)),$$

and therefore

$$(a_u, b_v) \in (S_1 \otimes S_2)^{-1}(V_A(\phi(y))) = (T_1 \otimes T_2)^{-1}(V_B(y)).$$

Thus $|((T_1 \otimes T_2)(a_u, b_v))(y)| = 1$, and consequently

$$\begin{aligned}|((T_1 \otimes T_2)(a,b))(y)| &= |((T_1 \otimes T_2)(a,b))(y)|\,|((T_1 \otimes T_2)(a_u,b_v))(y)| \\ &= |(T_1(a) T_2(b_v))(y)|\,|(T_1(a_u) T_2(b))(y)|\end{aligned}$$

$$\begin{aligned}&\le \|(T_1\otimes T_2)(a,b_v)\|\,\|(T_1\otimes T_2)(a_u,b)\|\\&=\|(S_1\otimes S_2)(a,b_v)\|\,\|(S_1\otimes S_2)(a_u,b)\|\\&=\|S_1(a)v\|\,\|S_2(b)u\|,\end{aligned}$$

where we made use of (3.1). Since $u\in P_{\mathcal{S}_1}(\phi(y))$ and $v\in P_{\mathcal{S}_2}(\phi(y))$ are chosen arbitrarily, Lemma 3.3 yields

$$\begin{aligned}|((T_1\otimes T_2)(a,b))(y)| &\le \inf_{v\in P_{\mathcal{S}_2}(\phi(y))}\|S_1(a)v\| \inf_{u\in P_{\mathcal{S}_1}(\phi(y))}\|S_2(b)u\|\\&=|S_1(a)(\phi(y))|\,|S_2(b)(\phi(y))|=|((S_1\otimes S_2)(a,b))(\phi(y))|,\end{aligned}$$

as claimed. Thus $|((T_1\otimes T_2)(a,b))(y)|=|((S_1\otimes S_2)(a,b))(\phi(y))|$ for every $a\in\mathcal{J}_1$, $b\in\mathcal{J}_2$ and $y\in\delta B$. $\square$

Lemma 3.14. *The bijection $\phi\colon \delta B\to\delta A$ is a homeomorphism.*

Proof. Let $y_0\in\delta B$. For each open neighborhood $O\subset\delta A$ of $\phi(y_0)$ there exists an open set $\widetilde{O}\subset X$ such that $O=\widetilde{O}\cap\delta A$, where X is the underlying space of A. The set $K=X\setminus\widetilde{O}$ is a closed subset of X and $\phi(y_0)\notin K$. By Lemma 3.2 there exist $h_1\in P_{\mathcal{S}_1}(\phi(y_0))$ and $h_2\in P_{\mathcal{S}_2}(\phi(y_0))$ such that $|h_i|<\frac{1}{2}$ on K for $i=1,2$. Choose $a\in\mathcal{J}_1$ and $b\in\mathcal{J}_2$ so that $S_1(a)=h_1$ and $S_2(b)=h_2$, and set

$$W=\left\{y\in\delta B : |((T_1\otimes T_2)(a,b))(y)|>\tfrac{1}{2}\right\}.$$

Then $y_0\in W$, since

$$|((T_1\otimes T_2)(a,b))(y_0)|=|((S_1\otimes S_2)(a,b))(\phi(y_0))|=|(h_1h_2)(\phi(y_0))|=1.$$

The continuity of $(T_1\otimes T_2)(a,b)$ implies that W is an open neighborhood of y_0 in δB. We assert that $\phi(W)\subset O$. In fact, for each $y\in W$ we have

$$|(h_1h_2)(\phi(y))|=|((S_1\otimes S_2)(a,b))(\phi(y))|=|((T_1\otimes T_2)(a,b))(y)|>\tfrac{1}{2}.$$

Since $|h_i|<\frac{1}{2}$ on K, we deduce that $\phi(y)\notin K=X\setminus\widetilde{O}$. Thus, $\phi(y)\in\widetilde{O}\cap\delta A=O$, and consequently $\phi(W)\subset O$, as claimed. This implies that the map ϕ is continuous at y_0. Since $y_0\in\delta B$ was arbitrarily chosen, we conclude that $\phi\colon\delta B\to\delta A$ is continuous on δB. The same argument, applied to $\phi^{-1}\colon\delta A\to\delta B$, shows that ϕ^{-1} is continuous too. Consequently, ϕ is a homeomorphism, as claimed. $\square$

Proposition 3.1 follows immediately from Lemmas 3.13 and 3.14. What it says is that the moduli of the maps $S_1 \otimes S_2$ and $T_1 \otimes T_2$ are equal up to the composition with a homeomorphism $\phi\colon \delta B \to \delta A$, i.e. that the diagram

$$\begin{tikzcd}
 & \mathcal{J}_1 \times \mathcal{J}_2 \arrow[dl, "|S_1 \otimes S_2|"'] \arrow[dr, "|T_1 \otimes T_2|"] & \\
|\mathcal{S}_1 \cdot \mathcal{S}_2|\big|_{\delta A} \arrow[rr, "C_\phi"'] & & |\mathcal{T}_1 \cdot \mathcal{T}_2|\big|_{\delta B}
\end{tikzcd}$$

is commutative. Here $C_\phi\colon C(\delta A) \to C(\delta B)$ is the composition operator induced by the homeomorphism $\phi\colon \delta B \to \delta A$, namely, $C_\phi(f) = f \circ \phi$, $f \in C(\delta A)$, and the mappings $|S_1 \otimes S_2|$ and $|T_1 \otimes T_2|$ are assumed to map $\mathcal{J}_1 \times \mathcal{J}_2$ into the restriction algebras $C(\delta A)$ and $C(\delta B)$, respectively. Note that since the mapping $\phi\colon \delta B \to \delta A$ is a homeomorphism, the equality (3.2) yields

$$|((T_1 \otimes T_2)(a,b))(\delta B)| = |((S_1 \otimes S_2)(a,b))(\delta A)|$$

for all $a \in \mathcal{J}_1$ and $b \in \mathcal{J}_2$.

4. Proofs of the main results

The property (3.1) alone is not sufficient for the equality

$$((T_1 \otimes T_2)(a,b))(y) = ((S_1 \otimes S_2)(a,b))(\phi(y)), \tag{4.1}$$

i.e. the equality (3.2) with the moduli deleted, to hold for every $a \in \mathcal{J}_1$, $b \in \mathcal{J}_2$ and $y \in \delta B$. In this section we establish spectral conditions on $\otimes$-products of two pairs of mappings, not necessarily linear, into multiplicative subsets of function algebras in order for (4.1) to hold, and together, the mappings in the first pair to equal the mappings in the second one up to compositions with certain operators. Recall that if $f \in A$, then the range of f is the set $\mathrm{Ran}(f) = f(X)$ and either $\mathrm{Ran}(f)$ or $\mathrm{Ran}(f) \cup \{0\}$ are compact subsets of $\mathbb{C}$.

We precede the proof of Theorem 2.2 by the following useful result, mentioned in Section 2, the proof of which we provide for reference purposes.

Lemma 4.1. *If A is a function algebra on a locally compact Hausdorff space X then*

$$\sigma_\pi(f) = \{z \in f(\delta A) : |z| = \|f\|\}, \quad f \in A.$$

Proof. We consider only the case when $f\in A\setminus\{0\}$. Since $f(\delta A)\subset\operatorname{Ran}(f)$,

$$\{z\in f(\delta A):|z|=\|f\|\}\subset\{z\in\operatorname{Ran}(f):|z|=\|f\|\}=\sigma_\pi(f).$$

Conversely, if $z_0\in\sigma_\pi(f)$, then there is an $x_0\in X$ such that $z_0=f(x_0)$ and $|f(x_0)|=|z_0|=\|f\|$. Let $f_0=f/\|f\|\in A$ and $\lambda=\overline{f_0(x_0)}$. Then $f_1=\frac{1}{2}((\lambda f_0)^2+\lambda f_0)$ is a peaking function of A. Hence $f_1^{-1}(1)=f_0^{-1}(\overline{\lambda})$ is a peak set of A. As peak sets of function algebras meet their Choquet boundaries (see e.g. [21, Proposition 2.1]), there is an $x_1\in f_0^{-1}(\overline{\lambda})\cap\delta A$. Therefore $f_0(x_1)=\overline{\lambda}=f_0(x_0)$, and hence $f(x_1)=f(x_0)=z_0$. Thus $z_0=f(x_1)\in f(\delta A)$, which implies that $z_0\in\{z\in f(\delta A):|z|=\|f\|\}$. Consequently, $\sigma_\pi(f)\subset\{z\in f(\delta A):|z|=\|f\|\}$. $\square$

4.1. Proof of Theorem 2.2

Let $\phi\colon\delta B\to\delta A$ be the homeomorphism from Proposition 3.1 such that

$$|((T_1\otimes T_2)(a,b))(y)|=|((S_1\otimes S_2)(a,b))(\phi(y))| \tag{4.2}$$

for all $a\in\mathcal{J}_1$, $b\in\mathcal{J}_2$ and $y\in\delta B$. Conditions (i), (ii) and Lemma 4.1 imply that

$$\sigma_\pi((S_1\otimes S_2)(a,b))\subset\sigma_\pi((T_1\otimes T_2)(a,b)) \tag{4.3}$$

for all $a\in\mathcal{J}_1$ and $b\in\mathcal{J}_2$. Let $y\in\delta B$. Note that $P_{\mathcal{T}_i}(y)\neq\varnothing$ as $p(\mathcal{T}_i)=\delta B$ for $i=1,2$. We will show that $(S_1(a)S_2(b))(\phi(y))=1$ for $a\in T_1^{-1}(P_{\mathcal{T}_1}(y))$ and $b\in T_2^{-1}(P_{\mathcal{T}_2}(y))$. Indeed, if $a\in T_1^{-1}(P_{\mathcal{T}_1}(y))$ and $b\in T_2^{-1}(P_{\mathcal{T}_2}(y))$, then, according to (4.3),

$$\sigma_\pi((S_1\otimes S_2)(a,b))\subset\sigma_\pi((T_1\otimes T_2)(a,b))=\sigma_\pi(T_1(a)T_2(b))=\{1\}.$$

Therefore, $(S_1\otimes S_2)(a,b)$ is a peaking function of A. The equality (4.2) yields

$$|((S_1\otimes S_2)(a,b))(\phi(y))|=|((T_1\otimes T_2)(a,b))(y)|=|T_1(a)(y)T_2(b)(y)|=1.$$

Thus the peaking function $(S_1\otimes S_2)(a,b)$ attains its maximum modulus at $\phi(y)$. Hence $(S_1(a)S_2(b))(\phi(y))=((S_1\otimes S_2)(a,b))(\phi(y))=1$, as desired. Consequently, the function $\alpha\colon\delta B\to\mathbb{C}\setminus\{0\}$ defined as

$$\alpha(y)=\frac{1}{S_1(a)(\phi(y))}=S_2(b)(\phi(y)) \tag{4.4}$$

for any $a\in T_1^{-1}(P_{\mathcal{T}_1}(y))$ and any $b\in T_2^{-1}(P_{\mathcal{T}_2}(y))$ is well defined.

Next we show that $T_1(a)(y)=\alpha(y)S_1(a)(\phi(y))$ for all $a\in\mathcal{J}_1$ and $y\in\delta B$. Let $a_0\in\mathcal{J}_1$ and $y_0\in\delta B$. For any $b\in T_2^{-1}(P_{\mathcal{T}_2}(y_0))$, equality (4.2) yields

$$\begin{aligned}|T_1(a_0)(y_0)| &= |T_1(a_0)(y_0)|\,|T_2(b)(y_0)| = |((T_1\otimes T_2)(a_0,b))(y_0)| \\ &= |((S_1\otimes S_2)(a_0,b))(\phi(y_0))| = |\alpha(y_0)S_1(a_0)(\phi(y_0))|.\end{aligned}$$

It follows that

$$|T_1(a_0)(y_0)| = |\alpha(y_0)S_1(a_0)(\phi(y_0))|. \tag{4.5}$$

Therefore, we only need to consider the case when $T_1(a_0)(y_0)\neq 0$, or, equivalently, $S_1(a_0)(\phi(y_0))\neq 0$. In the proof of this case we use some reasoning similar to the ones when $\mathcal{S}_i$ and $\mathcal{T}_i$ are Banach function algebras (cf. [9, Lemma 3.11]). A discussion with F. Sady is gratefully acknowledged. Since $p(\mathcal{T}_2)=\delta B$ and $\mathcal{T}_2$ is an algebra, there is a $v\in P_{\overline{\mathcal{T}}_2}(y_0)$ with $\sigma_\pi(T_1(a_0)v)=\{T_1(a_0)(y_0)\}$, and such that there are elements $v_n\in\mathcal{T}_2$ converging uniformly to v of type $v_n=r_nk_n$ for some $k_n\in P_{\mathcal{T}_2}(y_0)$ and $r_n=1-2^{-n}$, where $\overline{\mathcal{T}}_2$ is the uniform closure of $\mathcal{T}_2$ (see [9], and also [6, Lemma 2.3] and [26, Proposition 3.1]). Clearly, $r_n\to 1$ as $n\to\infty$. Let $T_2(b_n)=v_n$ for some $b_n\in\mathcal{J}_2$. We claim that

$$S_2(b_n)(\phi(y_0)) = \alpha(y_0)r_n \tag{4.6}$$

for all $n\in\mathbb{N}$. For each $a\in T_1^{-1}(P_{\mathcal{T}_1}(y_0))$, we have $T_1(a)k_n\in P_B(y_0)$, and thus equality (4.3) implies $\sigma_\pi((S_1\otimes S_2)(a,b_n))\subset\sigma_\pi((T_1\otimes T_2)(a,b_n))=\sigma_\pi(T_1(a)v_n)=\{r_n\}$. By (4.2),

$$|((S_1\otimes S_2)(a,b_n))(\phi(y_0))| = |((T_1\otimes T_2)(a,b_n))(y_0)| = r_n$$

for every $a\in T_1^{-1}(P_{\mathcal{T}_1}(y_0))$. Consequently, the function $(S_1\otimes S_2)(a,b_n)$ attains its maximum modulus, r_n, at the point $\phi(y_0)$. Hence

$$(S_1(a)S_2(b_n))(\phi(y_0)) = ((S_1\otimes S_2)(a,b_n))(\phi(y_0)) = r_n.$$

Since $a\in T_1^{-1}(P_{\mathcal{T}_1}(y_0))$, then, by the definition, $S_1(a)(\phi(y_0))=1/\alpha(y_0)$ and therefore, $S_2(b_n)(\phi(y_0))=r_n/S_1(a)(\phi(y_0))=\alpha(y_0)r_n$ as claimed. Equality (4.6) implies that $S_2(b_n)(\phi(y_0))=\alpha(y_0)r_n\to\alpha(y_0)$ as $n\to\infty$. Note that, by (ii),

$$(S_1(a_0)S_2(b_n))(\phi(y_0)) \in ((S_1\otimes S_2)(a_0,b_n))(\delta A) \subset \mathrm{Ran}((T_1\otimes T_2)(a_0,b_n)).$$

Hence there are points $y_n\in Y$ such that

$$(T_1(a_0)T_2(b_n))(y_n) = (S_1(a_0)S_2(b_n))(\phi(y_0)).$$

If $\lambda_n = (T_1(a_0)v)(y_n)$ and $\lambda = \alpha(y_0)S_1(a_0)(\phi(y_0))$, then we have

$$\begin{aligned}|\lambda_n - \lambda| &\le |\lambda_n - (T_1(a_0)T_2(b_n))(y_n)| + |(T_1(a_0)T_2(b_n))(y_n) - \lambda| \\ &= |T_1(a_0)(y_n)|\,|v(y_n) - T_2(b_n)(y_n)| + |(S_1(a_0)S_2(b_n))(\phi(y_0)) - \lambda| \\ &= |T_1(a_0)(y_n)|\,|v(y_n) - v_n(y_n)| + |S_1(a_0)(\phi(y_0))|\,|S_2(b_n)(\phi(y_0)) - \alpha(y_0)| \\ &\le \|T_1(a_0)\|\,\|v - v_n\| + |S_1(a_0)(\phi(y_0))|\,|\alpha(y_0)r_n - \alpha(y_0)| \\ &\to 0\end{aligned}$$

as $n \to \infty$. Hence $\lambda_n \to \lambda$, i.e. $(T_1(a_0)v)(y_n) \to \alpha(y_0)S_1(a_0)(\phi(y_0)) \neq 0$. Therefore, $\alpha(y_0)S_1(a_0)(\phi(y_0)) \in \mathrm{Ran}(T_1(a_0)v)$, as $\mathrm{Ran}(T_1(a_0)v) \cup \{0\}$ is a compact set in $\mathbb{C}$. The equality (4.5) implies that $|\alpha(y_0)S_1(a_0)(\phi(y_0))| = |T_1(a_0)(y_0)|$. Since $\sigma_\pi(T_1(a_0)v) = \{T_1(a_0)(y_0)\}$ it follows that $\alpha(y_0)S_1(a_0)(\phi(y_0)) = T_1(a_0)(y_0)$, as claimed.

We next prove that the function $\alpha\colon \delta B \to \mathbb{C} \setminus \{0\}$ is continuous. Let $y_0 \in \delta B$. Since $p(\mathcal{S}_1) = \delta A$, we have $P_{\mathcal{S}_1}(\phi(y_0)) \neq \varnothing$ and therefore there exists an $a_0 \in \mathcal{J}_1$ such that $S_1(a_0)(\phi(y_0)) \neq 0$. Since $S_1(a_0)$ and ϕ are continuous, there exists an open neighborhood $O \subset \delta B$ of y_0 such that $S_1(a_0)(\phi(y)) \neq 0$ for all $y \in O$. Thus $\alpha = T_1(a_0)/(S_1(a_0) \circ \phi)$ on O. The continuity of $T_1(a_0)$ and $S_1(a_0) \circ \phi$ at y_0 implies that α is continuous at $y_0 \in \delta B$. Since $y_0 \in \delta B$ was chosen arbitrarily, we see that α is continuous on δB.

It remains to show that $T_2(b)(y) = S_2(b)(\phi(y))/\alpha(y)$ for all $b \in \mathcal{J}_2$ and $y \in \delta B$. Since the conditions on S_1, T_1 and S_2, T_2 are symmetric, we get, by (4.4), $T_2(b)(y) = \beta(y)S_2(b)(\phi(y))$ for all $b \in \mathcal{J}_2$ and $y \in \delta B$ with $\beta(y) = 1/S_2(c)(\phi(y))$, where c is an arbitrary element in $T_2^{-1}(P_{\mathcal{T}_2}(y))$. So $\beta(y) = 1/\alpha(y)$ by definition, and therefore $T_2(b)(y) = S_2(b)(\phi(y))/\alpha(y)$.

4.2. Proof of Theorem 2.4

Recall that if $(\mathcal{S}, \mathcal{S}')$ is a *Bishop pair*, then for every $x \in p(\mathcal{S})$ and each $f' \in \mathcal{S}'$ with $f'(x) \neq 0$ there is a peaking function $h \in P_{\mathcal{S}}(x)$ such that $\sigma_\pi(f'h) = \{f'(x)\}$, and for every $x' \in p(\mathcal{S}')$ and each $f \in \mathcal{S}$ with $f(x') \neq 0$ there is a peaking function $h' \in P_{\mathcal{S}'}(x')$ such that $\sigma_\pi(fh') = \{f(x')\}$. A set $\mathcal{S}$ is a *Bishop set* if $(\mathcal{S}, \mathcal{S})$ is a Bishop pair.

The inclusion (2.2) implies that $\|(T_1 \otimes T_2)(a,b)\| = \|(S_1 \otimes S_2)(a,b)\|$ for all $a \in \mathcal{J}_1$ and $b \in \mathcal{J}_2$. Let $\phi\colon \delta B \to \delta A$ be the homeomorphism from Proposition 3.1 such that

$$|((T_1 \otimes T_2)(a,b))(y)| = |((S_1 \otimes S_2)(a,b))(\phi(y))| \tag{4.7}$$

for all $a\in\mathcal{J}_1$, $b\in\mathcal{J}_2$ and $y\in\delta B$. As in the proof of Theorem 2.2 we define a function $\alpha\colon \delta B\to\mathbb{C}\setminus\{0\}$ by

$$\alpha(y)=\frac{1}{S_1(a)(\phi(y))}=S_2(b)(\phi(y))$$

for any $a\in T_1^{-1}(P_{\mathcal{T}_1}(y))$ and $b\in T_2^{-1}(P_{\mathcal{T}_2}(y))$, such that

$$|T_1(a)(y)|=|\alpha(y)S_1(a)(\phi(y))|,$$

where $a\in\mathcal{J}_1$ and $y\in\delta B$ (see (4.5)). Let $a\in\mathcal{J}_1$ and $y\in\delta B$. Therefore to prove that $T_1(a)(y)=\alpha(y)S_1(a)(\phi(y))$ we only need to consider the case when $T_1(a)(y)\neq 0$, or, equivalently, $S_1(a)(\phi(y))\neq 0$. Since $(\mathcal{T}_1,\mathcal{T}_2)$ is a Bishop pair, there is an $h\in P_{\mathcal{T}_2}(y)$ such that $\sigma_\pi(T_1(a)h)=\{T_1(a)(y)\}$. If $b_h\in\mathcal{J}_2$ is such that $T_2(b_h)=h$, then we have $\sigma_\pi((T_1\otimes T_2)(a,b_h))=\{T_1(a)(y)\}$. The inclusion (2.2) yields $\sigma_\pi((S_1\otimes S_2)(a,b_h))\subset\sigma_\pi((T_1\otimes T_2)(a,b_h))=\{T_1(a)(y)\}$. By equality (4.7),

$$|((S_1\otimes S_2)(a,b_h))(\phi(y))|=|((T_1\otimes T_2)(a,b_h))(y)|=|T_1(a)(y)|.$$

Consequently, the function $(S_1\otimes S_2)(a,b_h)$ attains its maximum modulus at $\phi(y)$, and therefore, $((S_1\otimes S_2)(a,b_h))(\phi(y))=T_1(a)(y)$. As $b_h\in T_2^{-1}(P_{\mathcal{T}_2}(y))$ and, by definition, $\alpha(y)=S_2(b_h)(\phi(y))$, we have $T_1(a)(y)=\alpha(y)S_1(a)(\phi(y))$ as claimed. The continuity of α and the equality $T_2(b)(y)=S_2(b)(\phi(y))/\alpha(y)$ can be proven in the same way as in Theorem 2.2. This completes the proof of Theorem 2.4.

By using arguments similar to the ones in the proof of Theorem 2.4 (and of Theorem 2.5 below) one can prove the following result.

Proposition 4.2. *Let $\mathcal{S}_1,\mathcal{S}_2\subset A$ and $\mathcal{T}_1,\mathcal{T}_2\subset B$ be multiplicative sets in function algebras A and B on locally compact Hausdorff spaces X and Y such that $p(\mathcal{S}_i)=\delta A$, $p(\mathcal{T}_i)=\delta B$ and $(\mathcal{T}_1,\mathcal{T}_2)$ is a Bishop pair. If $\mathcal{J}_i$ are arbitrary sets of parameters and the pairs of surjective mappings $S_i\colon \mathcal{J}_i\to\mathcal{S}_i$ and $T_i\colon \mathcal{J}_i\to\mathcal{T}_i$, $i=1,2$, satisfy the conditions*

$$\sigma_\pi((S_1\otimes S_2)(a,b))\cap\sigma_\pi((T_1\otimes T_2)(a,b))\neq\varnothing \quad \textit{and} \quad \sigma_\pi(S_i(c))\subset\sigma_\pi(T_i(c))$$

for all $a\in\mathcal{J}_1$, $b\in\mathcal{J}_2$ and $c\in\mathcal{J}_i$, $i=1,2$, then there is a homeomorphism $\phi\colon \delta B\to\delta A$ so that

$$T_1(a)(y)=S_1(a)(\phi(y)) \quad \textit{and} \quad T_2(b)(y)=S_2(b)(\phi(y))$$

for all $a\in\mathcal{J}_1$, $b\in\mathcal{J}_2$ and $y\in\delta B$.

[14, Theorem 5], one of the main results in [14], is a consequence of this theorem. For subsets $\mathcal{T}_i$ with more specific properties we can remove the second spectral condition. Recall that $P^\circ_{\mathcal{S}}(x)$ is the set of all peak functions in a subset $\mathcal{S}$ of a function algebra A that peak at $x\in\delta A$, i.e. $P^\circ_{\mathcal{S}}(x)=\{h\in P_{\mathcal{S}}(x):|h|<1 \text{ on } \delta A\setminus\{x\}\}$. A point $x\in X$ is a peak point of $\mathcal{S}$, i.e. $x\in p^\circ(\mathcal{S})$ if and only if $P^\circ_{\mathcal{S}}(x)\neq\varnothing$.

4.3. Proof of Theorem 2.5

Condition (2.3) implies that $\|(T_1\otimes T_2)(a,b)\|=\|(S_1\otimes S_2)(a,b)\|$ for all $a\in\mathcal{J}_1$ and $b\in\mathcal{J}_2$. Note that $p(\mathcal{T}_i)=\delta B$, as $\delta B=p^\circ(\mathcal{T}_i)\subset p(\mathcal{T}_i)\subset\delta B$, $i=1,2$. Let $\phi\colon \delta B\to\delta A$ be the homeomorphism from Proposition 3.1 such that

$$|((T_1\otimes T_2)(a,b))(y)|=|((S_1\otimes S_2)(a,b))(\phi(y))| \tag{4.8}$$

for all $a\in\mathcal{J}_1$, $b\in\mathcal{J}_2$ and $y\in\delta B$. Let $y\in\delta B$. Since, by the hypothesis, y is a peak point of $\mathcal{T}_i$, we get that $P^\circ_{\mathcal{T}_i}(y)\neq\varnothing$ for $i=1,2$. We claim that $(S_1(a)S_2(b))(\phi(y))=1$ for all $a\in T_1^{-1}(P^\circ_{\mathcal{T}_1}(y))$ and $b\in T_2^{-1}(P_{\mathcal{T}_2}(y))$. Indeed, if both $a\in T_1^{-1}(P^\circ_{\mathcal{T}_1}(y))$ and $b\in T_2^{-1}(P_{\mathcal{T}_2}(y))$, then we have $(T_1\otimes T_2)(a,b)\in P^\circ_B(y)$, since $|T_1(a)|<1$ on $\delta B\setminus\{y\}$. Therefore $\sigma_\pi((T_1\otimes T_2)(a,b))=\{1\}$ and $((T_1\otimes T_2)(a,b))(y)=1$. By condition (2.3), $1\in\sigma_\pi((S_1\otimes S_2)(a,b))$, and hence, by Lemma 4.1, there exists an $x\in\delta A$ such that $((S_1\otimes S_2)(a,b))(x)=1$. Choose $y'\in\delta B$ so that $\phi(y')=x$. Therefore, by (4.8), $|((T_1\otimes T_2)(a,b))(y')|=1$. Hence, $(T_1\otimes T_2)(a,b)$ is a peaking function that attains its maximum modulus at y', and consequently $((T_1\otimes T_2)(a,b))(y')=1$. This implies that $y'=y$, since $|(T_1\otimes T_2)(a,b)|<1$ on $\delta B\setminus\{y\}$. Hence $x=\phi(y)$, and thus $(S_1(a)S_2(b))(\phi(y))=((S_1\otimes S_2)(a,b))(\phi(y))=1$ as claimed. Now $\alpha\colon \delta B\to\mathbb{C}\setminus\{0\}$ defined as

$$\alpha(y)=\frac{1}{S_1(a)(\phi(y))}=S_2(b)(\phi(y))$$

for any $a\in T_1^{-1}(P^\circ_{\mathcal{T}_1}(y))$ and for all $b\in T_2^{-1}(P_{\mathcal{T}_2}(y))$, is well defined.

We claim that $T_1(a)(y)=\alpha(y)S_1(a)(\phi(y))$ for all $a\in\mathcal{J}_1$ and $y\in\delta B$. Let $a\in\mathcal{J}_1$ and $y_0\in\delta B$. If $b\in T_2^{-1}(P_{\mathcal{T}_2}(y_0))$, then $T_2(b)(y_0)=1$, and therefore (4.8) implies

$$|T_1(a)(y_0)|=|((T_1\otimes T_2)(a,b))(y_0)|=|((S_1\otimes S_2)(a,b))(\phi(y_0))|=|\alpha(y_0)S_1(a)(\phi(y_0))|.$$

Therefore, we only need to consider the case when $T_1(a)(y_0)\neq 0$. Since $(\mathcal{T}_1,\mathcal{T}_2)$ is a Bishop pair, there is $u\in P_{\mathcal{T}_2}(y_0)$ such that $\sigma_\pi(T_1(a)u)=\{T_1(a)(y_0)\}$. As y_0 is a peak point of $\mathcal{T}_2$, we can choose $v\in P_{\mathcal{T}_2}(y_0)$ so that $|v(y)|<1$ for $y\neq y_0$. Thus, $w=uv\in P_{\mathcal{T}_2}(y_0)$ satisfies $\sigma_\pi(T_1(a)w)=\{T_1(a)(y_0)\}$ and $|T_1(a)(y)w(y)|<|T_1(a)(y_0)|$ for $y\neq y_0$. Let $b_w\in\mathcal{J}_2$ be such that $T_2(b_w)=w\in P_{\mathcal{T}_2}(y_0)$. Therefore, $\sigma_\pi((T_1\otimes T_2)(a,b_w))=\{T_1(a)(y_0)\}$. Condition (2.3) implies that $T_1(a)(y_0)\in\sigma_\pi((S_1\otimes S_2)(a,b_w))$.

By Lemma 4.1 there exists an $x_0 \in \delta A$ such that $((S_1 \otimes S_2)(a, b_w))(x_0) = T_1(a)(y_0)$. The equality (4.8) yields

$$|((T_1 \otimes T_2)(a, b_w))(\phi^{-1}(x_0))| = |((S_1 \otimes S_2)(a, b_w))(x_0)| = |T_1(a)(y_0)|.$$

Therefore, $\phi^{-1}(x_0) = y_0$, since $|((T_1 \otimes T_2)(a, b_w))(y)| = |T_1(a)(y)w(y)| < |T_1(a)(y_0)|$ for $y \neq y_0$. Therefore, $x_0 = \phi(y_0)$, and thus $T_1(a)(y_0) = ((S_1 \otimes S_2)(a, b_w))(\phi(y_0))$. Since $b_w \in T_2^{-1}(P_{\mathcal{T}_2}(y_0))$, we have $\alpha(y_0) = S_2(b_w)(\phi(y_0))$ by the definition, and consequently $T_1(a)(y_0) = \alpha(y_0) S_1(a)(\phi(y_0))$ as claimed. The continuity of α and the equality $T_2(b)(y) = S_2(b)(\phi(y))/\alpha(y)$ can be proven by the same arguments as in the proof of Theorem 2.2. This completes the proof of Theorem 2.5.

It is obvious that the spectral conditions in the main theorems can be written, equivalently, as $\|S_1(a)S_2(b)\| = \|T_1(a)T_2(b)\|$, $(S_1(a)S_2(b))(\delta A) \subset \operatorname{Ran}(T_1(a)T_2(b))$, $\sigma_\pi(S_1(a)S_2(b)) \subset \sigma_\pi(T_1(a)T_2(b))$, and $\sigma_\pi(S_1(a)S_2(b)) \cap \sigma_\pi(T_1(a)T_2(b)) \neq \varnothing$, respectively. The conclusions of the main theorems imply that the diagram

$$\begin{tikzcd}
\mathcal{J}_i \arrow[r, "T_i"] \arrow[d, "S_i"'] & \mathcal{T}_i|_{\delta B} \arrow[d, "M_i"'] \\
\mathcal{S}_i|_{\delta A} \arrow[r, "C_\phi"] & C(\delta B)
\end{tikzcd}$$

is commutative, where M_i, $i=1,2$, are the multiplicative operators $M_1(f) = (1/\alpha)f$, $M_2 = \alpha f$ and $C_\phi(S_i(a)) = S_i(a) \circ \phi$. Here the mappings S_i and T_i are assumed to map $\mathcal{J}_i$ onto the restriction sets $\mathcal{S}_i|_{\delta A}$ and $\mathcal{T}_i|_{\delta B}$, respectively. In particular, we see that $S_i(a) = 0$ if and only if $T_i(a) = 0$, and $S_i(a) = S_i(b)$ if and only if $T_i(a) = T_i(b)$, $i = 1, 2$. The main theorems imply also that $((T_1 \otimes T_2)(a, b))(y) = ((S_1 \otimes S_2)(a, b))(\phi(y))$ for all $a \in \mathcal{J}_1$, $b \in \mathcal{J}_2$ and $y \in \delta B$, i.e. the products $S_1 \otimes S_2$ and $T_1 \otimes T_2$ coincide up to the composition with the homeomorphism ϕ, or, equivalently, that the diagram

$$\begin{tikzcd}
& \mathcal{J}_1 \times \mathcal{J}_2 \arrow[dl, "S_1 \otimes S_2"'] \arrow[dr, "T_1 \otimes T_2"] & \\
(\mathcal{S}_1 \cdot \mathcal{S}_2)|_{\delta A} \arrow[rr, "C_\phi"'] & & (\mathcal{T}_1 \cdot \mathcal{T}_2)|_{\delta B}
\end{tikzcd}$$

is commutative.

Theorems 2.2 and 2.4 hold, say, in the case when $\mathcal{S}_i$ and $\mathcal{T}_i$ are function algebras, or, the unit balls of function algebras. Theorem 2.5 holds in the case when $\mathcal{S}_1=\mathcal{S}_2$ and $\mathcal{T}_1=\mathcal{T}_2$ are pointed Lipschitz algebras, since any pointed Lipschitz algebra A on a compact metric space X with a distinguished base point e is a Bishop set and a multiplicative set of $C(X)$ with $p(A)=p^\circ(A)=X\setminus\{e\}$. This yields the main result in [10].

5. Remarks and examples

Remark 5.1. It is clear that parts (1), (2) and (3) of Corollary 2.6 follow from Theorems 2.2, 2.4 and 2.5, respectively. Actually, the corresponding statements are equivalent. Let $\mathcal{S}_i$ and $\mathcal{T}_i$ be multiplicative sets in the function algebras A and B, $\mathcal{J}_i$ be arbitrary sets and let $S_i\colon \mathcal{J}_i\to\mathcal{S}_i$ and $T_i\colon \mathcal{J}_i\to\mathcal{T}_i$ be surjective mappings for $i=1,2$. We will show that, under the conditions of the main theorems, if $S_i(c)=S_i(c')$ then $T_i(c)=T_i(c')$ for all $c,c'\in\mathcal{J}_i$ and $i=1,2$. This will allow us to define surjections $U_i\colon \mathcal{S}_i\to\mathcal{T}_i$ as follows. Since S_i are surjective mappings, for any $f\in\mathcal{S}_1$ and $g\in\mathcal{S}_2$, there exist $a\in\mathcal{J}_1$ and $b\in\mathcal{J}_2$ such that $S_1(a)=f$ and $S_2(b)=g$. The mappings $U_i\colon \mathcal{S}_i\to\mathcal{T}_i$ we define by $U_1(f)=T_1(a)$ and $U_2(g)=T_2(b)$. U_1 and U_2 are well defined, since, as claimed, $S_i(c)=S_i(c')$ implies $T_i(c)=T_i(c')$, $i=1,2$. Since T_1 and T_2 are surjective, so are U_1 and U_2. Moreover, as we will show, under the conditions of Theorems 2.2, 2.4 and 2.5 the mappings U_i satisfy the conditions of parts (1), (2) and (3) of Corollary 2.6, respectively. Consequently, there exists a homeomorphism $\phi\colon \delta B\to\delta A$ and a continuous function $\alpha\colon \delta B\to\mathbb{C}\setminus\{0\}$ such that

$$U_1(f)(y)=\alpha(y)f(\phi(y)) \quad\text{and}\quad U_2(g)(y)=\frac{1}{\alpha(y)}g(\phi(y)) \tag{5.1}$$

for all $f\in\mathcal{S}_1$, $g\in\mathcal{S}_2$ and $y\in\delta B$. Let $S_1(a)=f$ and $S_2(b)=g$ for all $a\in\mathcal{J}_1$ and $b\in\mathcal{J}_2$. Then $U_1(f)=T_1(a)$ and $U_2(g)=T_2(b)$ by definition. Now equalities (5.1) imply that

$$T_1(a)(y)=\alpha(y)S_1(a)(\phi(y)) \quad\text{and}\quad T_2(b)(y)=\frac{1}{\alpha(y)}S_2(b)(\phi(y))$$

for all $a\in\mathcal{J}_1$, $b\in\mathcal{J}_2$ and $y\in\delta B$. This will complete the proof that the cases (1), (2) and (3) of Corollary 2.6 imply Theorems 2.2, 2.4 and 2.5, respectively, as claimed.

Claim 1. Theorem 2.2 follows from part (1) of Corollary 2.6.

We show that, under the conditions of Theorem 2.2, if $S_1(a)=S_1(b)$ then $T_1(a)=T_1(b)$ for all $a,b\in\mathcal{J}_1$, as claimed. Let $y_0\in\delta B$. Suppose that $T_1(b)(y_0)=0$.

By Lemma 3.3 for each $\varepsilon>0$ there is a $u_0\in P_{\mathcal{T}_2}(y_0)$ such that $\|T_1(b)u_0\|<\varepsilon$. Let $c\in\mathcal{J}_2$ be such that $T_2(c)=u_0$. Since $S_1(a)=S_1(b)$, we get that

$$\begin{aligned}\|T_1(a)u_0\| &= \|T_1(a)T_2(c)\| = \|S_1(a)S_2(c)\| \\ &= \|S_1(b)S_2(c)\| = \|T_1(b)T_2(c)\| = \|T_1(b)u_0\| < \varepsilon.\end{aligned}$$

Thus, $\inf_{u\in P_{\mathcal{T}_2}(y_0)}\|T_1(a)u\|=0$. By Lemma 3.3, $T_1(a)(y_0)=0=T_1(b)(y_0)$ as desired. Similarly, if $T_1(a)(y_0)=0$ then $T_1(b)(y_0)=0$. Suppose now that $T_1(a)(y_0)\neq0$ and $T_1(b)(y_0)\neq0$. By the same reasoning as in the proof of Theorem 2.2, there are $u,v\in P_{\overline{\mathcal{T}}_2}(y_0)$ with $\sigma_\pi(T_1(a)u)=\{T_1(a)(y_0)\}$ and $\sigma_\pi(T_1(b)v)=\{T_1(b)(y_0)\}$, and there are $k_n,h_n\in P_{\mathcal{T}_2}(y_0)$ and $r_n>0$ so that r_nk_n and r_nh_n converge uniformly to u and v, respectively. Then $w=uv\in P_{\overline{\mathcal{T}}_2}(y_0)$, $w_n=r_n^2k_nh_n$ converges uniformly to w, $\sigma_\pi(T_1(a)w)=\{T_1(a)(y_0)\}$ and $\sigma_\pi(T_1(b)w)=\{T_1(b)(y_0)\}$. Let $w_n=T_2(c_n)$ for some $c_n\in\mathcal{J}_2$. By (4.3) we have $\sigma_\pi(S_1(a)S_2(c_n))\subset\sigma_\pi(T_1(a)T_2(c_n))=\sigma_\pi(T_1(a)w_n)$ and $\sigma_\pi(S_1(b)S_2(c_n))\subset\sigma_\pi(T_1(b)T_2(c_n))=\sigma_\pi(T_1(b)w_n)$. If $S_1(a)=S_1(b)$, it follows that $\sigma_\pi(S_1(a)S_2(c_n))\subset\sigma_\pi(T_1(a)w_n)\cap\sigma_\pi(T_1(b)w_n)$. Hence $\sigma_\pi(T_1(a)w_n)\cap\sigma_\pi(T_1(b)w_n)\neq\varnothing$ for every n. As $T_1(a)w_n$ and $T_1(b)w_n$ converge uniformly to $T_1(a)w$ and $T_1(b)w$, respectively, we get that $\sigma_\pi(T_1(a)w)\cap\sigma_\pi(T_1(b)w)\neq\varnothing$. Therefore, $T_1(a)(y_0)=T_1(b)(y_0)$. Since y_0 was an arbitrary element in δB it follows that $T_1(a)=T_1(b)$, as desired. By the same argument we see that if $S_2(a)=S_2(b)$ then $T_2(a)=T_2(b)$ for all $a,b\in\mathcal{J}_2$. Let $f\in\mathcal{S}_1$, $g\in\mathcal{S}_2$, $a\in\mathcal{J}_1$ and $b\in\mathcal{J}_2$ be such that $S_1(a)=f$ and $S_2(b)=g$. Then $U_1(f)=T_1(a)$ and $U_2(g)=T_2(b)$ by the definition at the beginning of the remark, and conditions (i), (ii) of Theorem 2.2 imply that $\|fg\|=\|S_1(a)S_2(b)\|=\|T_1(a)T_2(b)\|=\|U_1(f)U_2(g)\|$ and

$$(fg)(\delta A)=(S_1(a)S_2(b))(\delta A)\subset\mathrm{Ran}(T_1(a)T_2(b))=\mathrm{Ran}(U_1(f)U_2(g))$$

for all $f\in\mathcal{S}_1$ and $g\in\mathcal{S}_2$. Hence the mappings U_1 and U_2 satisfy the condition (1) of Corollary 2.6, as desired. Consequently, Theorem 2.2 follows from Corollary 2.6, as claimed.

Claim 2. Theorem 2.4 follows from part (2) of Corollary 2.6.

We show again that, under the conditions of Theorem 2.4, if $S_1(a)=S_1(b)$ then $T_1(a)=T_1(b)$ for $a,b\in\mathcal{J}_1$. Let $y_0\in\delta B$. By the same arguments as in the proof of Claim 1, we see that $T_1(a)(y_0)=0$ if and only if $T_1(b)(y_0)=0$. Suppose now that $T_1(a)(y_0)\neq0$ and $T_1(b)(y_0)\neq0$. As $(\mathcal{T}_1,\mathcal{T}_2)$ is a Bishop pair there are $u,v\in P_{\mathcal{T}_2}(y_0)$ such that $\sigma_\pi(T_1(a)u)=\{T_1(a)(y_0)\}$ and $\sigma_\pi(T_1(b)v)=\{T_1(b)(y_0)\}$. Then for $w=uv\in P_{\mathcal{T}_2}(y_0)$, clearly, $\sigma_\pi(T_1(a)w)=\{T_1(a)(y_0)\}$ and $\sigma_\pi(T_1(b)w)=\{T_1(b)(y_0)\}$. If $c\in\mathcal{J}_2$ is such that $T_2(c)=w$, then (2.2) yields

$$\sigma_\pi(S_1(a)S_2(c))\subset\sigma_\pi(T_1(a)T_2(c))=\sigma_\pi(T_1(a)w)=\{T_1(a)(y_0)\}.$$

Therefore, $\sigma_\pi(S_1(a)S_2(c))=\{T_1(a)(y_0)\}$. By the same reasoning, we also have $\sigma_\pi(S_1(b)S_2(c))=\{T_1(b)(y_0)\}$. If $S_1(a)=S_1(b)$, then, clearly, $T_1(a)(y_0)=T_1(b)(y_0)$. Since this holds for every $y_0\in\delta B$, we have $T_1(a)=T_1(b)$, as desired. By the same arguments, we see that if $S_2(a)=S_2(b)$ then $T_2(a)=T_2(b)$ for all $a,b\in\mathcal{J}_2$. Let $f\in\mathcal{S}_1$, $g\in\mathcal{S}_2$, $a\in\mathcal{J}_1$ and $b\in\mathcal{J}_2$ be such that $S_1(a)=f$ and $S_2(b)=g$. Then $U_1(f)=T_1(a)$ and $U_2(g)=T_2(b)$ by the definition at the beginning of the remark. Condition (2.2) of Theorem 2.4 implies that

$$\sigma_\pi(fg)=\sigma_\pi(S_1(a)S_2(b))\subset\sigma_\pi(T_1(a)T_2(b))=\sigma_\pi(U_1(f)U_2(g)).$$

Therefore, the mappings U_1 and U_2 satisfy condition (2) of Corollary 2.6. Consequently, Theorem 2.4 follows from Corollary 2.6, as claimed.

Claim 3. Theorem 2.5 follows from part (3) of Corollary 2.6.

We show that under the conditions of Theorem 2.5, if $S_1(a)=S_1(b)$ then $T_1(a)=T_1(b)$ for $a,b\in\mathcal{J}_1$. Let $y_0\in\delta B$. As in the proof of Claim 1 we see that $T_1(a)(y_0)=0$ if and only if $T_1(b)(y_0)=0$. Let $T_1(a)(y_0)\neq 0$ and $T_1(b)(y_0)\neq 0$. By the hypotheses of Theorem 2.5 there are $u,v\in P^\circ_{\mathcal{T}_2}(y_0)$ such that

$$\sigma_\pi(T_1(a)u)=\{T_1(a)(y_0)\}\quad\text{and}\quad\sigma_\pi(T_1(b)v)=\{T_1(b)(y_0)\}.$$

For the function $w=uv\in P^\circ_{\mathcal{T}_2}(y_0)$ we have

$$\sigma_\pi(T_1(a)w)=\{T_1(a)(y_0)\}\quad\text{and}\quad\sigma_\pi(T_1(b)w)=\{T_1(b)(y_0)\}.$$

Let $c\in\mathcal{J}_2$ be such that $T_2(c)=w$. By the property (2.3),

$$\sigma_\pi(T_1(a)w)\cap\sigma_\pi(S_1(a)S_2(c))=\sigma_\pi(T_1(a)T_2(c))\cap\sigma_\pi(S_1(a)S_2(c))\neq\varnothing.$$

Therefore, we get $T_1(a)(y_0)\in\sigma_\pi(S_1(a)S_2(c))$. Similarly, $T_1(b)(y_0)\in\sigma_\pi(S_1(b)S_2(c))$. As $w\in P^\circ_{\mathcal{T}_2}(y_0)$, (3.2) implies that

$$|(S_1(a)S_2(c))(\phi(y))|=|(T_1(a)w)(y)|<|T_1(a)(y_0)|\quad\text{for } y\neq y_0.$$

By Lemma 4.1, $\sigma_\pi(S_1(a)S_2(c))=\{T_1(a)(y_0)\}$. The same arguments show that $\sigma_\pi(S_1(b)S_2(c))=\{T_1(b)(y_0)\}$. Hence if $S_1(a)=S_1(b)$, then $T_1(a)(y_0)=T_1(b)(y_0)$. Since this holds for every $y_0\in\delta B$, $T_1(a)=T_1(b)$, as desired. By the same arguments, we see that if $S_2(a)=S_2(b)$ then $T_2(a)=T_2(b)$ for all $a,b\in\mathcal{J}_2$. Let $f\in\mathcal{S}_1$, $g\in\mathcal{S}_2$, $a\in\mathcal{J}_1$ and $b\in\mathcal{J}_2$ be such that $S_1(a)=f$ and $S_2(b)=g$. Therefore, $U_1(f)=T_1(a)$ and $U_2(g)=T_2(b)$ by the definition at the beginning of the remark, and condition (2.3) of Theorem 2.5 implies that

$$\sigma_\pi(fg)\cap\sigma_\pi(U_1(f)U_2(g))=\sigma_\pi(S_1(a)S_2(b))\cap\sigma_\pi(T_1(a)T_2(b))\neq\varnothing.$$

Thus the mappings U_1 and U_2 satisfy condition (3) of Corollary 2.6. Consequently, Theorem 2.5 follows from Corollary 2.6, as claimed.

Example 5.2. Let $A(\overline{\mathbb{D}})$ be the disc algebra and $\mathbb{T}=\{z\in\mathbb{C}:|z|=1\}$. For each $f\in A(\overline{\mathbb{D}})$ define $f^*(z)=\overline{f(\bar z)}$ for $z\in\overline{\mathbb{D}}$. If $T\colon A(\overline{\mathbb{D}})\to A(\overline{\mathbb{D}})$ is a surjective map such that

$$\sigma_\pi(fg^*)\cap\sigma_\pi(T(f)T(g)^*)\neq\varnothing,\quad f,g\in A(\overline{\mathbb{D}}), \tag{5.2}$$

then Theorem 2.5 (with the choices $\mathcal{J}_1=\mathcal{J}_2=\mathcal{S}=\mathcal{T}=A(\overline{\mathbb{D}})$, $S_1=\mathrm{id}(A(\overline{\mathbb{D}}))$, $S_2(g)=g^*$, $T_1=T$ and $T_2(g)=T(g)^*$) yields that there exist a homeomorphism $\phi\colon\mathbb{T}\to\mathbb{T}$ and a continuous function $\alpha\colon\mathbb{T}\to\mathbb{C}\setminus\{0\}$ such that

$$T(f)(z)=\alpha(z)f(\phi(z))\quad\text{and}\quad T(f)^*(z)=\frac{1}{\alpha(z)}f^*(\phi(z))$$

for all $f\in A(\overline{\mathbb{D}})$ and $z\in\mathbb{T}$. Then $\alpha=T(1)$ on $\mathbb{T}$, and hence we may regard $\alpha=T(1)\in A(\overline{\mathbb{D}})$. By definition, $T(f)^*(z)=\overline{T(f)(\bar z)}$ and $f^*(\phi(z))=\overline{f(\overline{\phi(z)})}$, and thus

$$\alpha(z)f(\phi(z))=T(f)(z)=\overline{T(f)^*(\bar z)}=\frac{1}{\alpha^*(z)}f(\overline{\phi(\bar z)}).$$

Consequently, $\alpha(z)\alpha^*(z)f(\phi(z))=f(\overline{\phi(\bar z)})$ for all $f\in A(\overline{\mathbb{D}})$ and $z\in\mathbb{T}$. Taking $f=1$, we have $\alpha\alpha^*=1$ on $\overline{\mathbb{D}}$, since $\mathbb{T}$ is a boundary of $A(\overline{\mathbb{D}})$. For example, if $g\in A(\overline{\mathbb{D}})$ has no zeros in $\overline{\mathbb{D}}$, then $\alpha=g^*/g\in A(\overline{\mathbb{D}})$ satisfies $\alpha\alpha^*=1$ on $\overline{\mathbb{D}}$. As $f(\phi(z))=f(\overline{\phi(\bar z)})$, we get that $\phi(z)=\overline{\phi(\bar z)}$ for all $z\in\mathbb{T}$. Since α^*T is an algebra isomorphism on $A(\overline{\mathbb{D}})$, there exists a homeomorphism $\psi\colon\overline{\mathbb{D}}\to\overline{\mathbb{D}}$ such that $\alpha^*T(f)=f\circ\psi$ for all $f\in A(\overline{\mathbb{D}})$, and thus $\psi\in A(\overline{\mathbb{D}})$. Since $f\circ\phi=\alpha^*T(f)=f\circ\psi$ on $\mathbb{T}$, we see that ψ is an extension of ϕ. As is well known, such a ψ is a Möbius transform of the form $\psi(z)=\lambda(a-z)/(1-\bar a z)$, where $|\lambda|=1$ and $|a|<1$. For each $z\in\mathbb{T}$,

$$\lambda\frac{a-z}{1-\bar a z}=\psi(z)=\phi(z)=\overline{\phi(\bar z)}=\overline{\psi(\bar z)}=\overline{\lambda}\frac{\bar a-z}{1-az}.$$

Since $\mathbb{T}$ is a boundary of $A(\overline{\mathbb{D}})$, the above equalities hold for all $z\in\overline{\mathbb{D}}$, and hence $a=\bar a$ and $\lambda=\overline{\lambda}$. Consequently, if $T\colon A(\overline{\mathbb{D}})\to A(\overline{\mathbb{D}})$ is a surjective map that satisfies (5.2), then there exists an $\alpha\in A(\overline{\mathbb{D}})$ with $\alpha\alpha^*=1$ on $\overline{\mathbb{D}}$ such that $T(f)(z)=\alpha(z)f(\psi(z))$ for all $f\in A(\overline{\mathbb{D}})$ and $z\in\overline{\mathbb{D}}$, where $\psi(z)=\lambda(a-z)/(1-az)$ for $\lambda=\pm1$ and some real number a with $|a|<1$.

Remark 5.3. In the main theorems if $\mathcal{S}_1=\mathcal{S}_2=\mathcal{S}$ and $\mathcal{T}_1=\mathcal{T}_2=\mathcal{T}$, then the set $\mathcal{S}$ contains the constant function 1 if and only if $\mathcal{T}$ does. In fact, if $1\in\mathcal{S}$, then there exist $e_i\in\mathcal{J}_i$, $i=1,2$, such that $S_1(e_1)=1=S_2(e_2)$, and thus $T_1(e_1)=\alpha$ and

$T_2(e_2)=1/\alpha$ on δB. Hence $T_1(e_1)T_2(e_2)\in\mathcal{T}\subset B$ satisfies $T_1(e_1)T_2(e_2)=1$ on δB. Since δB is a boundary of B, we get that $T_1(e_1)T_2(e_2)$ is the unit of B, i.e. the constant function 1. Therefore, $1\in\mathcal{T}$. A similar argument shows that $1\in\mathcal{T}$ implies $1\in\mathcal{S}$.

Example 5.4. The function α in the main theorems is not necessarily unimodular. Indeed, let $A(\overline{\mathbb{D}})$ be the disc algebra and set $\overline{\mathbb{D}}_0=\overline{\mathbb{D}}\setminus\{0\}$. Then $A=\{f|_{\overline{\mathbb{D}}_0}:f\in A(\overline{\mathbb{D}})\text{ with } f(0)=0\}$ is a function algebra on the locally compact Hausdorff space $\overline{\mathbb{D}}_0$, whose Choquet boundary δA is the unit circle. Set $\alpha(z)=(2z+3)/(z+3)$, and then $\alpha, 1/\alpha\in A(\overline{\mathbb{D}})$ but $\alpha, 1/\alpha\notin A$, since $\alpha(0)=1$. Define $T_1, T_2\colon A\to A$ by

$$T_1(f)(z)=\alpha(z)f(z) \quad \text{and} \quad T_2(f)(z)=\frac{1}{\alpha(z)}f(z)$$

for $f\in A$ and $z\in\overline{\mathbb{D}}_0$. Both T_1 and T_2 are surjective mappings so that $T_1(f)T_2(g)=fg$ for all $f,g\in A$. Therefore, $\mathcal{J}_i=A=\mathcal{S}_i=\mathcal{T}_i$ for $i=1,2$, $S_1=S_2=\mathrm{id}(A)$, and T_1 and T_2 satisfy the hypotheses of the main theorems. Note that $|\alpha|$ is not constant on δA since $\alpha(1)=\frac{5}{4}$ and $\alpha(-1)=\frac{1}{2}$. This example shows that, in general, neither $\alpha\in B$ nor $|\alpha|=1$ on δB in the main theorems. Since A has no unit, then, by Remark 5.3, there are no surjective maps $S_1, S_2\colon A\to A$ and $T_1, T_2\colon A\to A(\overline{\mathbb{D}})$ that satisfy condition (2.3).

Example 5.5. Let $A(\overline{\mathbb{D}})$ be the disc algebra and

$$A=\{f|_{\overline{\mathbb{D}}_0}: f\in A(\overline{\mathbb{D}}) \text{ with } f(0)=0\},$$

where $\overline{\mathbb{D}}_0=\overline{\mathbb{D}}\setminus\{0\}$. For each $f\in A$ define $f^*(z)=\overline{f(\bar{z})}$ for every $z\in\overline{\mathbb{D}}_0$. If $T_1\colon A(\overline{\mathbb{D}})\to\exp(A(\overline{\mathbb{D}}))$ and $T_2\colon A\to A$ are surjective mappings such that

$$\sigma_\pi(e^f g^*)\cap\sigma_\pi(T_1(f)T_2(g))\neq\varnothing, \quad f\in A(\overline{\mathbb{D}}),\ g\in A, \tag{5.3}$$

then Theorem 2.5 (with the choices $\mathcal{J}_1=A(\overline{\mathbb{D}})$, $\mathcal{S}_1=\mathcal{T}_1=\exp(A(\overline{\mathbb{D}}))$, $\mathcal{J}_2=A=\mathcal{S}_2=\mathcal{T}_2$, $S_1(f)=e^f$ and $S_2(g)=g^*$) shows that there is a homeomorphism $\phi\colon\mathbb{T}\to\mathbb{T}$ and a continuous function $\alpha\colon\mathbb{T}\to\mathbb{C}\setminus\{0\}$ such that

$$T_1(f)(z)=\alpha(z)e^{f(\phi(z))} \quad \text{and} \quad T_2(g)(z)=\frac{1}{\alpha(z)}g^*(\phi(z))$$

for all $f\in A(\overline{\mathbb{D}})$, $g\in A$ and $z\in\mathbb{T}$. Since $\alpha=T_1(0)$ on $\mathbb{T}$, we may regard $\alpha=T_1(0)\in\exp(A(\overline{\mathbb{D}}))$. As $\mathbb{T}$ is a boundary of A, the mapping $g\rightsquigarrow\alpha T_2(g^*)$, $g\in A$, is an algebra isomorphism on A. Thus there is a homeomorphism $\psi\colon\overline{\mathbb{D}}_0\to\overline{\mathbb{D}}_0$ such that $\alpha T_2(g^*)=g\circ\psi$ for every $g\in A$. Therefore, $\psi\in A$ and hence we may regard $\psi\in A(\overline{\mathbb{D}})$

with $\psi(0)=0$. Since $\psi\colon\overline{\mathbb{D}}\to\overline{\mathbb{D}}$ is bijective and continuous, we see that ψ is a homeomorphism on $\overline{\mathbb{D}}$. Thus ψ is a Möbius transform with $\psi(0)=0$, and consequently $\psi(z)=\lambda z$, $z\in\overline{\mathbb{D}}$, for some $\lambda\in\mathbb{C}$ with $|\lambda|=1$. As $g\circ\phi=\alpha T_2(g^*)=g\circ\psi$ on $\mathbb{T}$, ψ is an extension of ϕ to $\overline{\mathbb{D}}$. Since $\mathbb{T}$ is a boundary of $A(\overline{\mathbb{D}})$ and A, if $T_1\colon A(\overline{\mathbb{D}})\to\exp(A(\overline{\mathbb{D}}))$ and $T_2\colon A\to A$ are surjective mappings that satisfy (5.3), then there is an $h\in A(\overline{\mathbb{D}})$ and a $\lambda\in\mathbb{C}$ with $|\lambda|=1$ such that

$$T_1(f)(z)=e^{h(z)}e^{f(\lambda z)},\quad T_2(g)(z)=e^{-h(z)}g^*(\lambda z)$$

for all $f\in A(\overline{\mathbb{D}})$, $g\in A$ and $z\in\overline{\mathbb{D}}$.

Remark 5.6. If the mappings $T_i\colon\mathcal{J}_i\to\mathcal{T}_i$ and $S_i\colon\mathcal{J}_i\to\mathcal{S}_i$ satisfy the conclusion of the main theorems for $i=1,2$, namely, $T_1(a)(y)=\alpha(y)S_1(a)(\phi(y))$ and $T_2(b)(y)=S_2(b)(\phi(y))/\alpha(y)$ for all $a\in\mathcal{J}_1$, $b\in\mathcal{J}_2$ and $y\in\delta B$, then

$$((T_1\otimes T_2)(a,b))(\delta B)=((S_1\otimes S_2)(a,b))(\delta A)$$

since $\phi(\delta B)=\delta A$. Consequently, the maps S_1, S_2, T_1 and T_2 satisfy the spectral conditions (i) and (ii) of Theorem 2.2. Therefore, under the conditions of Theorem 2.2, the following properties are equivalent:

- $\|(T_1\otimes T_2)(a,b)\|=\|(S_1\otimes S_2)(a,b)\|$ and
$$((S_1\otimes S_2)(a,b))(\delta A)\subset\operatorname{Ran}((T_1\otimes T_2)(a,b))$$
for all a $\in\mathcal{J}_1$ and $b\in\mathcal{J}_2$;
- $((T_1\otimes T_2)(a,b))(\delta B)=((S_1\otimes S_2)(a,b))(\delta A)$ for all $a\in\mathcal{J}_1$ and $b\in\mathcal{J}_2$.

Observe that the equality $((T_1\otimes T_2)(a,b))(\delta B)=((S_1\otimes S_2)(a,b))(\delta A)$ implies, by Lemma 4.1, that $\sigma_\pi((T_1\otimes T_2)(a,b))=\sigma_\pi((S_1\otimes S_2)(a,b))$ for all $a\in\mathcal{J}_1$ and $b\in\mathcal{J}_2$. As a consequence, under the conditions of Theorem 2.4, the following properties are equivalent:

- $\sigma_\pi((S_1\otimes S_2)(a,b))\subset\sigma_\pi((T_1\otimes T_2)(a,b))$ for all $a\in\mathcal{J}_1$ and $b\in\mathcal{J}_2$;
- $((T_1\otimes T_2)(a,b))(\delta B)=((S_1\otimes S_2)(a,b))(\delta A)$ for all $a\in\mathcal{J}_1$ and $b\in\mathcal{J}_2$;
- $\sigma_\pi((T_1\otimes T_2)(a,b))=\sigma_\pi((S_1\otimes S_2)(a,b))$ for all $a\in\mathcal{J}_1$ and $b\in\mathcal{J}_2$.

Similarly, under the conditions of Theorem 2.5, the following properties are equivalent:

- $\sigma_\pi((T_1\otimes T_2)(a,b))\cap\sigma_\pi((S_1\otimes S_2)(a,b))\neq\varnothing$ for all $a\in\mathcal{J}_1$ and $b\in\mathcal{J}_2$;
- $((T_1\otimes T_2)(a,b))(\delta B)=((S_1\otimes S_2)(a,b))(\delta A)$ for all $a\in\mathcal{J}_1$ and $b\in\mathcal{J}_2$;
- $\sigma_\pi((T_1\otimes T_2)(a,b))=\sigma_\pi((S_1\otimes S_2)(a,b))$ for all $a\in\mathcal{J}_1$ and $b\in\mathcal{J}_2$;
- $\sigma_\pi((S_1\otimes S_2)(a,b))\subset\sigma_\pi((T_1\otimes T_2)(a,b))$ for all $a\in\mathcal{J}_1$ and $b\in\mathcal{J}_2$.

Remark 5.7. If $\mathcal{S}$ is a subset of a function algebra A, then, clearly, $p(\mathcal{S})\subset\delta A$ and in order to have $p(\mathcal{S})=\delta A$ it is enough that δA be a subset of $p(\mathcal{S})$.

Remark 5.8. The assumption that a function algebra A is a subset of a space of type $C_0(X)$ is not restrictive. We may assume, equivalently, that a function algebra A is a uniformly closed subalgebra of the space $C_b(X)$ of bounded continuous functions on a locally compact Hausdorff space X that separates strongly the points of X. All results in this paper hold also under this assumption for function algebras if, in addition, the underlying space X contains the Shilov boundary of A.

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