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Modulus in Banach function spaces

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Abstract. Moduli of path families are widely used to mark curves which may be neglected for some applications. We introduce ordinary and approximation modulus with respect to Banach function spaces. While these moduli lead to the same result in reflexive spaces, we show that there are important path families (like paths tangent to a given set) which can be labeled as negligible by the approximation modulus with respect to the Lorentz $L^{p,1}$-space for an appropriate p, in particular, to the ordinary L^1-space if $p=1$, but not by the ordinary modulus with respect to the same space.

1. Introduction

For $p\geq 1$ the M_p-modulus of a curve family Γ in $\mathbb{R}^n$, $n\geq 1$, is defined as

$$M_p(\Gamma) = \inf \int_{\mathbb{R}^n} \rho^p \, dx$$

where the infimum is taken over all non-negative Borel functions ρ such that

$$\int_\gamma \rho \, ds \geq 1$$

for every curve $\gamma\in\Gamma$.

The M_p-modulus is used in the theory of function spaces. B. Fuglede [5] showed that a function u in the Sobolev space $W^{1,p}(\mathbb{R}^n)$ is not only absolutely continuous on almost every line segment parallel to a coordinate axis but satisfies

$$|u(\gamma(b))-u(\gamma(a))| \leq \int_\gamma |\nabla u| \, ds \tag{1}$$

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for every curve $\gamma:[a,b]\to\mathbb{R}^n$ except for a family of M_p-modulus zero. In a metric measure space X, the inequality (1) has been taken as a defining property to create the Newtonian space $N^{1,p}(X)$, which has many properties similar to $W^{1,p}(\mathbb{R}^n)$, see [2], [6] and [17]. Due to the conformal invariance, the M_n-modulus has turned out to be a basic tool to study conformal and quasiconformal mappings in $\mathbb{R}^n$.

The approximation modulus, AM_p-modulus, is defined as

$$AM_p(\Gamma)=\inf_{(\rho_i)}\liminf_{i\to\infty}\int_{\mathbb{R}^n}\rho_i^p\,dx$$

where the infimum is taken over all sequences (ρ_i) of Borel functions $\rho_i:\mathbb{R}^n\to[0,\infty]$ such that

$$\liminf_{i\to\infty}\int_\gamma\rho_i\,ds\geq 1$$

for every $\gamma\in\Gamma$. The AM_1-modulus was introduced in [14] to study functions of bounded variation in $\mathbb{R}^n$ and in metric measure spaces, see also [15]. The purpose of this paper is to study an analog of the M_p- and AM_p-modulus in the more general framework of Banach function spaces than in the L^p-spaces. We introduce $M_{\mathcal{F}}$-modulus and $AM_{\mathcal{F}}$-modulus for an arbitrary Banach function space $\mathcal{F}$. The moduli in Lebesgue scale are obtained as $M_p=M_{L^p}^p$ and $AM_p=AM_{L^p}^p$. The main novelty is the investigation of the $AM_{\mathcal{F}}$-modulus. The $M_{\mathcal{F}}$-modulus has been already studied in an even broader generality by L. Malý [12]. Of course, we need both moduli for comparison purposes.

Although we are mostly concerned in distinction between M- and AM-moduli, we first study when they are equal.

Theorem 1. *Let X be a metric measure space and $\mathcal{F}$ be a reflexive Banach function space on X. Then*

$$M_{\mathcal{F}}(\Gamma)=AM_{\mathcal{F}}(\Gamma) \tag{2}$$

for every curve family Γ in X. In particular,

$$M_p(\Gamma)=AM_p(\Gamma)\quad \textit{if } 1<p<\infty.$$

To find situation in which M- and AM-moduli differ, we must turn our attention to nonreflexive spaces. Besides L^1 and L^∞, the most important examples of nonreflexive Banach function spaces are the Lorentz spaces $L^{p,1}(\mathbb{R}^n)$ and their duals. Note that L^1 is just the limiting case $p=1$, indeed $L^{1,1}(\mathbb{R}^n)=L^1(\mathbb{R}^n)$. We focus our attention to the Lorentz spaces $L^{p,1}(\mathbb{R}^n)$ rather than to their duals, as the spaces $L^{p,1}(\mathbb{R}^n)$ are intimately connected with Hausdorff measures, see [8], [9] and [10]. We prove the following theorem:

Theorem 2. *Let* $1\le p\le n$*. Then there exists a curve family* Γ *in* $\mathbb{R}^n$ *such that*

$$AM_{L^{p,1}}(\Gamma)=0 \quad \text{but} \quad M_{L^{p,1}}(\Gamma)=\infty. \tag{3}$$

In particular, if $n\ge 1$*, then there exists a curve family* Γ *in* $\mathbb{R}^n$ *such that*

$$AM_1(\Gamma)=0 \quad \text{but} \quad M_1(\Gamma)=\infty. \tag{4}$$

In the course of proofs of Theorem 2 and related results, we construct various curve families which are self-interesting. Let $E\subset\mathbb{R}^n$. We denote the family of all curves γ which meet E by $\Gamma(E)$. Already this family can distinguish between AM- and M-moduli. However, to construct examples in which $AM_{L^{k,1}}(\Gamma)=0<M_{L^{k,1}}(\Gamma)$, we need slightly more sofisticated, but important families:

Definition 3. Let X be a metric space and $E\subset X$. We define $\Gamma_i(E)$ as the family of all rectifiable curves γ which meet E infinitely times, this means that the set $\{t\colon \gamma(t)\in E\}$ is infinite.

Definition 4. If E is an $(n-k)$-dimensional C^1 surface in $\mathbb{R}^n$, we say that γ is *right tangential* to E if there exists t such that $\gamma(t)\in E$ and $\gamma'_+(t)$ belongs to the tangent space $T_{\gamma(t)}(E)$. The family of all right tangential curves to E is denoted by $\Gamma_t(E)$.

Then we can state the following theorem. Note that estimates (6) and (8) have applications to the fine setting of the Stokes theorem, see [7].

Theorem 5. *Let* $1\le k\le n-1$ *and* $E\subset\mathbb{R}^n$ *be an* $(n-k)$*-dimensional* C^1 *surface. Then*

$$AM_{L^{k,1}}(\Gamma_t(E))=0, \quad \text{but} \quad M_{L^{k,1}}(\Gamma_t(E))=\infty. \tag{5}$$

In particular, if the dimension of E *is* $n-1$*, then*

$$AM_1(\Gamma_t(E))=0, \quad \text{but} \quad M_1(\Gamma_t(E))=\infty. \tag{6}$$

If, moreover, the closure of E *is contained in an* $(n-k)$*-dimensional* C^1 *surface* E'*, then*

$$AM_{L^{k,1}}(\Gamma_i(E))=0, \quad \text{but} \quad M_{L^{k,1}}(\Gamma_i(E))=\infty, \tag{7}$$

which for $k=1$ *yields*

$$AM_1(\Gamma_i(E))=0, \quad \text{but} \quad M_1(\Gamma_i(E))=\infty. \tag{8}$$

Remark 6. Note that by definition, an $(n-k)$ C^1 surface is always nonempty. The assumption on existence of the "supersurface" E' is not much restrictive, we need only to avoid some pathological surfaces like the one described in Remark 39.

We prove our results in a broader generality than indicated in Theorems above. Namely, we can formulate most results in the setting of metric measure spaces. Also, there are other versions of tangential behavior and, in particular, we present versions of tangential behavior in the setting of metric spaces.

The plan of the paper is following. After preliminaries, the properties of the $AM_{\mathcal{F}}$-modulus in general Banach function spaces $\mathcal{F}$ are studied in Section 3. Section 4 is devoted to the method of path truncation which plays a crucial role in the differences between the AM- and M-moduli. The equivalence of these moduli is proved in the reflexive function spaces in Section 5. Tangential type behavior of paths and their AM-modulus is studied in Lorentz spaces in Section 6 and this leads to the proof of the AM part of Theorem 5. We also introduce a closely related density tangential property for curves in Section 7 and in Section 8 we obtain estimates for the M-modulus of tangential type paths and these, together with the results in Section 9, lead to the proof of the M-modulus part of Theorem 5. Finally in Section 10 a set in $\mathbb{R}^n$ with noninteger Minkowski dimension is constructed and this completes the proof of Theorem 2 in the Lorentz spaces $L^{p,1}$.

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2. Preliminaries

First, we introduce Banach function spaces as in [1].

Definition 7. Let (X,ν) be a measure space, let $\mathcal{M}_+$ be the cone of all ν-measurable functions on X with values in $[0,\infty]$. A mapping $\Xi{:}\mathcal{M}_+\to[0,\infty]$ is called a *(Banach) function norm* if, for all $f, g, f_n\in\mathcal{M}_+$, for all $a\geq 0$ and for all ν-measurable sets $E\subset X$, it holds

(1) $\Xi(f)=0\Leftrightarrow f=0$ ν-a.e.; $\Xi(af)=a\Xi(f)$; $\Xi(f+g)\leq\Xi(f)+\Xi(g)$;
(2) $0\leq g\leq f$ ν-a.e. $\Rightarrow\Xi(g)\leq\Xi(f)$;
(3) $0\leq f_n\uparrow f$ ν-a.e. $\Rightarrow\Xi(f_n)\uparrow\Xi(f)$;
(4) $\nu(E)<\infty\Rightarrow\Xi(\chi_E)<\infty$;
(5) $\nu(E)<\infty\Rightarrow\int_E f\,d\nu\leq C_E\Xi(f)$ for C_E depending on E, ν and Ξ.

Definition 8. Let $\mathcal{M}$ be the set of all extended scalar-valued ν-measurable functions on X. Let Ξ be a function norm. The collection $\mathcal{F}$ of all functions in $\mathcal{M}$ for which $\Xi(|f|)<\infty$ is called a *Banach function space.* We define a norm on $\mathcal{F}$ by

$$\|f\|_{\mathcal{F}}=\Xi(|f|).$$

Important examples of Banach function spaces are provided by the scale of Lorentz spaces.

Definition 9. Let (X,ν) be a σ-finite measure space and $1\le p<\infty$, $1\le q\le\infty$. The Lorentz space $L^{p,q}=L^{p,q}(X,\nu)$ consists of all ν-measurable functions f with finite values ν-a.e. for which the quantity

$$\|f\|_{L^{p,q}}=\begin{cases}\left\{\int_0^\infty [t^{\frac{1}{p}}f^*(t)]^q\frac{dt}{t}\right\}^{\frac{1}{q}}, & q\in[1,\infty),\\ \sup_{t\in(0,\infty)}\left\{t^{\frac{1}{p}}f^*(t)\right\}, & q=\infty,\end{cases}$$

is finite. Here f^* is the decreasing rearrangement

$$f^*(t)=\inf\{s>0\colon \nu(\{x\in X: |f(x)|>s\})\le t\}.$$

We present there some basic properties of Lorentz spaces. For details, see [1, Chapter 4.4].

The expression $\|\cdot\|_{L^{p,q}}$ is a genuine norm (after identifying functions which are equal a.e.) if $1\le q\le p$; for $1<p<q$ it is equivalent to a genuine norm.

The spaces $L^{p,1}(X,\nu)$ and $L^{p',\infty}(X,\nu)$ are in duality, in particular, the Hölder type inequality

$$\int_X fg\,d\nu\le\|f\|_{L^{p,1}}\,\|g\|_{L^{p',\infty}} \tag{9}$$

holds.

It is easy to compute that for a measurable set,

$$\|\chi_E\|_{L^{p,q}}=c\,\nu(E)^{1/p}\quad\text{with } c=c(p,q)=(p/q)^{1/q}. \tag{10}$$

For us, it is enough to remember that $c(p,1)=p$.

In typical situations (like in $\mathbb{R}^n$ with the Lebesgue measure), the spaces $L^{p,q}(X,\nu)$ are reflexive if and only if $1<p,q<\infty$; the reflexivity part follows from the characterization of the dual space in [1, Corollary 4.4.7], the nonreflexivity part can be derived from [1, Corollary 1.4.4] and [16, Theorem 9.5.]

The Lebesgue spaces $L^p(X,\nu)$ are included in the scale of Lorentz spaces as $L^p(X,\nu)=L^{p,p}(X,\nu)$, $p\in[1,\infty]$.

For $1\le p\le\infty$ and $1\le q\le s\le\infty$, we have the embedding

$$\|f\|_{L^{p,s}}\le C\|f\|_{L^{p,q}},\quad f\in L^{p,q}(X,\nu).$$

Definition 10. In what follows, we restrict ourselves to rectifiable curves, which will be called paths. So, a *path* will be a non-constant Lipschitz continuous mapping $\gamma\colon [a,b]\to X$. Every path can be parametrized by its arclength and we assume that it is done as so, if not specified otherwise. The domain of γ will be $[0,\ell]$, where $\ell=\ell(\gamma)$ is the total length of γ.

Note that the curvelinear integral

$$\int_\gamma \rho\, ds = \int_0^{\ell(\gamma)} \rho(\gamma(t))\, dt$$

is well defined whenever γ is a path and ρ is a non-negative Borel function on X.

Now, we can define a modulus and approximation modulus with respect to a Banach function space $\mathcal{F}$.

Definition 11. Let (X,ν) be a metric space with a Borel regular measure ν. Let $\mathcal{F}$ be a Banach function space on X and Γ a family of paths in X. A Borel measurable function $\rho\colon X\to[0,\infty]$ is called *admissible* for Γ if $\int_\gamma \rho\, ds\ge 1$ for every $\gamma\in\Gamma$. Define the *$M_{\mathcal{F}}$-modulus* of Γ as

$$M_{\mathcal{F}}(\Gamma) = \inf\{\|\rho\|_{\mathcal{F}}\colon\ \rho \text{ is admissible for } \Gamma\}$$

where the infimum is taken over all admissible functions for Γ.

Remark 12. In case of $\mathcal{F}=L^p(\mathbb{R}^n)$, $p>1$, the $M_{\mathcal{F}}$-modulus and the standard M_p-modulus are almost the same – the only difference is that the M_p-modulus uses the p-th-power of the L^p-norm, i.e. $M_p(\Gamma)=(M_{L^p}(\Gamma))^p$. In particular, the zero families are the same. For $p=1$ the moduli are obviously exactly the same.

Definition 13. Let (X,ν) be a metric space with a Borel regular measure ν. Let $\mathcal{F}$ be a Banach function space and Γ a family of paths in X. A sequence of non-negative Borel functions $\rho_i\colon X\to[0,\infty]$ is *admissible* for Γ if

$$\liminf_{i\to\infty} \int_\gamma \rho_i\, ds \ge 1$$

for every $\gamma\in\Gamma$. The approximation modulus with respect to $\mathcal{F}$, shortly *$AM_{\mathcal{F}}$-modulus* of Γ is defined as

$$AM_{\mathcal{F}}(\Gamma) = \inf(\liminf_{i\to\infty} \|\rho_i\|_{\mathcal{F}})$$

where the infimum is taken over all admissible sequences (ρ_i) for Γ. Note that $AM_p(\Gamma)=(AM_{L^p}(\Gamma))^p$.

3. Basic properties

In this section, we show some basic properties of $M_{\mathcal{F}}$-modulus and $AM_{\mathcal{F}}$-modulus. The ideas are analogous to [5] and [14].

Remark 14. It holds that $AM_{\mathcal{F}}(\Gamma) \le M_{\mathcal{F}}(\Gamma)$. It is easy to see this inequality because if ρ is admissible for Γ, then $\rho_i = \rho$, $i = 1, 2, \ldots$, is admissible for Γ and

$$AM_{\mathcal{F}}(\Gamma) \le \|\rho\|_{\mathcal{F}}.$$

Taking infimum over all admissible ρ, the assertion follows.

Theorem 15. ([12]) *The $M_{\mathcal{F}}$-modulus is an outer measure on the set of paths in X, i.e.*

$$M_{\mathcal{F}}(\varnothing) = 0; \tag{11}$$

$$\Gamma_1 \subset \Gamma_2 \Longrightarrow M_{\mathcal{F}}(\Gamma_1) \le M_{\mathcal{F}}(\Gamma_2); \tag{12}$$

$$M_{\mathcal{F}}\Big(\bigcup_{j=1}^{\infty} \Gamma_j\Big) \le \sum_{j=1}^{\infty} M_{\mathcal{F}}(\Gamma_j). \tag{13}$$

Theorem 16. *The $AM_{\mathcal{F}}$-modulus is an outer measure on the set of paths in X, i.e.*

$$AM_{\mathcal{F}}(\varnothing) = 0; \tag{14}$$

$$\Gamma_1 \subset \Gamma_2 \Longrightarrow AM_{\mathcal{F}}(\Gamma_1) \le AM_{\mathcal{F}}(\Gamma_2); \tag{15}$$

$$AM_{\mathcal{F}}\Big(\bigcup_{j=1}^{\infty} \Gamma_j\Big) \le \sum_{j=1}^{\infty} AM_{\mathcal{F}}(\Gamma_j). \tag{16}$$

Proof. The property (14) is obvious since $\rho_i = 0$ is admissible for the empty set. Similarly, (15) follows from the fact that each admissible sequence for Γ_2 is also admissible for Γ_1. To prove (16), assume that $\sum_{j=1}^{\infty} AM_{\mathcal{F}}(\Gamma_j) < \infty$. Then fix $\varepsilon > 0$ and for each $j = 1, 2, \ldots$ pick an admissible sequence $(\rho_i^j)_{i=1}^{\infty}$ for Γ_j such that

$$\|\rho_i^j\|_{\mathcal{F}} \le AM_{\mathcal{F}}(\Gamma_j) + 2^{-j}\varepsilon \tag{17}$$

for every $i = 1, 2, \ldots$.

Set $\rho_i = \sum_{j=1}^{\infty} \rho_i^j$. Then ρ_i is admissible for $\Gamma = \bigcup_{j=1}^{\infty} \Gamma_j$ as if $\gamma \in \Gamma$, then there exists j_0 such that $\gamma \in \Gamma_{j_0}$ and

$$\liminf_{i\to\infty} \int_\gamma \rho_i \, ds = \liminf_{i\to\infty} \int_\gamma \sum_{j=1}^{\infty} \rho_i^j \, ds \ge \liminf_{i\to\infty} \int_\gamma \rho_i^{j_0} \, ds \ge 1.$$

Now, using [1, Theorem 1.6] we can estimate

$$AM_{\mathcal{F}}(\Gamma) \le \liminf_{i\to\infty} \|\rho_i\|_{\mathcal{F}} = \liminf_{i\to\infty} \Big\|\sum_{j=1}^{\infty} \rho_i^j\Big\|_{\mathcal{F}} \le \liminf_{i\to\infty} \sum_{j=1}^{\infty} \|\rho_i^j\|_{\mathcal{F}}$$
$$\le \sum_{j=1}^{\infty} \big(AM_{\mathcal{F}}(\Gamma_j)+2^{-j}\varepsilon\big) = \sum_{j=1}^{\infty} AM_{\mathcal{F}}(\Gamma_j)+\varepsilon.$$

Letting $\varepsilon\to 0$, we obtain (16). □

Definition 17. A set of paths Γ_2 is *minorised* by Γ_1 if for every path $\gamma\in\Gamma_2$ there exists a subpath of γ in Γ_1. It is denoted as $\Gamma_1<\Gamma_2$.

Proposition 18. *If* $\Gamma_1<\Gamma_2$, *then* $M_{\mathcal{F}}(\Gamma_2)\le M_{\mathcal{F}}(\Gamma_1)$.

Proposition 19. *If* $\Gamma_1<\Gamma_2$, *then* $AM_{\mathcal{F}}(\Gamma_2)\le AM_{\mathcal{F}}(\Gamma_1)$.

Proof. For both theorems, the proofs are easy and similar. If $\rho\in\mathcal{F}$ ($\rho_i\in\mathcal{F}$ respectively) is admissible for Γ_1, it is admissible for Γ_2 too. Thus a set of admissible functions for Γ_2 is the same or larger then for Γ_1 and the infimum over a larger set is smaller or the same. □

Proposition 20. *Let* Γ *be a family of paths in* X. *Then* $M_{\mathcal{F}}(\Gamma)=0$ *if and only if there is an admissible sequence* (ρ_i) *for* Γ *such that*

$$\liminf_{i\to\infty} \|\rho_i\|_{\mathcal{F}} = 0. \tag{18}$$

Proof. If $M_{\mathcal{F}}(\Gamma)=0$, then there exist admissible functions ρ_i such that $\|\rho_i\|_{\mathcal{F}}\le\frac{1}{i}$ for each $i=1,2,\dots$ and this is the required sequence.

For the other direction, let (ρ_i) be as in (18). We can choose a subsequence (ω_i) of (ρ_i) such that $\|\omega_i\|_{\mathcal{F}}\le 2^{-i-1}\varepsilon$ for $i=1,2,\dots$.

Now define $\overline{\rho}=\sum_{i=1}^{\infty}\omega_i$. Since (ω_i) is admissible for Γ, there exists $k\in\mathbb{N}$ such that for all $i\ge k$ it holds that $\int_\gamma \omega_i\,ds\ge\frac{1}{2}$. We can use the Lebesgue monotone convergence theorem to infer that

$$\int_\gamma \overline{\rho}\,ds = \sum_{i=1}^{\infty}\int_\gamma \omega_i\,ds = \infty > 1$$

and thus $\overline{\rho}$ is admissible for Γ.

Using again [1, Theorem 1.6], we get

$$M_{\mathcal{F}}(\Gamma) \le \|\overline{\rho}\|_{\mathcal{F}} \le \sum_{i=1}^{\infty} \|\omega_i\|_{\mathcal{F}} \le \varepsilon.$$

Letting $\varepsilon\to 0$, $M_{\mathcal{F}}(\Gamma)$ is zero as desired. □

Remark 21. The zero set of $AM_{\mathcal{F}}$-modulus can be a set of positive $M_{\mathcal{F}}$-modulus only if (18) fails for any admissible sequence (ρ_i), i.e. there exists no minimizing sequence (ρ_i).

The following lemma is a version of Fuglede's theorem [5, Theorem 3(f)] adapted to the setting of Banach functions spaces.

Lemma 22. *If a sequence $\rho_j \in \mathcal{F}$ converges strongly in $\mathcal{F}$ to ρ, then there is a subsequence ρ_j such that for almost every path*

$$\lim_{j\to\infty} \int_\gamma |\rho-\rho_j|\,ds = 0$$

except for a set of paths of a zero $M_{\mathcal{F}}$-modulus.

Remark 23. Since the $AM_{\mathcal{F}}$-modulus is smaller, the lemma holds for the $AM_{\mathcal{F}}$-modulus as well.

Proof. We can choose a subsequence ρ_j such that $\|\rho-\rho_j\|_{\mathcal{F}} < 2^{-2j}$. Denote

$$\Gamma_j = \left\{\gamma : \int_\gamma |\rho-\rho_j|\,ds > 2^{-j}\right\},$$

$$\Psi_k = \bigcup_{j>k} \Gamma_j, \quad \Psi = \bigcap_k \Psi_k.$$

The function $2^j|\rho-\rho_j|$ is admissible for Γ_j, thus

$$M_{\mathcal{F}}(\Gamma_j) \le 2^j \|\rho-\rho_j\|_{\mathcal{F}} < 2^{-j}.$$

By Theorem 15, for every $k \in \mathbb{N}$, we have

$$M_{\mathcal{F}}(\Psi) \le M_{\mathcal{F}}(\Psi_k) \le \sum_{j>k} M_{\mathcal{F}}(\Gamma_j) < 2^{-k}.$$

Since the previous inequality holds for every k, $M_{\mathcal{F}}(\Psi)=0$. For every $\gamma \notin \Psi$ there exists an index k such that $\gamma \notin \Psi_k$, i.e. $\int_\gamma |\rho-\rho_j|\,ds \le 2^{-j}$ for every $j>k$. Hence

$$\lim_{j\to\infty} \int_\gamma |\rho-\rho_j|\,ds = 0. \quad \square$$

The next theorem is also due to Fuglede [5, Theorem 2]. We omit the proof; it is easy to observe that Fuglede's argument holds also in the setting of Banach function spaces. The idea is also clear from the (more complicated) proof of Theorem 25.

Theorem 24. *Let* Γ *be a family of paths in* X. *Then* $M_{\mathcal{F}}(\Gamma)=0$ *if and only if there is a non-negative Borel function* $\rho\in\mathcal{F}(X)$ *such that*

$$\int_\gamma \rho\, ds = \infty \tag{19}$$

for each $\gamma\in\Gamma$.

For the $AM_{\mathcal{F}}$-modulus, there is also a corresponding result.

Theorem 25. *Let* Γ *be a family of paths in* X. *Then* $AM_{\mathcal{F}}(\Gamma)=0$ *if and only if there is a sequence* $(\tilde\rho_i)$ *of non-negative Borel functions* $\tilde\rho_i$ *such that*

$$\liminf_{i\to\infty} \|\tilde\rho_i\|_{\mathcal{F}} < \infty \tag{20}$$

and for each $\gamma\in\Gamma$

$$\lim_{i\to\infty} \int_\gamma \tilde\rho_i\, ds = \infty. \tag{21}$$

Proof. The conditions (20) and (21) are clearly sufficient to show that $AM_{\mathcal{F}}(\Gamma)=0$. For the converse, suppose that $AM_{\mathcal{F}}(\Gamma)=0$ and let $\varepsilon>0$. For each $j\in\mathbf{N}$, we can choose a sequence $(\rho_i^j)_i$ such that $(\rho_i^j)_i$ is admissible for Γ and

$$\| \rho_i^j \|_{\mathcal{F}} \le 2^{-j} \tag{22}$$

for each i and j. For every i set

$$\tilde\rho_i = \sum_{j=1}^\infty \rho_i^j$$

Then [1, Theorem 1.6] and (22) yield

$$\| \tilde\rho_i \|_{\mathcal{F}} \le \sum_{j=1}^\infty \| \rho_i^j \|_{\mathcal{F}} \le \sum_{j=1}^\infty 2^{-j} = 1$$

and this gives (20).

It remains to show that $(\tilde\rho_i)$ satisfies (21). Fix a path $\gamma\in\Gamma$ and let $k\in\mathbf{N}$. Since each sequence $(\rho_i^j)_i$ is admissible for Γ, there is i_0 such that for $i\ge i_0$

$$\int_\gamma \rho_i^j\, ds > 1/2$$

for $j=1, 2, ..., k$. Now

$$\int_\gamma \tilde\rho_i\, ds \ge \sum_{j=1}^k \int_\gamma \rho_i^j\, ds \ge k/2$$

for $i \geq i_0$ and hence

$$\liminf_{i\to\infty} \int_\gamma \tilde{\rho}_i \, ds \geq k/2.$$

Letting $k \to \infty$ we obtain

$$\lim_{i\to\infty} \int_\gamma \tilde{\rho}_i \, ds = \infty$$

as required. □

Remark 26. The left hand part of (20) can be made arbitrarily small.

4. Truncation

Here we consider a phenomenon which clarifies the difference between the $M_{\mathcal{F}}$- and $AM_{\mathcal{F}}$-modulus.

Let Γ be a family of paths in X. We say that a family Γ_T is a *truncated family associated with* Γ if $\Gamma \subset \Gamma_T$ and for each $\gamma \in \Gamma$, Γ_T contains some family of arbitrary short subpaths of γ. Note that there are many truncated families associated with Γ.

Theorem 27. *If* Γ *is a family of paths in* X *with* $M_{\mathcal{F}}(\Gamma) > 0$*, then* $M_{\mathcal{F}}(\Gamma_T) = \infty$ *for every truncated family* Γ_T *associated with* Γ.

Proof. Assume that $M_{\mathcal{F}}(\Gamma) < \infty$. Let $\rho \in \mathcal{F}(X)$ be admissible for Γ_T. Then ρ is admissible for Γ as well and hence there is a path $\gamma \in \Gamma$ such that

$$\int_\gamma \rho \, ds < \infty \tag{23}$$

because if

$$\int_\gamma \rho \, ds = \infty$$

for every $\gamma \in \Gamma$, then $M_{\mathcal{F}}(\Gamma) = 0$ by the Fuglede Theorem 24 which is a contradiction. Now the absolute continuity of integral shows that ρ is not admissible for Γ_T since the path γ in (23) contains paths of arbitrary small length and we can find a subpath $\widetilde{\gamma}$ of γ such that $\int_{\widetilde{\gamma}} \rho \, ds < 1$. The theorem follows. □

Remark 28. Note that Theorem 27 is not true for the $AM_{\mathcal{F}}$-modulus in general. This is an important difference between these two concepts. Consider the family Γ of all segments connecting rz with z where $r \in (0,1)$ and $z \in \partial B(0,1)$. Then the sequence (ρ_i) defined as $\rho_i = 2^i \chi_{B(0,1) \setminus B(0,1-2^{-i})}$ is admissible for Γ and shows that the AM_1-modulus of Γ is finite, although Γ is a truncated family associated with itself and its AM_1-modulus is strictly positive, see also [14, Example 3.14].

Remark 29. As the following section shows, the $M_{\mathcal{F}}$- and $AM_{\mathcal{F}}$-modulus are the same for every reflexive space $\mathcal{F}$, and thus the property of Theorem 27 transmits to the $AM_{\mathcal{F}}$-modulus in reflexive spaces. In particular, it holds for the AM_p-modulus, $1<p<\infty$.

5. Equivalence of moduli in reflexive spaces

In this section, we will prove that $M_{\mathcal{F}}$-modulus and $AM_{\mathcal{F}}$-modulus are the same if the space $\mathcal{F}$ is reflexive. In particular, Theorem 1 shows that M_p-modulus and AM_p-modulus are the same for $1<p<\infty$.

Proof of Theorem 1. Since $AM_{\mathcal{F}}(\Gamma)\le M_{\mathcal{F}}(\Gamma)$ by Remark 14, it suffices to prove the reverse inequality.

We may assume that $AM_{\mathcal{F}}(\Gamma)<\infty$.

Let $\delta>0$. Find an admissible sequence (ρ_i) for Γ such that

$$\liminf_{i\to\infty} \|\rho_i\|_{\mathcal{F}} \le AM_{\mathcal{F}}(\Gamma)+\delta. \tag{24}$$

Since the space $\mathcal{F}$ is reflexive and the sequence (ρ_i) is bounded, by [18, Section V.2, Theorem 1], there exist $\rho\in\mathcal{F}$ and a subsequence of (ρ_i), denoted again by (ρ_i), such that $\rho_i\to\rho$ weakly in $\mathcal{F}$ and satisfies (24). This subsequence is still admissible for Γ. Since $\rho_i\to\rho$ weakly in $\mathcal{F}$, by the Mazur lemma [18, Section V.1, Theorem 2], for each $k\in\mathbb{N}$, there exists a convex combination ν_k of $\rho_k, \rho_{k+1}, \dots$ such that

$$\|\nu_k-\rho\|_{\mathcal{F}} < 1/k.$$

By Lemma 22, a subsequence, denoted again by (ν_k), satisfies

$$\int_\gamma \nu_k\,ds \longrightarrow \int_\gamma \rho\,ds$$

for every $\gamma\in\overline{\Gamma}\subset\Gamma$, where $M_{\mathcal{F}}(\Gamma\setminus\overline{\Gamma})=0$. Let $\gamma\in\overline{\Gamma}$. Then by convexity,

$$\int_\gamma \nu_k\,ds \ge \inf_{i\ge k}\int_\gamma \rho_i\,ds, \quad k=1,2,\dots,$$

so that

$$\int_\gamma \rho\,ds = \lim_{k\to\infty}\int_\gamma \nu_k\,ds \ge \liminf_{i\to\infty}\int_\gamma \rho_i\,ds = 1.$$

It follows that ρ is admissible for $\overline{\Gamma}$. Then by Fatou Lemma [1, Lemma 1.5],

$$M_{\mathcal{F}}(\overline{\Gamma}) \le \|\rho\|_{\mathcal{F}} \le \liminf_{i\to\infty}\|\rho_i\|_{\mathcal{F}} \le AM_{\mathcal{F}}(\Gamma)+\delta.$$

Letting $\delta\to 0$, we obtain $M_{\mathcal{F}}(\overline{\Gamma})\le AM_{\mathcal{F}}(\Gamma)$. Since $M_{\mathcal{F}}(\Gamma\setminus\overline{\Gamma})=0$, we conclude that

$$M_{\mathcal{F}}(\Gamma)\le AM_{\mathcal{F}}(\Gamma). \quad \square$$

6. Estimates of approximation modulus

In this section we assume that the measure ν is doubling.

Definition 30. For $0\leq q$ let

$$co\underline{M}^q(E)=\liminf_{t\to 0}\frac{\nu(\{x\in X:d(x,E)<t\})}{t^q}$$

denote the lower Minkowski content of codimension q of a set $E\subset X$. By $coH^q(E)$ we denote the Hausdorff measure of codimension q of E defined as

$$coH^q(E)=\sup_{\delta>0} coH^q_\delta(E)$$

where for $\delta>0$

$$coH^q_\delta(E)=\inf\Big\{\sum_{j=1}^\infty \frac{\nu(B(x_j,r_j))}{r_j^q}: E\subset\bigcup_{j=1}^\infty B(x_j,r_j),\ \sup_j r_j<\delta\Big\}$$

is the δ-content associated with $coH^q(E)$.

It easily follows from the 5-covering lemma, see e.g. [2, Lemma 1.7], that $coH^q(E)\leq c_0\, co\underline{M}^q(E)$ where c_0 depends only on the doubling constant of ν.

Definition 31. We say that a path $\gamma:[0,\ell(\gamma)]\to X$ has a *meeting of order* $\alpha>0$ with a set $E\subset X$ if

$$\lim_{\delta\to 0}\frac{|\{s\in[0,\ell(\gamma)]:d(\gamma(s),E)<\delta\}|}{\delta^{1/\alpha}}=\infty.$$

Here $|A|$ stands for the Lebesgue measure of $A\subset\mathbf{R}$.

Theorem 32. *Suppose that* $co\underline{M}^q(E)<\infty$ *and that* Γ *is a family of paths in* X *such that each* $\gamma\in\Gamma$ *has a meeting of order* α *with* E. *Then* $AM_{L^{p,1}}(\Gamma)=0$ *provided that* $1\leq p\leq\alpha q$.

Proof. Pick first a sequence $1\geq\delta_1>\delta_2>...$ such that $\lim_{i\to\infty}\delta_i=0$ and for some $M<\infty$

$$\frac{\nu(\{x\in X:d(x,E)<\delta_i\})}{\delta_i^q}\leq M.$$

Let $\varepsilon>0$ and for $i=1,2,...$ let

$$\rho_i=\varepsilon\delta_i^{-1/\alpha}\chi_{E(\delta_i)}$$

where $E(t)=\{x\in X:d(x,E)<t\}$. Then by (10),

$$\|\rho_i\|_{L^{p,1}(X)}=\varepsilon\delta_i^{-1/\alpha}p\,(\nu(E(\delta_i)))^{\frac{1}{p}}\,.$$

We show that the sequence (ρ_i) is admissible for Γ.

To this end, fix $\gamma\in\Gamma$. Since γ has a meeting of order α with E, there is i_0 such that

$$\varepsilon\,|\{s\in[0,\ell(\gamma)]:\, d(\gamma(s),E)<\delta_i\}|\ge\delta_i^{1/\alpha}$$

for $i\ge i_0$. Now for $i\ge i_0$

$$\int_\gamma \rho_i\,ds=\varepsilon\delta_i^{-1/\alpha}\,|\{s\in[0,\ell(\gamma)]:d(\gamma(s),E)<\delta_i\}|\ge 1,$$

and so the sequence (ρ_i) is admissible for Γ.

Since $p/\alpha\le q$ and since (ρ_i) is admissible for Γ, we can use the sequence (ρ_i) to obtain

$$\begin{aligned} AM_{L^{p,1}}(\Gamma) &\le \liminf_{i\to\infty}\|\rho_i\|_{L^{p,1}(X)}\le\varepsilon\liminf_{i\to\infty}\frac{p\,\nu(E(\delta_i))^{1/p}}{\delta_i^{1/\alpha}} \\ &\le\varepsilon\liminf_{i\to\infty}\left(\frac{\nu(E(\delta_i))}{\delta_i^q}\right)^{1/p}\le M^{1/p}\varepsilon. \end{aligned}$$

Since M is independent of ε, we have $AM_{L^{p,1}}(\Gamma)=0$ as required. □

Remark 33. The above theorem is not true if the lower Minkowski content of codimension q is replaced by the Hausdorff measure of codimension q. Easy examples can be constructed by adding a countable set to the set E. This has no effect on $coH^q(E)$ but the meeting property holds for a much larger family Γ of paths than before and so the condition $AM_p(\Gamma)=0$ need not hold for $p\le\alpha q$.

Corollary 34. *Let $E\subset\mathbb{R}^n$ be an $(n-k)$-dimensional C^1 surface, $k\in\{1,2,...,n\}$, and Γ be a family of paths such that each $\gamma\in\Gamma$ has a meeting of order* 1 *with a compact part of E (depending on γ). Then $AM_{L^{k,1}}(\Gamma)=0$.*

Proof. Obviously, for each compact part K of E, $co\underline{M}^k(K)<\infty$. We can write E as $\bigcup_j W_j=\bigcup_j\overline{W}_j$, where W_j are open in E and with compact closures in E. Then $\Gamma=\bigcup_j\Gamma_j$, where each $\gamma\in\Gamma_j$ has a meeting of order 1 with the compact set $\overline{W}_j\subset E$. Now, we can use the countable subadditivity of the approximation modulus (Theorem 16). □

Proposition 35. *Let $E\subset X$ and $\Gamma_i(E)$ be as in Definition 3. Then each $\gamma\in\Gamma_i(E)$ has a meeting of order* 1 *with E.*

Proof. Choose $m\in\mathbb{N}$ and pick distinct points $t_1,...,t_m\in(0,\ell(\gamma))$ such that $\gamma(t_i)\in E$, $i=1,...,m$. Find $\delta>0$ such that the intervals $(t_i-\delta,t_i+\delta)$, $i=1,...,m$, are pairwise disjoint and contained in $(0,\ell(\gamma))$. Let $0<s<\delta$. Then for each $i=1,...,m$

and $t\in(t_i-s,t_i+s)$ we have $d(\gamma(t),E)<s$ due to the 1-Lipschitz property of γ. Hence

$$|\{t\in[0,\ell(\gamma)]\colon d(\gamma(t),E)<s\}|\geq 2ms.$$

Letting $m\to\infty$ we obtain the assertion. □

Corollary 36. *Let $E\subset X$ and $co\underline{M}^p(E)<\infty$. Then $AM_{L^{p,1}}(\Gamma_i(E))=0$.*

Remark 37. If we replace the assumption $co\underline{M}^p(E)<\infty$ by $coH^p(E)<\infty$, it can be proved that $AM_{L^p}(\Gamma_i(E))=0$, see [7]. Thus, for $p=1$ the conclusion of Corollary 36 can be obtained under a weaker assumption, whereas for $p>1$ the conclusion is also weaker (Lebesgue spaces instead of Lorentz spaces).

Corollary 38. *Let $k\in\{1,...,n-1\}$, $E\subset\mathbb{R}^n$ and $1\leq p\leq k$. Suppose that there exists an $(n-k)$-dimensional C^1 surface E' in $\mathbb{R}^n$ such that $\overline{E}\subset E'$. Then*

$$AM_{L^{p,1}}(\Gamma_i(E))=0.$$

Proof. Let $\gamma\in\Gamma_i(E)$. Then there exists an ball B which contains the locus of γ. The set $K=\overline{E}\cap\overline{B}$ is a compact subset of E' such that $\gamma\in\Gamma_i(K)$. By Proposition 35, γ has a meeting of order 1 with K. Therefore the assumptions of Corollary 34 are satisfied. □

Remark 39. The set

$$E=\bigcup_j\partial B(0,1-2^{-j})$$

is a C^1-surface. Consider the family Γ of all segments connecting points of $\partial B(0,1)$ with the origin. Then by [14, 3.12], $AM_1(\Gamma)>0$, although $\Gamma\subset\Gamma_i(E)$. Hence we cannot drop the assumption regarding the "supersurface" E' in Corollary 38.

Definition 40. For $E\subset X$ we define

$$\Gamma_\tau(E)=\Big\{\gamma\in\Gamma(E)\colon\gamma(0)\in E,\ \lim_{t\to0_+}\frac{d(\gamma(t),E)}{t}=0\Big\}.$$

Remark 41. If γ is right tangential in the sense of Definition 4, then there is a subpath of γ in $\Gamma_\tau(E)$.

Proposition 42. *Let $E\subset X$ and $\gamma\in\Gamma_\tau(E)$. Then γ has a meeting of order 1 with E.*

Proof. Choose $m\in\mathbb{N}$ and find $\delta>0$ such that

$$0<t<\delta\Longrightarrow d(\gamma(t),E)\leq\frac{t}{m}.$$

Let $s<\frac{\delta}{m}$. Then for each $t\in(0,ms)$ we have $t<\delta$ and $d(\gamma(t),E)\le s$. Thus,

$$|\{t\in[0,\ell(\gamma)]: d(\gamma(t),E)<s\}|\ge ms.$$

Letting $m\to\infty$ we obtain the assertion. □

The converse of Proposition 42 is not true. For example, consider the set $E=(-\infty,\infty)\times\{0\}$ in $\mathbf{R}^2$ and the curve γ with locus

$$[0,1]\times\{0\}\cup\cup_{i=1}^{\infty}\{2^{-i}\}\times[0,2^{-i}].$$

Corollary 43. *Let* $co\underline{M}^q(E)<\infty$. *Then* $AM_{L^{p,1}}(\Gamma_\tau(E))=0$ *provided that* $1\le p\le q$.

Corollary 44. *Let* $k\in\{1,...,n-1\}$, E *be an* $(n-k)$*-dimensional* C^1 *surface in* $\mathbb{R}^n$ *and* $1\le p\le k$. *Then*

$$AM_{L^{p,1}}(\Gamma_t(E))=AM_{L^{p,1}}(\Gamma_\tau(E))=0.$$

Proof. In view of Remark 41, $AM_{L^{p,1}}(\Gamma_t(E))\le AM_{L^{p,1}}(\Gamma_\tau(E))$. Let $\gamma\in\Gamma_\tau(E)$. If V is a neighborhood of $\gamma(0)$ in E such that $\overline{V}\subset E$, then $\gamma\in\Gamma_\tau(\overline{V})$ and by Proposition 42, γ has a meeting of order 1 with $\overline{V}$. Therefore $\Gamma_\tau(E)$ satisfies the assumptions of Corollary 34. □

7. Density-tangential curves

In this section, X will be a metric space with a doubling measure ν. We derive an estimate which will be needed in applications (see [7]). We are able to handle sets of finite Hausdorff measure, but the result does not seem to extend easily to Lorentz spaces.

Definition 45. Let μ be a finite Borel measure on X, $x\in X$ and $\tau>0$. We denote

$$R_\tau(x,\mu)=\inf\{r>0: r\mu(B(x,r))\ge\tau\nu(B(x,r))\}.$$

We say that γ is τ-density tangential to μ if

$$\lim_{t\to0+}\frac{R_\tau(\gamma(t),\mu)}{d(\gamma(t),\gamma(0))}=0.$$

Remark 46. Typically μ is the Hausdorff measure of codimension 1 restricted to some set E.

Theorem 47. *Let* $\Gamma_{\mu,\tau}$ *be the family of all paths which are* τ*-density tangential to* μ. *Then* $AM_1(\Gamma_{\mu,\tau})=0$.

Proof. For each $k\in\mathbb{N}$, consider the family of all balls $B(x,r)$ with the properties that $r\mu(B(x,r))\ge\tau\nu(B(x,r))$ and $0<r<\frac{1}{k}$. Using the Vitali type theorem, there exists a pairwise disjoint subfamily $\{B(x_i^k,r_i^k)\}_i$ of this family such that

$$F_k:=\left\{x\in X:\, R_\tau(x,\mu)<\frac{1}{k}\right\}\subset\bigcup_i B(x_i^k,5r_i^k).$$

Set

$$\rho_k(x)=\sum_i\frac{\chi_{B(x_i^k,6r_i^k)}}{r_i^k}.$$

Using the doubling property of ν we estimate

$$\begin{aligned}\int_X\rho_k\,d\nu&=\sum_i\frac{\nu(B(x_i^k,6r_i^k))}{r_i^k}\le\sum_i\frac{\nu(B(x_i^k,r_i^k))}{r_i^k}\\&\le\frac{C}{\tau}\sum_i\mu(B(x_i^k,r_i^k))\le\frac{C}{\tau}\,\mu(X).\end{aligned}$$

Pick $\gamma\in\Gamma_{\mu,\tau}$. Choose $m\in\mathbb{N}$ and find $\delta>0$ such that

$$0<t<\delta\implies\frac{R_\tau(\gamma(t),\mu)}{d(\gamma(t),\gamma(0))}<\frac{1}{20m}. \tag{25}$$

Choose $T<\delta$ such that $d(\gamma(T),\gamma(0))>0$. Find $k_m\in\mathbb{N}$ such that $\frac{20m}{k_m}<d(\gamma(T),\gamma(0))$. Let $k\ge k_m$. Find $0<t_1<\ldots<t_m$ such that

$$d(\gamma(t_j),\gamma(0))=\frac{20j}{k},\quad j=1,\ldots,m.$$

By (25),

$$R_\tau(\gamma(t_j),\mu)<\frac{1}{20m}d(\gamma(t_j),\gamma(0))=\frac{20j}{20mk}\le\frac{1}{k}.$$

Thus, for each j it holds that $\gamma(t_j)\in F_k$ and there exists $i(j)$ such that

$$\gamma(t_j)\in B(x_{i(j)}^k,5r_{i(j)}^k).$$

Since $r_i^k\le\frac{1}{k}$ and the mutual distance of $\gamma(t_j)$ is estimated below by $\frac{20}{k}$, the indices $i(j)$ are distinct. The path γ travels in each $B(x_{i(j)}^k,6r_{i(j)}^k)$ at least distance $r_{i(j)}^k$ and thus

$$\int_\gamma\rho_k\ge m,\quad k\ge k_m.$$

By Theorem 25 it follows that $AM_1(\Gamma_{\mu,\tau})=0$. $\square$

8. Estimates of M-modulus

In this section we assume that X is locally compact and ν is doubling.

We first recall some notions from analysis on metric measure spaces, see [2] and [6].

Definition 48. A Borel measurable function $\rho\colon X\to[0,\infty]$ is said to be an *upper gradient* to $u\colon X\to\mathbb{R}$ if

$$|u(y)-u(x)|\le\int_\gamma \rho\,ds$$

for each $x,y\in X$ and each path γ connecting x to y.

Let $B=B(z,R)$ be a ball and $E\subset B$. We say that a function $u\colon X\to\mathbb{R}$ is a cap-competitor for (E,B) if $u\ge1$ on E and $u=0$ on $X\setminus B$. The Dirichlet-Lorentz $DL^{p,1}$ seminorm of a ν-measurable function $u\colon X\to\mathbb{R}$ is defined by

$$\|u\|_{DL^{p,1}}:=\inf\{\|\rho\|_{L^{p,1}}: \rho \text{ is an upper gradient to } u\}.$$

The *Dirichlet-Lorentz $DL^{p,1}$-capacity* of a set $E\subset B$ is defined as

$$\operatorname{cap}_{DL^{p,1}}(E;B)=\inf\Big\{\|u\|^p_{DL^{p,1}}: u \text{ is a cap-competitor for } (E,B)\Big\}.$$

Remark 49. The Dirichlet type seminorms provide a homogeneous counterpart of more familiar Newtonian type norms, see [2], [6] and [12].

Theorem 50. *Let $p\ge1$ and let E be a subset of a relatively compact ball $B\subset X$ with $\operatorname{cap}_{DL^{p,1}}(E,B)>0$. Then $M_{L^{p,1}}(\Gamma_i(E)\cap\Gamma_\tau(E))=\infty$.*

Proof. (For a similar construction see [2, Lemma 5.25].) Let $\rho\in L^{p,1}(B)$, $\rho\ge0$, be a lower semicontinuous function. We may assume that ρ is bounded away from 0. For $x\in B$, let Γ_x be the family of all paths $\gamma\colon[0,\ell]\to\overline{B}$ such that $\gamma(0)=x$, $\gamma(\ell)\notin B$ and $\gamma((0,\ell))\subset B$. Set

$$u(x)=\inf\Big\{\int_\gamma\rho\,ds\colon \gamma\in\Gamma_x\Big\}.$$

Let x, $x_i\in\overline{B}$ be such that $x_i\to x$ and $\sup_i u(x_i)<\infty$. Consider a sequence (γ_i) such that $\gamma_i\in\Gamma_{x_i}$ and

$$\int_{\gamma_i}\rho\,ds<u(x_i)+2^{-i}.$$

Then, as ρ is bounded away from zero, the lengths of γ_i are bounded and we can reparametrize these paths to be defined on the same interval with a bounded Lipschitz constant. Then by the Arzela-Ascoli argument, we can find a subsequence converging to a limit path γ such that (as ρ is lower semicontinuous)

$$\int_\gamma\rho\,ds\le\liminf_i\int_{\gamma_i}\rho\,ds\le\liminf_i u(x_i).$$

Hence the function u is lower semicontinuous. Also we easily verify that ρ is an upper gradient of u. Assume first that $u=\infty$ on $E\cap B(z,R)$. Then all the functions u/j serve as cap-competitors for (E,B) and it follows that $\operatorname{cap}_{DL^{p,1}}(E,B)=0$, a contradiction. Thus, there exists $y\in E$ such that, under the notation above, $u(y)<\infty$. We can find a path parametrized by its arc length $\gamma\colon[0,\ell]\to X$ such that $\gamma\in\Gamma_y$ and $\int_\gamma\rho\,ds<\infty$.

Now, we can construct a path $\widetilde{\gamma}\in\Gamma_i(E)\cap\Gamma_\tau(E)$. The path $\widetilde{\gamma}$ has the same locus as the path γ but goes to $\gamma(0)$ infinitely often and has, at the same time, the required tangential property.

Find a sequence $\delta_m\searrow 0$ such that

$$\delta_1=\ell,$$

and

$$\sum_{m=1}^{\infty} m\int_0^{\delta_m}\rho(\gamma(t))\,dt<\infty. \tag{26}$$

Let q_m be positive integers such that

$$q_m\ge\frac{\delta_m}{\delta_{m+1}}. \tag{27}$$

Set

$$\begin{aligned} t_m&=\sum_{n>m}(2n+2)\delta_n, && m=0,1,2,...,\\ h_m&=\frac{\delta_m}{q_m}, && m=1,2,...\end{aligned}$$

Our plan is to find the new curve $\widetilde{\gamma}$ as

$$\widetilde{\gamma}(t)=\gamma(\xi(t)),\quad t\in[0,\tilde{\ell}],$$

where

$$\tilde{\ell}=t_0=\sum_{n=1}^{\infty}(2n+2)\delta_n$$

and $\xi\colon[0,\tilde{\ell}]\to[0,\ell]$ is defined as follows. Fix $m\in\mathbb{N}$ and set

$$\xi(t_{m-1}-s)=s,\quad 0\le s\le\delta_m.$$

Now, we define ξ on $[t_m,t_{m-1}-\delta_m]$ to be linear on each

$$[t_m+(i-1)h_m,t_m+ih_m],\quad i=1,...,(2m+1)q_m$$

and attaining the values

$$\left.\begin{aligned} \xi(t_m+((k-1)(2m+1)+2j)h_m) &= (k-1)h_m, \\ \xi(t_m+((k-1)(2m+1)+(2j+1))h_m) &= kh_m, \end{aligned}\right\} \quad \begin{aligned} j &= 1,\dots,m, \\ k &= 1,\dots,q_m. \end{aligned}$$

For completeness we set $\xi(0)=0$. Observe that $|\xi'|=1$ at all points except for the partition ones, so that $\widetilde{\gamma}$ is parametrized by its arclength.

Since $\widetilde{\gamma}(t_m)=\gamma(0)=y$ for each $m\in\mathbb{N}$, we verify that $\widetilde{\gamma}\in\Gamma_i(E)$.

Next, we want to show that $\widetilde{\gamma}$ is tangential according to the definition of $\Gamma_\tau(E)$. This reduces to the property

$$\lim_{t\to 0_+}\frac{\xi(t)}{t}=0. \tag{28}$$

Let $t\in[t_m,t_{m-1}]$ Then (using (27))

$$\frac{\xi(t)}{t}\le\frac{\frac{t-t_m}{2m+1}+h_m}{t-t_m+(2m+3)\delta_{m+1}}\le\frac{1}{2m+1}$$

which tends to 0 as $m\to\infty$ and (28) is verified.

By (26)

$$\int_{\widetilde{\gamma}}\rho\,ds=\sum_{m=1}^{\infty}(2m+2)\int_0^{\delta_m}\rho(\gamma(t))\,dt<\infty.$$

Making $\widetilde{\gamma}$ shorter, we can achieve that $\int_{\widetilde{\gamma}}\rho\,ds<1$ and still $\widetilde{\gamma}\in\Gamma_i(E)\cap\Gamma_\tau(E)$, so that ρ is not admissible for $\Gamma_i(E)\cap\Gamma_\tau(E)$. We have shown that there does not exist any lower semicontinuous admissible function for $\Gamma_i(E)\cap\Gamma_\tau(E)$ in $L^{p,1}(X)$. Since admissible functions for a curve family can be approximated by lower semicontinuous admissible functions, it follows that

$$M_{L^{p,1}}(\Gamma_i(E)\cap\Gamma_\tau(E))=\infty. \quad \square$$

Corollary 51. *Let $k\in\{1,\dots,n\}$ and $E\subset\mathbb{R}^n$ be a $(n-k)$-dimensional C^1 surface. Then $M_{L^{k,1}}(\Gamma_t(E)\cap\Gamma_i(E))=\infty$.*

Remark 52. In Euclidean spaces, the Dirichlet-Lorentz capacity $\operatorname{cap}_{DL^{p,1}}$ is just the homogeneous Sobolev-Lorentz capacity. By [10, Theorem 8.19 and Corollary 9.6],

$$H^{n-p}_\infty(E)\le C\operatorname{cap}_{DL^{p,1}}(E), \quad E\subset\mathbb{R}^n. \tag{29}$$

A similar estimate holds also in metric measure spaces satisfying the $(1,1)$-Poincaré inequality [8], see also [13]. For the Newtonian-Lorentz capacity see also [3] and [11].

9. Estimates of modulus: smooth tangential paths

In this section we provide an elementary proof of Corollary 51. In fact, we prove a stronger assertion, as the paths considered in the proof of Theorem 50 are not smooth. Since null sets for the modulus in consideration are obviously invariant with respect to smooth deformations, we may consider that our surface is flat.

Definition 53. Let $E\subset\mathbb{R}^n$ be a C^1 surface and $T_x(E)$ denote the tangent space to E at a point $x\in E$. Then $\Gamma_s(E)$ is the family of all paths $\gamma\colon [0,\ell]\to\mathbb{R}^n$ which are C^1-smooth in $[0,\ell]$ and satisfy

$$\gamma'_+(0)\in T_{\gamma(0)}(E).$$

Definition 54. Let $k\in\{1,...,n\}$ and

$$\mathbb{H}_{n-k}=\{x=(x_1,...,x_n)\in\mathbb{R}^n: x_{n-k+1}=...=x_n=0\}.$$

Define $D_k=\mathbb{H}_{n-k}\cap B_n(0,1)$.

Theorem 55. *Let $k\in\{1,...,n-1\}$. Then $M_{L^{k,1}}(\Gamma_s(D_k))=\infty$.*

Proof. Write $x\in\mathbb{R}^n$ as $x=(y,z)$, where $y\in\mathbb{R}^{n-k}$ and $z\in\mathbb{R}^k$. Denote

$$B_k=B_k(0,1),\quad S_{k-1}=\partial B_k,$$
$$B_{n-k}=B_{n-k}(0,1),\quad P=B_{n-k}\times B_k.$$

Then $D_k=B_{n-k}\times\{0_k\}$,

$$|z|=d((y,z),\,D_k)\quad\text{in } P$$

and a routine calculation shows that the function

$$(y,z)\longmapsto|z|^{1-k}$$

belongs to $L^{k',\infty}(P)=(L^{k,1}(P))^*$. Pick $\rho\in L^{k,1}(\mathbb{R}^n)$. Then the duality argument (see (9)) yields

$$\iint_P \rho(y,z)|z|^{1-k}\,dy\,dz<\infty.$$

Find a continuous function $f\colon[0,\infty)\to\mathbb{R}$ such that

$$\lim_{t\to 0_+}\frac{f(t)}{t^{k-1}}=0, \tag{30}$$

and still

$$\iint_P \frac{\rho(y,z)}{f(|z|)}\,dy\,dz<\infty. \tag{31}$$

Further assume that

$$\int_0^1 \frac{s^{k-1}}{f(s)}\,ds<\infty \tag{32}$$

and

$$0\le f(t)\le t^{k-1},\quad t\ge 0. \tag{33}$$

Let $\beta\colon [0,1]\to\mathbb{R}$ be the positive solution of the initial value problem

$$\begin{aligned}&\beta'(t)=\big(\beta(t)\big)^{1-k}f(\beta(t)),\\ &\beta(0)=0.\end{aligned} \tag{34}$$

The existence of such a solution is guaranteed by (32). By (33), we have $0\le\beta'(t)\le 1$. Thus we may set

$$\alpha(t)=\int_0^t \sqrt{1-(\beta'(s))^2}\,ds,\quad t\in[0,1]. \tag{35}$$

Given $y\in B_{n-k}$ and $\zeta\in S_{k-1}$, set

$$\gamma_{y,\zeta}(t)=(y+\alpha(t)\mathbf{e}_1,\ \beta(t)\,\zeta),\quad t\in[0,1],$$

where

$$\mathbf{e}_1=(1,0,...,0).$$

By (35), each $\gamma_{y,\zeta}$ is parametrized by its arclength and from (30) and (34) it follows that $\gamma_{y,\zeta}\in\Gamma_s(D_k)$. We consider the transformation of variables

$$(\tilde y,\ \tilde z)=\Phi(y,z)=\gamma_{y,\frac{z}{|z|}}(|z|),$$

so that

$$\Phi(y,t\zeta)=(y+\alpha(t)\mathbf{e}_1,\ \beta(t)\,\zeta),\quad y\in B_{n-k},\ \zeta\in S_{k-1},\ t\in(0,1).$$

Then

$$\Phi'(y,z)=\begin{pmatrix} I, & F\\ 0, & A\end{pmatrix},$$

where I is the $(n-k)\times(n-k)$ unit matrix, F is a $(n-k)\times k$ matrix and A is the derivative of the radial deformation

$$z\longmapsto \beta(|z|)\frac{z}{|z|}.$$

Now the Jacobian determinant $J\Phi$ of Φ has the form

$$J\Phi(y,z)=\det(A)=\Big(\frac{\beta(|z|)}{|z|}\Big)^{k-1}\beta'(|z|)$$

Using (34) we obtain

$$|z|^{1-k}=\frac{J\Phi(y,z)}{f(|\beta(z)|)}.$$

It follows

$$\begin{aligned}
&\int_{B_{n-k}}\Big(\int_{S_{k-1}}\Big(\int_{\gamma_{y,\zeta}}\rho\,ds\Big)\,d\mathcal{H}^{k-1}(\zeta)\Big)\,dy\\
&=\int_{B_{n-k}}\Big(\int_{S_{k-1}}\Big(\int_0^1\rho\big(y+\alpha(t)\mathbf{e}_1,\,\beta(t)\zeta\big)\,dt\Big)\,d\mathcal{H}^{k-1}(\zeta)\Big)\,dy\\
&=\int_{B_{n-k}}\Big(\int_0^1\Big(\int_{S_{k-1}}\rho\big(y+\alpha(t)\mathbf{e}_1,\,\beta(t)\zeta\big)\,d\mathcal{H}^{k-1}(\zeta)\Big)\,dt\Big)\,dy\\
&=\int_{B_{n-k}}\Big(\int_0^1\Big(\int_{\partial B_k}\rho\big(\Phi(y,t\zeta)\big)\,d\mathcal{H}^{k-1}(\zeta)\Big)\,dt\Big)\,dy\\
&=\int_{B_{n-k}}\Big(\int_0^1\Big(t^{1-k}\int_{\partial B_k(0,t)}\rho\big(\Phi(y,\xi)\big)\,d\mathcal{H}^{k-1}(\xi)\Big)\,dt\Big)\,dy\\
&=\int_{B_{n-k}}\Big(\int_{B_k}|z|^{1-k}\rho\big(\Phi(y,z)\big)\,dz\Big)\,dy\\
&=\int_{B_{n-k}}\Big(\int_{B_k}\rho\big(\Phi(y,z)\big)\frac{J\Phi(y,z)}{f(|\beta(z)|)}\,dz\Big)\,dy\\
&=\iint_{B_{n-k}\times B_k}\frac{\rho(\tilde y,\tilde z)}{f(|\tilde z|)}\,d\tilde y\,d\tilde z<\infty,
\end{aligned}$$

where the last integral is finite by (31). Therefore there must exist $(y,\frac{z}{|z|})\in B_{n-k}\times S_{k-1}$ such that

$$\int_{\gamma_{y,\frac{z}{|z|}}}\rho\,ds<\infty.$$

If we truncate the domain of $\gamma_{y,\frac{z}{|z|}}$, we obtain a subpath $\widetilde{\gamma}_{y,\frac{z}{|z|}}\in\Gamma_s(D_k)$ such that

$$\int_{\widetilde{\gamma}_{y,\frac{z}{|z|}}}\rho\,ds<1.$$

Therefore there is no admissible function for $\Gamma_s(D_k)$ in $L^{k,1}(P)$. We conclude that

$$M_{L^{k,1}}(\Gamma_s(D_k))=\infty.\quad\square$$

Remark 56. In contrast to Corollary 51, this result does not hold for $k=n$.

10. Comparison results

In this section, we show that in some instances, AM-modulus gives other results than the corresponding M-modulus. We prove Theorems 2 and 5 from the introduction.

Proof of Theorem 5. The assertion is a mere combination of Corollaries 44, 38 and 51. □

To show examples for p noninteger, we seek for a fractal set E in $\mathbb{R}^n$ such that

$$0 < \mathrm{cap}_{DL^{p,1}}(E) \quad \text{and} \quad co\underline{M}^p(E) < \infty.$$

By (29), the capacitary inequality is verified whenever $H^{n-p}(E) > 0$. (See Remark 52 which clarifies the situation.) On the other hand, for the simplest examples of sets with

$$H^{n-p}(E) < \infty$$

it also holds that $co\underline{M}^p(E) < \infty$, but it is not easy to find a proof in literature. Thus, we look for a fractal set K with

$$0 < H^{n-p}(K) < \infty$$

but the upper Hausdorff measure estimate must be refined to an estimate of the Minkowski content.

Definition 57. Let $0 < \lambda < 1$. Let K_0 be the unit interval $[0,1]$. If K_m is a disjointed union of 2^m intervals $I_1^m, \ldots I_{2^m}^m$ of length $2r_m = \left(\frac{1}{2}(1-\lambda)\right)^m$, we produce K_{m+1} by removing a concentric open interval of length $2\lambda r_m$ from each I_i^m. The resulting fractal

$$K = \bigcap_m K_m$$

is called the *middle λ Cantor set.*

Lemma 58. ([4, Example 4.5, Proposition 7.1]) *Let K be the middle λ Cantor set, K^n be the Cartesian product of n copies of K and*

$$s = \frac{\log 2}{\log\left(\frac{2}{1-\lambda}\right)}. \tag{36}$$

Then $H^{ns}(K^n) > 0$.

Lemma 59. *Let K be the middle λ Cantor set and s be as in (36). Then $co\underline{M}^{n-ns}(K^n) < \infty$.*

Proof. Let $0<r<1$ and find m such that $r_{m+1}\le r<r_m$. If x_i^m are the centers of I_i^m, then

$$\{x\in\mathbb{R}^n : d(x,K^n)<r\}\subset\Big(\bigcup_{i=1}^{2^m}(x_i^m-r_m-r,\, x_i^m+r_m+r)\Big)^n$$

and thus

$$|\{x\in\mathbb{R}^n : d(x,K^n)<r\}|\le(2^{m+1}(r_m+r))^n\le(2^{m+2}r_m)^n\lesssim(1-\lambda)^{nm},$$

whereas

$$r^{n-ns}\ge r_{m+1}^{n(1-s)}\gtrsim\big(\tfrac{1}{2}(1-\lambda)\big)^{nm(1-s)}=(1-\lambda)^{nm}.$$

Here symbols $\lesssim$, $\gtrsim$ mean inequalities up to a positive multiplicative constant independent of m. Now

$$\frac{|\{x\in\mathbb{R}^n : d(x,K^n)<r\}|}{r^{n-ns}}\le c$$

where the constant c is independent of r and letting $r\to 0$ we see that $co\underline{M}^{n-ns}(K^n)<\infty$ as required. □

Proof of Theorem 2. Let K be the middle λ Cantor set and s be given by (36). If λ is chosen so that $ns=n-p$ and $E=K^n$, then $co\underline{M}^p(E)<\infty$ and $coH^p(E)>0$. By Corollary 36, (29) and Theorem 50,

$$AM_{L^{p,1}}(\Gamma_i(E))=0 \quad\text{but}\quad M_{L^{p,1}}(\Gamma_i(E))=\infty. \quad\square$$

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