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Irregular sets of two-sided Birkhoff averages and hyperbolic sets

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Abstract. For two-sided topological Markov chains, we show that the set of points for which the two-sided Birkhoff averages of a continuous function diverge is residual. We also show that the set of points for which the Birkhoff averages have a given set of accumulation points other than a singleton is residual. A nontrivial consequence of our results is that the set of points for which the local entropies of an invariant measure on a locally maximal hyperbolic set does not exist is residual. This strongly contrasts to the Shannon–McMillan–Breiman theorem in the context of ergodic theory, which says that local entropies exist on a full measure set.

1. Introduction

We show that the irregular set of the points for which the two-sided Birkhoff averages of a given continuous function diverge is often residual. More precisely, for topologically mixing topological Markov chains we show that the irregular set is either empty or residual, even though in the context of ergodic theory it has zero measure with respect to any invariant measure.

Let $f\colon X\to X$ be a homeomorphism on a compact metric space. Given a continuous function $\varphi\colon X\to\mathbb{R}$, we consider the irregular set

$$X_\varphi=\left\{x\in X:\liminf_{n\to\infty}\frac{1}{2n}\sum_{i=-n}^{n}\varphi\big(f^i(x)\big)<\limsup_{n\to\infty}\frac{1}{2n}\sum_{i=-n}^{n}\varphi\big(f^i(x)\big)\right\}.$$

It follows from Birkhoff's ergodic theorem that X_φ has zero measure with respect to any finite f-invariant measure on X. It turns out that from the topological point of view quite the contrary happens; namely, X_φ is typically as large as the whole space. The following statement illustrates well this phenomenon. It is combination of our results with work of Barreira and Schmeling in [4].

Theorem 1.1. *Let $f\colon X\to X$ be a topologically mixing topological Markov chain. Then for a C^0-dense set of continuous functions $\varphi\colon X\to\mathbb{R}$ the set X_φ is residual.*

More precisely, we show in the present paper that for each continuous function φ the set X_φ is either empty or residual (see Theorem 2.4). On the other hand, it is shown in [4] that there exists a C^0-dense set S of continuous functions for which the irregular set is nonempty. For example, S can be taken to be the class of (Hölder) continuous functions φ such that:

1. φ is a linear combination of characteristic functions of cylinder sets;
2. φ is not cohomologous to a constant.

We recall that φ is said to be cohomologous to a constant if there exist a bounded function $\psi\colon X\to\mathbb{R}$ and a constant c such that

$$\varphi=\psi-\psi\circ f+c \quad \text{on } X.$$

Clearly, if φ is cohomologous to a constant, then X_φ is the empty set.

More generally, we consider the following refined version of the irregular set. Given an interval $I\subset\mathbb{R}$, let

$$X_I=\big\{x\in X: A_\varphi(x)=I\big\},$$

where $A_\varphi(x)$ is the set of accumulation points of the sequence

$$S_\varphi(x,n)=\frac{1}{2n}\sum_{i=-n}^{n}\varphi\big(f^i(x)\big).$$

Again for topologically mixing topological Markov chains, we show that when I is not a singleton and φ is an arbitrary continuous function, the set X_I is either empty or residual (see Theorem 2.1). This is our main result. Roughly speaking, the proof consists of bridging together strings of sufficiently large length corresponding to limits of two-sided Birkhoff averages. We note that X_I is a subset of the irregular set X_φ when I is not a singleton and so Theorem 1.1 is in fact a consequence of the corresponding result for the sets X_I.

As an application, we obtain corresponding statements for the averages $S_\varphi(x,n)$ when X is a locally maximal hyperbolic set (see Theorem 3.1). Here we describe only a nontrivial application of Theorem 3.1 to the local entropies.

We recall that if μ is a Gibbs measure of a continuous function ψ, assumed without loss of generality to have zero topological pressure, then the limit

$$h_\mu(x):=\lim_{n\to\infty}-\frac{1}{2n}\log\mu\big(B_n(x,\varepsilon)\big)=\lim_{n\to\infty}S_{-\psi}(x,n) \tag{1}$$

exists for μ-almost every $x\in X$, where

$$B_n(x,\varepsilon)=\bigcap_{k=-n}^{n} f^{-k}B\big(f^k(x),\varepsilon\big)$$

and ε is any sufficiently small constant. This is an immediate consequence of the invariance of the measure μ and the Shannon–McMillan–Breiman theorem (that usually is formulated for one-sided iterates). The number $h_\mu(x)$ (when defined) is called the local entropy of μ at the point x (with respect to f). For the definitions and basic results of the theory we refer the reader to [2, Chapters 1–2].

The following result is a consequence of Theorem 3.1 and identity (1).

Theorem 1.2. *If μ is a Gibbs measure on a locally maximal hyperbolic set, then the set of points for which the local entropies do not exist, that is, the set*

$$\left\{x\in X:\liminf_{n\to\infty}-\frac{1}{2n}\log\mu\big(B_n(x,\varepsilon)\big)<\limsup_{n\to\infty}-\frac{1}{2n}\log\mu\big(B_n(x,\varepsilon)\big)\right\},$$

is either empty or residual.

We note that the irregular sets can also be very large from the point of view of dimension theory. In particular, it was shown in [4] that for a locally maximal hyperbolic set X of a $C^{1+\varepsilon}$ map f that is topologically mixing and conformal on X (this means that the derivative of f is a multiple of an isometry at each point of X), if φ is Hölder continuous and is not cohomologous to a constant, then the one-sided irregular set

$$Y_\varphi=\left\{x\in X:\liminf_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\varphi\big(f^i(x)\big)<\limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\varphi\big(f^i(x)\big)\right\}$$

of the set X_φ is as large as the whole space from the points of view of topological entropy and Hausdorff dimension, that is,

$$h(f|Y_\varphi)=h(f|X)\quad\text{and}\quad \dim_H Y_\varphi=\dim_H X, \tag{2}$$

where $h(f|Z)$ is the topological entropy of f on the set $Z\subset X$ and $\dim_H Z$ is the Hausdorff dimension of Z. Since the proof is based on the construction of noninvariant measures obtained from concatenating Gibbs measures, the same argument applies with minor modifications to show that

$$h(f|X_\varphi)=h(f|X)\quad\text{and}\quad \dim_H X_\varphi=\dim_H X$$

for the two-sided irregular set. Now let $Y=\bigcup_\varphi Y_\varphi$ be the union over all Hölder continuous functions. Under the same hypotheses, we have

$$h(f|Y)=h(f|X) \quad \text{and} \quad \dim_H Y=\dim_H X. \tag{3}$$

The first identity in (3) was first established by Pesin and Pitskel in [10] for the full shift on two symbols. In a related direction, Shereshevsky [11] proved that for a generic C^2 surface diffeomorphism with a locally maximal hyperbolic set X and an equilibrium measure μ of a Hölder continuous C^0-generic function, the set of points for which the pointwise dimension does not exist has positive Hausdorff dimension, that is,

$$\dim_H\left\{x\in X:\liminf_{r\to 0}\frac{\log\mu(B(x,r))}{\log r}<\limsup_{r\to 0}\frac{\log\mu(B(x,r))}{\log r}\right\}>0.$$

It was shown in [4] that the identities in (2) also hold for topologically mixing topological Markov chains and for repellers of $C^{1+\varepsilon}$ maps. We refer the reader to [1, Chapter 8] for a detailed discussion of some of these results.

For topological Markov chains, the first identity in (2) was extended by Fan, Feng and Wu in [7] to arbitrary continuous functions. For repellers of $C^{1+\varepsilon}$ conformal maps, the second identity was extended by Feng, Lau and Wu in [8] to arbitrary continuous functions. For further related work, we refer the reader to [3] for analogous results for hyperbolic flows, to [5] and [12] for the study of the entropy of irregular sets of continuous functions for maps with the specification property and to [9] for an extension of some of these results to the general case of the sets X_I.

2. Main result

Let σ be the shift map on $\Sigma=\{1,\dots,k\}^{\mathbb{Z}}$, where $k\geq 2$ is an integer. We equip Σ with the distance d defined by

$$d(\omega,\omega')=2^{-\inf\{|n|:\omega_n\neq\omega'_n\}},\quad \omega=(\omega_i)_{i\in\mathbb{Z}},\ \omega'=(\omega'_i)_{i\in\mathbb{Z}}.$$

Given a $k\times k$ matrix $A=(a_{ij})$ with entries in $\{0,1\}$, let

$$\Sigma_A=\left\{(\dots\omega_{-1}\omega_0\omega_1\dots)\in\Sigma:a_{\omega_n\omega_{n+1}}=1 \text{ for } n\in\mathbb{Z}\right\}.$$

The restriction of the shift map $\sigma|\Sigma_A\colon\Sigma_A\to\Sigma_A$ is called the (two-sided) topological Markov chain with transition matrix A. We recall that $\sigma|\Sigma_A$ is topologically mixing if and only if some power of A has only positive entries.

Given continuous functions $\varphi,\psi\colon\Sigma_A\to\mathbb{R}$, we consider the level sets

$$B^-_\varphi(\alpha)=\left\{\omega\in\Sigma_A:\lim_{n\to\infty}S^-_\varphi(\omega,n)=\alpha\right\}$$

and

$$B_\psi^+(\alpha)=\Big\{\omega\in\Sigma_A : \lim_{n\to\infty} S_\psi^+(\omega,n)=\alpha\Big\},$$

where

$$(4)\qquad S_\varphi^-(\omega,n)=\frac{1}{2n-1}\sum_{i=-n+1}^{n-1}\varphi\big(\sigma^{-i}(\omega)\big),\quad S_\psi^+(\omega,n)=\frac{1}{2n-1}\sum_{i=-n+1}^{n-1}\psi\big(\sigma^{i}(\omega)\big).$$

Following arguments in [6], one can show that the sets

$$\mathcal{L}_\varphi^-=\big\{\alpha\in\mathbb{R} : B_\varphi^-(\alpha)\neq\varnothing\big\}\quad\text{and}\quad\mathcal{L}_\psi^+=\big\{\alpha\in\mathbb{R} : B_\psi^+(\alpha)\neq\varnothing\big\}$$

are nonempty closed intervals. For each $\omega\in\Sigma_A$, we denote by $A_\varphi^-(\omega)$ and $A_\psi^+(\omega)$, respectively, the sets of accumulation points of the sequences

$$n\longmapsto S_\varphi^-(\omega,n)\quad\text{and}\quad n\longmapsto S_\psi^+(\omega,n).$$

One can easily verify that

$$A_\varphi^-(\omega)=\Big[\liminf_{n\to\infty} S_\varphi^-(\omega,n),\limsup_{n\to\infty} S_\varphi^-(\omega,n)\Big]$$

and

$$A_\psi^+(\omega)=\Big[\liminf_{n\to\infty} S_\psi^+(\omega,n),\limsup_{n\to\infty} S_\psi^+(\omega,n)\Big].$$

The following is our main result.

Theorem 2.1. *Let $\sigma|\Sigma_A$ be a topologically mixing topological Markov chain and let $\varphi,\psi\colon\Sigma_A\to\mathbb{R}$ be continuous functions. Given closed intervals $I\subset\mathcal{L}_\varphi^-$ and $J\subset\mathcal{L}_\psi^+$ that are not singletons, if the set*

$$\Sigma_{I,J}^{\varphi,\psi}:=\big\{\omega\in\Sigma_A : A_\varphi^-(\omega)=I \text{ and } A_\psi^+(\omega)=J\big\}$$

is nonempty, then it is residual.

Proof. We first introduce some notation. For each $n\in\mathbb{N}$, let

$$S_n=\big\{(\omega_{-n}\dots\omega_0\dots\omega_n) : a_{\omega_i\omega_{i+1}}=1 \text{ for } -n\le i<n\big\}$$

and let $\Sigma^*=\bigcup_{n\in\mathbb{N}} S_n$. Given $\omega=(\dots\omega_{-1}\omega_0\omega_1\dots)\in\Sigma_A$, we write $\omega^+=(\omega_0\omega_1\dots)$. Moreover, given $\omega=(\dots\omega_{-1}\omega_0\omega_1\dots)\in\Sigma_A$ and $m\in\mathbb{N}$ or given $\omega=(\omega_{-n}\dots\omega_n)\in S_n$ and $m\in\mathbb{N}$ with $m\le n$, we write $\omega|_m=\omega_{-m}\dots\omega_m$. For each $\omega\in S_n$, we write $|\omega|=2n+1$ and we define the cylinder set

$$[\omega]=\{\rho\in\Sigma_A : \rho|n=\omega\}.$$

Given $\omega=(\omega_{-n}\dots\omega_n)\in S_n$ and $\omega'=(\omega'_{-m}\dots\omega'_m)\in S_m$, we write

$$\omega\omega'=\big(\omega_{-n}\dots\omega_n\omega'_{-m}\dots\omega'_m\big).$$

Since all entries of $A^{\tau+1}$ are positive, for any two admissible strings $\omega,\omega'\in\Sigma^*$ there exists $\rho=\rho(\omega,\omega')\in S_\tau$ such that $\omega\rho\omega'\in\Sigma^*$. We say that $\rho(\omega,\omega')$ is a *bridge* between ω and ω', and we write $\omega\rho\omega'=\omega\bowtie\omega'$ (although we emphasize that ρ need not be unique). Moreover, given sets $W,W_1,\dots,W_n\subset\Sigma^*$ and a string $\omega\in\Sigma^*$, we write

$$W_1\bowtie\dots\bowtie W_n=\{\omega_1\bowtie\omega_2\bowtie\dots\bowtie\omega_n:\omega_i\in W_i,1\le i\le n\}$$

and

$$\omega\bowtie W=\{\omega\bowtie\eta:\eta\in W\},$$

where each symbol $\bowtie$ runs over all admissible bridges. Finally, we write

$$W^{\bowtie n}=W_1\bowtie\dots\bowtie W_n$$

when $W_1=\dots=W_n=W$.

We proceed with the proof of the theorem. It consists of constructing a dense G_δ set $E\subset\Sigma_A$ such that $E\subset\Sigma_{I,J}^{\varphi,\psi}$. Given $\alpha\in\mathbb{R}$, $\varepsilon>0$ and $n\in\mathbb{N}$, let

$$L(\alpha,n,\varepsilon)=\big\{\omega|_{n-1}:\omega\in\Sigma_A\text{ and }\big|S_\varphi^-(\omega,n)-\alpha\big|<\varepsilon\big\}$$

and

$$R(\alpha,n,\varepsilon)=\big\{\omega|_{n-1}:\omega\in\Sigma_A\text{ and }\big|S_\psi^+(\omega,n)-\alpha\big|<\varepsilon\big\},$$

with $S_\varphi^-(\omega,n)$ and $S_\psi^+(\omega,n)$ as in (4). Clearly, for each $\varepsilon>0$, we have

$$L(\alpha,n,\varepsilon)\neq\varnothing\quad\text{and}\quad R(\beta,n,\varepsilon)\neq\varnothing$$

for $\alpha\in\mathcal{L}_\varphi^-,\beta\in\mathcal{L}_\psi^+$ and all sufficiently large n.

Now take $k\in\mathbb{N}$ and choose $\alpha_{k,1},\dots,\alpha_{k,q_k}\in I$ and $\beta_{k,1},\dots,\beta_{k,q_k}\in J$ such that

$$I\subset\bigcup_{i=1}^{q_k}B\left(\alpha_{k,i},\frac{1}{k}\right),\quad J\subset\bigcup_{i=1}^{q_k}B\left(\beta_{k,i},\frac{1}{k}\right)$$

and

$$\begin{aligned}&|\alpha_{k,i+1}-\alpha_{k,i}|<\frac{1}{k},\quad|\alpha_{k,q_k}-\alpha_{k+1,1}|<\frac{1}{k},\\&|\beta_{k,i+1}-\beta_{k,i}|<\frac{1}{k},\quad|\beta_{k,q_k}-\beta_{k+1,1}|<\frac{1}{k}\end{aligned}\tag{5}$$

for $i=0,\dots,q_k-1$. Moreover, we consider a sequence of positive real numbers

$$\varepsilon_{1,1}>\varepsilon_{1,2}>\dots>\varepsilon_{1,q_1}>\varepsilon_{2,1}>\varepsilon_{2,2}>\dots>\varepsilon_{2,q_2}>\dots$$

decreasing to zero and a sequence

$$n_{1,1} < n_{1,2} < \dots < n_{1,q_1} < n_{2,1} < n_{2,2} < \dots < n_{2,q_2} < \dots$$

of positive integers such that

$$L_{k,i} := (\alpha_{k,i}, n_{k,i}, \varepsilon_{k,i}) \neq \varnothing \quad \text{and} \quad R_{k,i} := R(\beta_{k,i}, n_{k,i}, \varepsilon_{k,i}) \neq \varnothing$$

for $k \in \mathbb{N}$ and $1 \leq i \leq q_k$.

Let $\Omega_0 = \Sigma^*$. For each $\omega \in \Omega_0$, we choose positive integers $N_{k,i} = N_{k,i}(\omega)$ for $k \in \mathbb{N}$ and $i = 1, \dots, q_k$ such that:

(i) $N_{1,i} \geq 2^{m_{1,i+1}+\tau}$ for $2 \leq i \leq q_1 - 1$,
$N_{k,i} \geq 2^{m_{k,i+1}+\tau}$ for $k \geq 2$, $1 \leq i \leq q_k - 1$,
$N_{k,q_k} \geq 2^{m_{k+1,1}+\tau}$ for $k \geq 1$;

(ii) $N_{k,i+1} \geq 2^{|\omega|+\tau+N_{1,1}(m_{1,1}+\tau)+N_{1,2}(m_{1,2}+\tau)+\dots+N_{k,i}(m_{k,i}+\tau)}$,
$N_{k+1,1} \geq 2^{|\omega|+\tau+N_{1,1}(m_{1,1}+\tau)+N_{1,2}(m_{1,2}+\tau)+\dots+N_{k,q_k}(m_{k,q_k}+\tau)}$ for $k \geq 1$,
$1 \leq i < q_k$.

Here $m_{i,k} = 2n_{i,k} - 1$ and τ is some fixed integer such that $A^{\tau+1}$ has only positive entries. We define recursively sets $\Omega_{k,i} \subset \Sigma^*$ for $k \in \mathbb{N}$ and $i = 1, \dots, q_k$ by

$$\Omega_{1,1} = \bigcup_{\omega \in \Omega_0} L_{1,1}^{\bowtie N_{1,1}(\omega)} \bowtie \omega \bowtie R_{1,1}^{\bowtie N_{1,1}(\omega)},$$

$$\Omega_{1,2} = \bigcup_{\eta \in \Omega_{1,1}} L_{1,2}^{\bowtie N_{1,2}(\omega)} \bowtie \eta \bowtie R_{1,2}^{\bowtie N_{1,2}(\omega)},$$

$$\dots$$

$$\Omega_{1,q_1} = \bigcup_{\eta \in \Omega_{1,q_1-1}} L_{1,q_1}^{\bowtie N_{1,q_1}(\omega)} \bowtie \eta \bowtie R_{1,q_1}^{\bowtie N_{1,q_1}(\omega)},$$

$$\Omega_{2,1} = \bigcup_{\eta \in \Omega_{1,q_1}} L_{2,1}^{\bowtie N_{2,1}(\omega)} \bowtie \eta \bowtie R_{2,1}^{\bowtie N_{2,1}(\omega)},$$

and so on. Finally, let

$$E_{k,i} = \bigcup_{\omega \in \Omega_{k,i}} [\omega]$$

and

$$E = \bigcap_{k=1}^{\infty} \bigcap_{i=1}^{q_k} E_{k,i}.$$

We note that E is a G_δ set since each cylinder set $[\omega]$ is open. Moreover, since $\Omega_0=\Sigma^*$, each set $E_{k,i}$ is dense and so, it follows from Baire's theorem that E is also dense.

Lemma 2.2. $E\subset\Sigma_{I,J}^{\varphi,\psi}$.

Proof. In order to prove that $E\subset\Sigma_{I,J}^{\varphi,\psi}$, we must show that $A_\varphi^-(\omega)=I$ and $A_\psi^+(\omega)=J$ for each $\omega\in E$. We only show that

$$A_\varphi^-(\omega)=I, \tag{6}$$

since the identity $A_\psi^+(\omega)=J$ can be proved in a similar manner.

We first show that

$$I\subset A_\varphi^-(\omega). \tag{7}$$

Given $\alpha\in I\subset\bigcup_{i=1}^{q_k}B(\alpha_{k,i},1/k)$, take $i_k\in\{1,\ldots,q_k\}$ such that $\alpha\in B(\alpha_{k,i_k},1/k)$. For simplicity of the exposition we assume that $i_k\notin\{1,q_k\}$. We recall that there exists $\omega^0\in\Omega_0$ such that

$$\omega\in\ldots\bowtie L_{1,1}^{\bowtie N_{1,1}}\bowtie\omega^0\bowtie R_{1,1}^{\bowtie N_{1,1}}\bowtie\ldots, \tag{8}$$

where $N_{k,i}=N_{k,i}(\omega^0)$. Let

$$s_{k,i_k}=\left|\omega^0\right|+\tau+\sum_{j=1}^{q_1}N_{1,j}(m_{1,j}+\tau)+\ldots+\sum_{j=1}^{i_k}N_{k,j}(m_{k,j}+\tau). \tag{9}$$

We will prove that

$$\left|S_\varphi^-(\omega,s_{k,i_k})-\alpha_{k,i_k}\right|\to 0\quad\text{when } k\to\infty. \tag{10}$$

It then follows from (10) that

$$\begin{aligned}\left|S_\varphi^-(\omega,s_{k,i_k})-\alpha\right|&\le\left|S_\varphi^-(\omega,s_{k,i_k})-\alpha_{k,i_k}\right|+|\alpha_{k,i_k}-\alpha|\\&<\left|S_\varphi^-(\omega,s_{k,i_k})-\alpha_{k,i_k}\right|+\frac{1}{k}\to 0\end{aligned}$$

when $k\to\infty$. Therefore, $\alpha\in A_\varphi^-(\omega)$ and (7) holds. Write

$$s_{k,i_k}=\tilde{s}_{k,i_k}+t_{k,i_k}, \tag{11}$$

where $t_{k,i_k}=N_{k,i_k}(m_{k,i_k}+\tau)$. Since $|\alpha_{k,i_k}|\le\|\varphi\|$, where $\|\varphi\|=\max_{\omega\in\Sigma_A}|\varphi(\omega)|$, writing $m_{k,i_k}+\tau=r$ we obtain

$$\begin{aligned}
&\left|\sum_{i=0}^{s_{k,i_k}-1}\varphi\big(\sigma^{-i}(\omega)\big)-s_{k,i_k}\alpha_{k,i_k}\right|\\
&\qquad\le\left|\sum_{i=0}^{\tilde{s}_{k,i_k}-1}\varphi\big(\sigma^{-i}(\omega)\big)-\tilde{s}_{k,i_k}\alpha_{k,i_k}\right|+\left|\sum_{i=\tilde{s}_{k,i_k}}^{s_{k,i_k}-1}\varphi\big(\sigma^{-i}(\omega)\big)-t_{k,i_k}\alpha_{k,i_k}\right|\\
&\qquad\le 2\tilde{s}_{k,i_k}\|\varphi\|+\left|\sum_{i=0}^{t_{k,i_k}-1}\varphi\big(\sigma^{-i}\big(\sigma^{-\tilde{s}_{k,i_k}}(\omega)\big)\big)-t_{k,i_k}\alpha_{k,i_k}\right|\\
&\qquad\le 2\tilde{s}_{k,i_k}\|\varphi\|+2\tau N_{k,i_k}\|\varphi\|\\
&\qquad\quad+\sum_{q=0}^{N_{k,i_k}-1}\left|\sum_{j=0}^{m_{k,i_k}-1}\varphi\big(\sigma^{-j}\big(\sigma^{-(\tilde{s}_{k,i_k}+qr)}(\omega)\big)\big)-m_{k,i_k}\alpha_{k,i_k}\right|\\
&\qquad= 2\tilde{s}_{k,i_k}\|\varphi\|+2\tau N_{k,i_k}\|\varphi\|\\
&\qquad\quad+\sum_{q=0}^{N_{k,i_k}-1}\left|\sum_{j=-m_{k,i_k}+1}^{m_{k,i_k}-1}\varphi\big(\sigma^{-j}\big(\sigma^{-(\tilde{s}_{k,i_k}+n_{k,i_k}-1+qr)}(\omega)\big)\big)-m_{k,i_k}\alpha_{k,i_k}\right|. \qquad (12)
\end{aligned}$$

We introduce the number $V_n(\varphi)=2\sum_{j=0}^{n-1}\operatorname{Var}_j(\varphi)$, where

$$\operatorname{Var}_k(\varphi)=\sup\big\{\big|\varphi(\omega)-\varphi\big(\omega'\big)\big|:\omega,\omega'\in\Sigma_A,\ \omega|_k=\omega'|_k\big\}.$$

By (8) and the definition of L_{k,i_k}, there exist $\overline{\omega}^0,\dots,\overline{\omega}^{N_{k,i_k}-1}\in\Sigma_A$ such that

$$\sigma^{-(\tilde{s}_{k,i_k}+n_{k,i_k}-1+qr)}(\omega)|_{n_{k,i_k}-1}=\overline{\omega}^q|_{n_{k,i_k}-1} \qquad (13)$$

and

$$\big|S_\varphi^-\big(\overline{\omega}^q,n_{k,i_k}\big)-\alpha_{k,i_k}\big|<\varepsilon_{k,i_k} \qquad (14)$$

for $q=0,\dots,N_{k,i_k}-1$.

It follows from (13) and (14) that

$$\begin{aligned}
&\left|\sum_{j=-n_{k,i_k}+1}^{n_{k,i_k}-1}\varphi\big(\sigma^{-j}\big(\sigma^{-(\tilde{s}_{k,i_k}+n_{k,i_k}-1+qr)}(\omega)\big)\big)-m_{k,i_k}\alpha_{k,i_k}\right|\\
&\qquad\le\left|\sum_{j=-n_{k,i_k}+1}^{n_{k,i_k}-1}\varphi\big(\sigma^{-j}\big(\sigma^{-(\tilde{s}_{k,i_k}+n_{k,i_k}-1+qr)}(\omega)\big)\big)-\sum_{j=-n_{k,i_k}+1}^{n_{k,i_k}-1}\varphi\big(\sigma^{-j}\big(\overline{\omega}^q\big)\big)\right|
\end{aligned}$$

$$+\left|\sum_{j=-n_{k,i_k}+1}^{n_{k,i_k}-1}\varphi\big(\sigma^{-j}(\overline{\omega}^q)\big)-m_{k,i_k}\alpha_{k,i_k}\right|$$
$$\le V_{n_{k,i_k}}(\varphi)+m_{k,i_k}\varepsilon_{k,i_k}$$

for $q=0,\dots,N_{k,i_k}-1$. Together with (12) this implies that

$$\left|\sum_{i=0}^{s_{k,i_k}-1}\varphi\big(\sigma^{-i}(\omega)\big)-s_{k,i_k}\alpha_{k,i_k}\right|$$
$$\le 2\tilde{s}_{k,i_k}\|\varphi\|+N_{k,i_k}\big(V_{n_{k,i_k}}(\varphi)+m_{k,i_k}\varepsilon_{k,i_k}\big)+2\tau N_{k,i_k}\|\varphi\|$$
$$=2\tilde{s}_{k,i_k}\|\varphi\|+N_{k,i_k}V_{n_{k,i_k}}(\varphi)+N_{k,i_k}\big(m_{k,i_k}\varepsilon_{k,i_k}+2\tau\|\varphi\|\big).$$

Using (9), (11) and condition (ii), we obtain

$$\frac{s_{k,i_k}}{\tilde{s}_{k,i_k}}-1\ge\frac{2^{\tilde{s}_{k,i_k}}}{\tilde{s}_{k,i_k}}(m_{k,i_k}+\tau)$$

and thus,

$$\frac{s_{k,i_k}}{\tilde{s}_{k,i_k}}\to+\infty\quad\text{when } k\to\infty.$$

Moreover, it follows from the uniform continuity of φ on the compact set Σ_A that $\mathrm{Var}_n(\varphi)\to 0$ when $n\to\infty$. Hence, $V_n(\varphi)/n\to 0$ when $n\to\infty$ and

$$\frac{N_{k,i_k}V_{n_{k,i_k}}(\varphi)}{s_{k,i_k}}\le\frac{V_{n_{k,i_k}}(\varphi)}{m_{k,i_k}}\to 0\quad\text{when } k\to\infty.$$

By the definition of s_{k,i_k}, we have $s_{k,i_k}>t_{k,i_k}$ and

$$\frac{N_{k,i_k}}{s_{k,i_k}}<\frac{N_{k,i_k}}{t_{k,i_k}}=\frac{1}{m_{k,i_k}+\tau}.$$

Therefore,

$$\big|S_\varphi^-(\omega,s_{k,i_k})-\alpha_{k,i_k}\big|<\frac{2\tilde{s}_{k,i_k}\|\varphi\|}{s_{k,i_k}}+\frac{V_{n_{k,i_k}}(\varphi)}{m_{k,i_k}}+\frac{m_{k,i_k}\varepsilon_{k,i_k}+2\tau\|\varphi\|}{n_{k,i_k}}\to 0$$

when $k\to\infty$, which completes the proof of (10).

Now we show that

$$A_\varphi^-(\omega)\subset I. \tag{15}$$

For each positive integer $n>|\omega^0|+\tau$ there exist $k\in\mathbb{N}$, $i_k\in\{1,2,\dots,q_k\}$ and $0\le p<N_{k,i_k+1}-1$ such that

$$s_{k,i_k}+p(m_{k,i_k+1}+\tau)<n\le s_{k,i_k}+(p+1)(m_{k,i_k+1}+\tau). \tag{16}$$

We claim that

$$\left|S_\varphi^-(\omega,n)-\alpha_{k,i_k}\right|\to 0 \quad \text{when } n\to\infty. \tag{17}$$

Again for simplicity of the exposition, we assume that $i_k\neq q_k$. If (17) holds, then it follows from (5) that

$$\operatorname{dist}\bigl(S_\varphi^-(\omega,n),I\bigr)\le\left|S_\varphi^-(\omega,n)-\alpha_{k,i_k}\right|+\operatorname{dist}(\alpha_{k,i_k},I)\to 0$$

when $n\to\infty$ (notice that $k\to\infty$ when $n\to\infty$). Since I is closed, we conclude that $A_\varphi^-(\omega)\subset I$ and (15) holds.

Now we establish (17). Writing $m_{k,i_k+1}+\tau=r$, we obtain

$$\begin{aligned}\left|\sum_{i=0}^{n-1}\varphi\bigl(\sigma^{-i}(\omega)\bigr)-n\alpha_{k,i_k}\right|\le&\left|\sum_{i=0}^{s_{k,i_k}-1}\varphi\bigl(\sigma^{-i}(\omega)\bigr)-s_{k,i_k}\alpha_{k,i_k}\right|\\&+\left|\sum_{i=s_{k,i_k}}^{s_{k,i_k}+pr-1}\varphi\bigl(\sigma^{-i}(\omega)\bigr)-pr\alpha_{k,i_k}\right|\\&+\left|\sum_{i=s_{k,i_k}+pr}^{n-1}\varphi\bigl(\sigma^{-i}(\omega)\bigr)-(n-s_{k,i_k}-pr)\alpha_{k,i_k}\right|.\end{aligned}$$

As in (13) and (14), one can choose $\overline{\omega}^0,\dots,\overline{\omega}^{p-1}\in\Sigma_A$ such that

$$\sigma^{-(s_{k,i_k}+n_{k,i_k+1}-1+qr)}(\omega)|_{n_{k,i_k+1}-1}=\overline{\omega}^q|_{n_{k,i_k+1}-1} \tag{18}$$

and

$$\left|S_\varphi^-\bigl(\overline{\omega}^q,n_{k,i_k+1}\bigr)-\alpha_{k,i_k+1}\right|<\varepsilon_{k,i_k+1} \tag{19}$$

for $q=0,\dots,p-1$. It follows from (5), (18) and (19) that

$$\begin{aligned}&\left|\sum_{j=0}^{m_{k,i_k+1}-1}\varphi\bigl(\sigma^{-j}\bigl(\sigma^{-(s_{k,i_k}+qr)}(\omega)\bigr)\bigr)-m_{k,i_k+1}\alpha_{k,i_k}\right|\\&\quad\le\left|\sum_{j=0}^{m_{k,i_k+1}-1}\varphi\bigl(\sigma^{-j}\bigl(\sigma^{-(s_{k,i_k}+qr)}(\omega)\bigr)\bigr)-m_{k,i_k+1}\alpha_{k,i_k+1}\right|\\&\quad\quad+\left|m_{k,i_k+1}\alpha_{k,i_k+1}-m_{k,i_k+1}\alpha_{k,i_k}\right|\end{aligned}$$

$$
\begin{aligned}
&\le \left|\sum_{j=-n_{k,i_k+1}+1}^{n_{k,i_k+1}-1} \varphi\big(\sigma^{-j}\big(\sigma^{-(s_{k,i_k}+n_{k,i_k+1}-1+qr)}(\omega)\big)\big) - \sum_{j=-n_{k,i_k}+1}^{n_{k,i_k+1}-1} \varphi\big(\sigma^{-j}\big(\overline{\omega}^q\big)\big)\right| \\
&\quad + \left|\sum_{j=-n_{k,i_k+1}+1}^{n_{k,i_k+1}-1} \varphi\big(\sigma^{-j}\big(\overline{\omega}^q\big)\big) - m_{k,i_k+1}\alpha_{k,i_k+1}\right| + \frac{m_{k,i_k+1}}{k} \\
&\le V_{n_{k,i_k+1}}(\varphi) + m_{k,i_k+1}\varepsilon_{k,i_k+1} + \frac{m_{k,i_k+1}}{k}
\end{aligned}
$$

for $q=0,\dots,p-1$. Therefore,

$$
\begin{aligned}
&\left|\sum_{i=s_{k,i_k}}^{s_{k,i_k}+pr-1} \varphi\big(\sigma^{-i}(\omega)\big) - pr\alpha_{k,i_k}\right| \\
&\qquad \le \sum_{q=0}^{p-1}\left|\sum_{j=-n_{k,i_k+1}+1}^{n_{k,i_k+1}-1} \varphi\big(\sigma^{-j}\big(\sigma^{-(s_{k,i_k}+qr)}(\omega)\big)\big) - m_{k,i_k+1}\alpha_{k,i_k}\right| + 2p\tau\|\varphi\| \\
&\qquad \le p\left(V_{n_{k,i_k+1}}(\varphi) + m_{k,i_k+1}\varepsilon_{k,i_k+1} + \frac{m_{k,i_k+1}}{k}\right) + 2p\tau\|\varphi\|. \qquad (20)
\end{aligned}
$$

Moreover, by (16), we have

$$
\begin{aligned}
&\left|\sum_{i=s_{k,i_k}+pr}^{n-1} \varphi\big(\sigma^{-i}(\omega)\big) - (n-s_{k,i_k}-pr)\alpha_{k,i_k}\right| \\
&\qquad \le 2(n-s_{k,i_k}-pr)\|\varphi\| \le 2r\|\varphi\| = 2(m_{k,i_k+1}+\tau)\|\varphi\|. \qquad (21)
\end{aligned}
$$

By (20) and (21), we obtain

$$
\begin{aligned}
\big|S_\varphi^-(\omega,n)-\alpha_{k,i_k}\big| &\le \big|S_\varphi^-(\omega,s_{k,i_k})-\alpha_{k,i_k}\big|\frac{s_{k,i_k}}{n} \\
&\quad + \frac{2(m_{k,i_k+1}+\tau)\|\varphi\|}{n} + \frac{pV_{m_{k,i_k+1}}(\varphi)}{n} \\
&\quad + \frac{pm_{k,i_k+1}}{kn} + \frac{p(m_{k,i_k+1}\varepsilon_{k,i_k+1}+2\tau\|\varphi\|)}{n}. \qquad (22)
\end{aligned}
$$

As in the proof of (10), one can show that the first term in the right-hand side of (22) tends to zero when $n\to\infty$ (notice that $s_{k,i_k}\le n$). Moreover, using (16) and condition (i), we obtain

$$
\frac{2(m_{k,i_k+1}+\tau)\|\varphi\|}{n} \le \frac{2(m_{k,i_k+1}+\tau)\|\varphi\|}{s_{k,i_k}} \le \frac{2(m_{k,i_k+1}+\tau)\|\varphi\|}{N_{k,i_k}} \to 0
$$

when $n \to \infty$. On the other hand, it follows from (16) that

$$\frac{pm_{k,i_k+1}}{kn} \le \frac{2}{k} \to 0 \quad \text{when } n \to \infty,$$

$$\frac{pV_{n_{k,i_k+1}}(\varphi)}{n} \le \frac{V_{n_{k,i_k+1}}(\varphi)}{m_{k,i_k+1}} \to 0 \quad \text{when } n \to \infty,$$

and

$$\frac{p(m_{k,i_k+1}\varepsilon_{k,i_k+1} + 2\tau\|\varphi\|)}{n} \le \frac{m_{k,i_k+1}\varepsilon_{k,i_k+1} + 2\tau\|\varphi\|}{m_{k,i_k+1}} \to 0 \quad \text{when } n \to \infty$$

(since $k \to \infty$ when $n \to \infty$). This shows that the right-hand side of (22) tends to zero when $n \to \infty$, which establishes (17).

Property (6) follows now readily from (7) and (15) and the proof of Lemma 2.2 is complete. □

This shows that the set E has the desired properties and the proof of the theorem is complete. □

The following is a simple consequence of Theorem 2.1.

Theorem 2.3. *Let $\sigma|\Sigma_A$ be a topologically mixing topological Markov chain and let $\psi\colon \Sigma_A \to \mathbb{R}$ be a continuous function. Given a closed interval $J \subset \mathcal{L}_\psi^+$ that is not a singleton, if the set*

$$\Sigma_I^\varphi := \left\{ \omega \in \Sigma_A : A_\psi^+(\omega) = I \right\}$$

is nonempty, then it is residual.

Proof. It suffices to observe that $\Sigma_{I,J}^{\varphi,\psi} \subset \Sigma_J^\psi$ and apply Theorem 2.1. □

As an application of Theorem 2.3, we obtain the following result.

Theorem 2.4. *Let $\sigma|\Sigma_A$ be a topologically mixing topological Markov chain and let $\varphi\colon \Sigma_A \to \mathbb{R}$ be a continuous function. If the irregular set*

$$\Sigma^\varphi := \left\{ \omega \in \Sigma_A : \liminf_{n\to\infty} S_\varphi^+(\omega, n) < \limsup_{n\to\infty} S_\varphi^+(\omega, n) \right\}$$

is nonempty, then it is residual.

Proof. If the set Σ^φ is nonempty, then there exists a closed interval $I\subset\mathcal{L}_\varphi$ with $\Sigma_I^\varphi\neq\varnothing$ that is not a singleton. Otherwise, if $\Sigma_I^\varphi=\varnothing$ for any closed interval, then Σ^φ would be empty (since $\Sigma_I^\varphi=\varnothing$ when I is not a closed interval), and if for any closed interval I such that $\Sigma_I^\varphi\neq\varnothing$ this last set was a singleton, then again Σ^φ would be empty. Since $\Sigma_I^\varphi\subset\Sigma^\varphi$, the desired result follows readily from Theorem 2.3. □

3. Results for hyperbolic sets

In this section we obtain corresponding results for the Birkhoff averages of a continuous function on a locally maximal hyperbolic set.

Let $f\colon M\to M$ be a C^1 diffeomorphism on a smooth manifold M and let $\Lambda\subset M$ be a compact f-invariant set. We say that f is a *hyperbolic set* for f if there exist $\tau\in(0,1)$, $c>0$ and a decomposition

$$T_xM=E^s(x)\oplus E^u(x)$$

for each $x\in\Lambda$, such that

$$d_xfE^s(x)=E^s\big(f(x)\big),\quad d_xfE^u(x)=E^u\big(f(x)\big),$$

$$\big\|d_xf^nv\big\|\le c\tau^n\|v\|\quad\text{whenever } v\in E^s(x),$$

and

$$\big\|d_xf^{-n}v\big\|\le c\tau^n\|v\|\quad\text{whenever } v\in E^u(x)$$

for every $x\in\Lambda$ and $n\in\mathbb{N}$. We say that Λ is *locally maximal* if there exists an open set $U\supset\Lambda$ such that

$$\Lambda=\bigcap_{n\in\mathbb{Z}}f^n(U).$$

Given continuous functions $\varphi,\psi\colon\Lambda\to\mathbb{R}$ and sets $I,J\subset\mathbb{R}$, let

$$\Lambda_{I,J}^{\varphi,\psi}=\big\{x\in X: A_\varphi^-(x)=I,\ A_\psi^+(x)=J\big\},$$

where $A_\varphi^-(x)$ and $A_\psi^+(x)$ are, respectively, the sets of accumulation points of the sequences

$$n\longmapsto S_\varphi^-(x,n)\quad\text{and}\quad n\longmapsto S_\psi^+(x,n),$$

with $S_\varphi^-(x,n)$ and $S_\psi^+(x,n)$ as in (4). Moreover, let

$$\mathcal{R}_\varphi^-=\big\{\alpha\in\mathbb{R}:B_\varphi^-(\alpha)\neq\varnothing\big\}\quad\text{and}\quad\mathcal{R}_\psi^+=\big\{\alpha\in\mathbb{R}:B_\psi^+(\alpha)\neq\varnothing\big\},$$

where

$$B_\varphi^-(\alpha)=\Big\{x\in\Lambda:\lim_{n\to\infty}S_\varphi^-(x,n)=\alpha\Big\}$$

and

$$B_\psi^+(\alpha)=\Big\{x\in\Lambda:\lim_{n\to\infty}S_\psi^+(x,n)=\alpha\Big\}.$$

The following is a version of Theorem 2.1 for the Birkhoff averages on a locally maximal hyperbolic set.

Theorem 3.1. *Let Λ be a locally maximal hyperbolic set for a topologically mixing C^1 diffeomorphism f. Given closed intervals $I\subset\mathcal{R}_\varphi^-$ and $J\subset\mathcal{R}_\psi^+$ that are not singletons, if the set $\Lambda_{I,J}^{\varphi,\psi}$ is nonempty, then it is residual.*

Proof. Since Λ is a locally maximal hyperbolic set, there exists $\delta>0$ such that the map

$$[\cdot,\cdot]\colon \big\{(x,y)\in\Lambda\times\Lambda: d(x,y)\le\delta\big\}\to\Lambda$$

is well defined. We recall that a closed set $R\subset\Lambda$ is called a rectangle if:

1. $\operatorname{diam} R<\delta$ and $R=\overline{\operatorname{int} R}$, with the interior taken with respect to the induced topology on Λ;
2. $[x,y]\in R$ whenever $x,y\in R$.

Moreover, a collection of rectangles $R_1,\dots,R_k\subset\Lambda$ is called a Markov partition of Λ (with respect to f) if:

1. $\operatorname{int} R_i\cap\operatorname{int} R_j=\varnothing$ whenever $i\ne j$;
2. if $x\in\operatorname{int} R_i$ and $f(x)\in\operatorname{int} R_j$, then

$$f\big(V_i^u(x)\big)\supset V_j^u\big(f(x)\big)\quad\text{and}\quad f^{-1}\big(V_j^s\big(f(x)\big)\big)\supset V_i^s(x),$$

where

$$V_i^s(x)=\big\{y\in B(x,\varepsilon): d\big(f^n(x),f^n(y)\big)<\varepsilon \text{ for } n\ge 0\big\}\cap R_i,$$
$$V_i^u(x)=\big\{y\in B(x,\varepsilon): d\big(f^n(x),f^n(y)\big)<\varepsilon \text{ for } n\le 0\big\}\cap R_i,$$

and where $B(x,\varepsilon)$ is the ball of radius ε centered at x (for some sufficiently small $\varepsilon>0$).

Any locally maximal hyperbolic set has Markov partitions of arbitrarily small diameter (see for example [2]). Let $A=(a_{ij})$ be a $k\times k$ matrix with entries $a_{ij}=1$ if $\operatorname{int} f(R_i)\cap\operatorname{int} R_j\ne\varnothing$ and $a_{ij}=0$ otherwise. Writing $X=\Sigma_A$, we obtain a coding map $\pi\colon X\to\Lambda$ for the set Λ letting

$$\pi(\omega)=\bigcap_{n\in\mathbb{Z}} f^{-n}R_{\omega_n},\quad \omega=(\dots\omega_{-1}\omega_0\omega_1\dots).$$

One can verify that π is continuous, onto and that $\pi\circ\sigma=f\circ\pi$ in X. This last identity implies that $\mathcal{L}_{\varphi\circ\pi}^-=\mathcal{R}_\varphi^-$ and $\mathcal{L}_{\psi\circ\pi}^+=\mathcal{R}_\psi^+$.

Now let

$$B=\bigcup_{n\in\mathbb{Z}} f^n\left(\bigcup_{i=1}^{k} \partial R_i\right)=\bigcup_{n\in\mathbb{Z}}\bigcup_{i=1}^{k} f^n(\partial R_i),$$

where ∂R_i is the boundary of R_i, be the set of points in Λ for which the coding is not unique. We consider the sets

$$S=X\setminus\pi^{-1}B \quad\text{and}\quad \Lambda^*=\Lambda\setminus B.$$

Clearly, the map $\pi\colon S\to\Lambda^*$ is bijective. Moreover, ∂R_i is closed and has empty interior. Since f is a diffeomorphism, each set $f^n(\partial R_i)$ is closed and has empty interior. Hence, B is an F_σ set and by Baire's theorem it has empty interior. Moreover, since π is continuous, $S=\pi^{-1}\Lambda^*$ is a G_δ set.

Now we show that S is dense. We proceed by contradiction. If S was not dense, then it would exist a cylinder set $[i_{-m}\dots i_m]\subset X\setminus S=\pi^{-1}B$. Therefore,

$$R:=\bigcap_{n=-m}^{m} f^{-n+1}R_{i_n}=\pi\big([i_{-m}\dots i_m]\big)\subset\pi\big(\pi^{-1}B\big)=B,$$

since π is onto. But by the properties of a Markov partition, this finite intersection has nonempty interior (indeed, $R=\overline{\operatorname{int} R}$), which contradicts the fact that B has empty interior. Hence, S is dense.

We note that $\Phi=\varphi\circ\pi$ and $\Psi=\psi\circ\pi$ are continuous functions on X. Let

$$I\subset\mathcal{R}_\varphi^-=\mathcal{L}_\Phi^- \quad\text{and}\quad J\subset\mathcal{R}_\psi^+=\mathcal{L}_\Psi^+$$

be closed intervals. It follows from Theorem 2.1 that there exists a dense G_δ set $E\subset X_{I,J}^{\Phi,\Psi}$. To complete the proof, it suffices to show that the set $F=\pi(E\cap S)\subset\Lambda^*$ satisfies the following properties:

1. $F\subset\Lambda_{I,J}^{\varphi,\psi}$;
2. F is dense in Λ;
3. F is a G_δ set.

It follows from the identity $\pi\circ\sigma=f\circ\pi$ that

$$F\subset\pi(E)\subset\pi\big(X_{I,J}^{\Phi,\Psi}\big)=\Lambda_{I,J}^{\varphi,\psi}.$$

Moreover, $E\cap S$ is a dense G_δ set since both E and S are dense G_δ sets. In particular,

$$\Lambda=\pi(X)=\pi(\overline{E\cap S})\subset\overline{\pi(E\cap S)}=\overline{F}$$

and F is dense in Λ. In order to show that F is a G_δ set, we observe that

$$\begin{aligned}\Lambda\setminus F &= \big(B\cup\Lambda^*\big)\setminus F = B\cup\big(\Lambda^*\setminus F\big) \quad \text{(since } B\cap F=\varnothing\text{)}\\ &= B\cup\big(\pi(S)\setminus\pi(E\cap S)\big)\\ &= B\cup\pi\big(S\setminus(E\cap S)\big) \quad \text{(since } \pi \text{ is bijective on } S\text{)}\\ &= \pi(X\setminus S)\cup\pi\big(S\setminus(E\cap S)\big)\\ &= \pi\big((X\setminus S)\cup\big(S\setminus(E\cap S)\big)\big)\\ &= \pi\big(X\setminus(E\cap S)\big).\end{aligned}$$

Finally, $X\setminus(E\cap S)$ is an F_σ set (since $E\cap S$ is a G_δ set) and writing $X\setminus(E\cap S)=\bigcup_i F_i$ as a countable union of closed sets $F_i\subset X$, we obtain

$$\Lambda\setminus F=\pi\big(X\setminus(E\cap S)\big)=\bigcup_i \pi(F_i),$$

where $\pi(F_i)$ is a closed set (since π is continuous and X is compact). This shows that F is a G_δ set and the proof of the theorem is complete. □

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